Transactions of the Nigerian Association of Mathematical Physics
Volume 5,(September and November, 2017), pp141 –144
© *Trans. of NAMP*

On the Influence of the Unit Inner Source of Heat onto the Thermoelastic Displacement in Green's Function.

*Udoh P. J. And **Nsien E.F.*

Department of Mathematics and Statistics, University of Uyo, Nigeria.

Abstract

This paper focuses on the influence function of the unit inner source of heat onto the thermoelastic displacement. The main function associated with the influence is the Green's function denoted by $U_k(x,\xi)$. Approaching this function from Maysel's formula, the result follows that the product of the dilatation caused by the unit concentrated body forces equals the displacement caused by internal temperature.

Keyword:

1. INTRODUCTION

In principle, to analyze properties and possibilities for the definition of the introduced influence functions in the theory of Green's elasticity, it seems quite enough to define this for the function $U_k(x,\xi)$ of the Green's family. [1]. Indeed, since the influence functions are deductions from the stated family of Green, functions, then the function $U_k(x,\xi)$ can be called the main function [2]. It shows the influence function in this case is due to the unit inner source of heat onto the thermoelastic displacements [3]. As discussed in [4], the Green's function in elastostatics gives the solution of the boundary value problems for the equation in terms of initially prescribed surface or displacements, body forces and respective components.
Many literature had discussed different methods in components of influence functions. But the issue of the unit source of heat has not been analyzed. In this paper we clarify the physical sense of the function $U_k(x,\xi)$ in a full measure using Maysel's formula [5-7]. The result follows that the product of the dilatation caused by the unit concentrated body forces equals the displacement caused by internal temperature.

2. MATHEMATICAL SETTINGS

The Green's function in variable separable is given as

$$G(x,\xi) = g(x,\xi) + f(x,\xi) \tag{1}$$

where $g(x,\xi) + f(x,\xi)$ are separable variables as in [8].
The influence function in question can be deduced from the equation [9] given by

$$U_i(\xi) = \gamma\int_V T_i(z)\Theta^{(k)}(z,\xi)dV(z) = \gamma\int_V G(x,z)\Theta^{(k)}(z,\xi)dV(z) = U_k(x,\xi) \tag{2}$$

where $\Theta^{(k)} = U^{(k)}_{j.j}$ is the thermoelastic bulk dilatation induced by the action of the body force $F = \delta(x-\xi),\ i,k = 1,2,3,$ γ is the Lame elastic constant. The inner heat source is given by

$$F = \delta(x-\xi) \tag{3}$$

onto the thermoelastic displacement.
Equation (2) is generated from the unit internal heat source with known boundary conditions as in [10].
The solution to this equation can also be given using Maysel's formula

$$U_i(\xi) = \gamma\int_V T_i(z)\Theta^{(k)}(z,\xi)dV(z) \tag{4}$$

From Eq.(4), we can clarify the physical sense of the function $U_k(x,\xi)$ in a full measure. It should be noted that from Maysel's formula in eqn (4) it follows that

Correspondence Author: Udoh P.J. E-mail: udohpattie@yahoo.com, Tel: +2348036683649

$$\gamma\,\Theta^{(k)}(x,\xi)=\hat{U}_k(x,\xi), \qquad (5)$$

which means that the product of the dilatation caused by the unit concentrated body forces $\delta_{ik}\delta(x-\xi)$ and γ equals to the displacements $\hat{U}_k(x,\xi)$, caused by the internal temperature

$$T_i=\delta(x-\xi). \qquad (6)$$

3. APPLICATION USING MAYSEL'S FORMULA

Upon substituting $T=\delta(x-z)$ into eqn (4) and accounting for the Dirac's function property given by the equation

$$\int_V f(z)\,\delta(z-\xi)\,dV(z) = f(\xi) \qquad (7)$$

and

$$\int_\Gamma f(y_0)\,\delta(y_0-y)\,d\Gamma(y_0) = f(y) \qquad (8)$$

we obtain the equality in (5) which allows eqn (2) to be re-written in the form

$$U_k(x,\xi)=\gamma\int_v G(x,z)\Theta^{(k)}(z,\xi)dV(z) = \int_v G(x,z)\hat{U}^{(k)}(z,\xi)dV(z). \qquad (9)$$

From eqn (9), it follows that the function $U_k(x,\xi)$ is determined by the integral over the body volume V of the product of the function describing the influence of the unit heat source, $F=\delta(x-z)$, onto the temperature $T=G(x,z)$ and the function describing the influence of the unit inner temperature $T=\delta(x-\xi)$ on the thermoelastic displacements $\hat{U}_k(x,\xi)$. Therefore, for the function $U_k(x,\xi)$ it is convenient to use a new term of a function of double transitive influence. Upon operating the variable x in eqn (9) with Laplacian and using the equation

$$\nabla^2 T(x)= a^{-1}F(x);\quad x,\xi\in V;\ \nabla_x^2\, G(x,\xi)=\delta(x-\xi) \qquad (10)$$

we have,

$$\nabla_x^2 U_k(x,\xi)=\gamma\int_v \nabla_x^2(x,z)\Theta^{(k)}(z,\xi)dV(z)=-\gamma\int_v \delta(x-z)\Theta^{(k)}(z,\xi)dV(z). \qquad (11)$$

together with the Dirac's function property given by eqns (7-8), one can get

$$\nabla_x^2 U_k(x,\xi)=-\gamma\,\Theta^{(k)}(x,\xi). \qquad (12)$$

Then, from eqn (11), it follows that with regard to the variables of the point $x\equiv(x_1,x_2,x_3)$ the boundary conditions for the function $U_k(x,\xi)$ are similar to ones for the function $G(x,\xi)$ determined by [Seremet] , i.e.

$$U_k(y,\xi)\big|_{\Gamma D}=0,\ y\in\Gamma_D;\ \partial U_k(y,\xi)/\partial n_y\big|_{\Gamma N}=0,\ y\in\Gamma_N;$$
$$\alpha U_k(y,\xi)+a\left(\partial U_k(y,\xi)/\partial n_y\right)\big|_{\Gamma M}=0,\ y\in\Gamma_M. \qquad (13)$$

Thus, from eqns (12), (13) and those in [11], it follows that to determine the influence functions $U_k(x,\xi)$ the corresponding boundary-value problem described by eqn (10) should be solved, the difference being that the unit heat source $a^{-1}F=\delta(x-\xi)$ should be replaced with the elastic dilatation $\Theta^{(k)}(x,\xi)$ multiplied by the coefficient γ.

Then, from the physical meaning of the function $\hat{U}_k(x,\xi)$ treated as displacements caused by the unit inner temperature, it follows that $\hat{U}_k(x,\xi)$ satisfies the respective boundary-conditions at $T=\delta(z-\xi)$. Indeed, from eqn (11) and from the boundary-value problem shown above for $\hat{U}_k(x,\xi)$ it follows that

$$\mu\nabla_\xi^2 U_k(x,\xi)+(\lambda+\mu)\Theta,_k(x,\xi)=\gamma\int_v G(x,z)\frac{\partial}{\partial\xi_k}\delta(z-\xi)dV(z). \qquad (14)$$

Then, by making use of the Dirac's function property given by eqns (7-8) we come to the conclusion that the influence function $U_k(x,\xi)$ determined by integral (11) satisfies the variables $\xi\equiv(\xi_1,\xi_2,\xi_3)$, the following boundary-value problem of thermoelastostatics:

$$\mu\nabla_{\xi}^{2}U_{k}(x,\xi)+(\lambda+\mu)\Theta,_{k}(x,\xi)-\gamma G,_{k}(x,\xi)=0;\qquad x,\xi\in V; \tag{15}$$

$$U_{k}(x,y)\big|_{\Gamma U}=0; y\in\Gamma_{U}=; i,k=1,2,3; \tag{16}$$

$$P_{k}(x,y)\big|_{\Gamma P}=0; P_{k}=\sigma_{jk}n_{j}; y\in\Gamma_{P}; \tag{17}$$

$$\beta_{1}U_{k}(x,y)+\beta_{2}P_{k}(x,y)\big|_{\Gamma UP}=0; y\in\Gamma_{UP}; \Gamma=\Gamma_{U}+\Gamma_{P}+\Gamma_{UP}. \tag{18}$$

4 ILLUSTRATIVE EXAMPLE

As an example, let us consider the generalization of a classical Poisson's formula for a half-space which is well known in the theory of harmonic potentials

$$T(\xi)=a^{-1}\int_{0}^{\infty}dx_{1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(x)\frac{1}{4\pi}\left[R^{-1}(x,\xi)-R_{1}^{-1}(x,\xi)\right]dx_{2}dx_{3} \tag{19}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}T(y)(2\pi)^{-1}\xi_{1}R^{-3}(y,\xi)dy_{2}dy_{3};\qquad y\equiv(0,y_{2},y_{3}), y\in\Gamma, \tag{20}$$

to a case of uncoupled thermoelasticity

EXAMPLE Determine the thermoelastic displacements in the elastic half-space $V(0\le x_{1}\le\infty,-\infty\le x_{2},x_{3}\le\infty)$ with the fixed boundary $\Gamma(y_{1}=0,-\infty\le y_{2},y_{3}\le\infty)$ caused by the surface temperature $T(y)$:

$$T=T(y);\qquad U_{i}(y)=0;\qquad y\equiv(0,y_{2},y_{3}),\qquad y\in\Gamma;\qquad i=1,2,3, \tag{21}$$

and the internal source of heat $F(x),\quad x\equiv(x_{1},x_{2},x_{3}),\quad x\in V.$

Computational Procedure: The desired thermoelastic displacements are determined by the following formula

$$U_{k}(\xi)=a^{-1}\int_{v}F(x)U_{k}(x,\xi)dV(x)-\int_{\Gamma}T(y_{0})\frac{\partial U_{k}(y_{0},\xi)}{\partial n_{0}}d\Gamma(y_{0}), \tag{22}$$

deduced from general formula (7.26) at $\Gamma_{N}\equiv\Gamma_{M}\equiv0$ and $\Gamma_{D}\equiv\Gamma$. Here, the influence functions $U_{k}(x,\xi)$ and $\partial U_{k}(y_{0},\xi)/\partial n_{0}$ are determined by eqn (2).. So, let us re-write formulae as in [12] in the form :

$$\Theta^{(k)}(x,\xi)=-(\lambda+2\mu)^{-1}\left[\frac{\partial}{\partial\xi_{k}}\frac{1}{4_{\pi}}\left(R^{-1}-R_{1}^{-1}\right)+\frac{1}{2\pi}L_{\Theta}^{(k)}\frac{\partial}{\partial\xi_{1}}R_{1}^{-1}\right]; \tag{23}$$

$$L_{\Theta}^{(k)}.f=\left(\delta_{1k}-B^{-1}\xi_{1}\frac{\partial}{\partial\xi_{k}}\right).f;\qquad k=1,2,3; \delta_{1k}=0, k\ne1; \delta_{1k}=1, k=1, \tag{24}$$

for dilatation and

$$G=G(x,\xi)=(4\pi)^{-1}\left(R^{-1}-R_{1}^{-1}\right);\qquad R=|x-\xi|; x\equiv(x_{1},x_{2},x_{3}); \xi\equiv(\xi_{1},\xi_{2},\xi_{3});$$
$$R_{1}=R(x,\xi_{1}^{\bullet});\qquad \xi_{1}^{\bullet}\equiv(-\xi_{1},\xi_{2},\xi_{3});\qquad B=(\lambda+3\mu)/(\lambda+\mu), \tag{25}$$

for the Green's function. Now, upon substituting them into eqn (9) and calculating the respective integrals we obtain the expression for the influence of a unit heat source on the thermoelastic displacements as following

$$U_{k}(x,\xi)=-\gamma\int_{0}^{\infty}dz_{1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\lambda+2\mu)^{-1}\left[\frac{1}{4_{\pi}}\frac{\partial}{\partial\xi_{k}}\left(R^{-1}(z,\xi)-R_{1}^{-1}(z,\xi)\right)\right.$$
$$\left.+L_{\Theta}^{(k)}\frac{1}{2_{\pi}}\frac{\partial}{\partial\xi_{1}}R_{1}^{-1}(z,\xi)\right]\frac{1}{4\pi}\left[R^{-1}(x,z)-R_{1}^{-1}(x,z)\right]dz_{2}dz_{3}$$
$$=\gamma[8\pi(\lambda+2\mu)]^{-1}\left\{\frac{\partial}{\partial\xi_{k}}\left[R(x,\xi)-R_{1}(x,\xi)\right]+2x_{1}L_{\Theta}^{(k)}R_{1}^{-1}(x,\xi)\right\}. \tag{26}$$

Now, having derived Eqn. (25) we are in a position to calculate the limit of the normal derivative with respect to the above displacements

4. DISCUSSION

The specific features of the introduced influence functions (functions of double transitive influence) are the consideration of both physical process (heat conduction and elasticity) in solids (in the sense of uncoupled thermoelasticity):

In the case of stationary heat conduction the function of the influence of the unit internal heat source onto the temperature field are the Green's functions of the respective boundary-value problems for the Laplacian. However it is known that for the domains of regular shape these functions either have already been constructed or can be constructed by the methods of mathematical physics [1, 2, 3, 8-12]. They satisfy the equations of the boundary-value problem in the theory of elasticity (16) to (18) (regarding the variables of the point ξ and the equations for the fictive boundary-value problem in the heat conduction (12), (13) (regarding to the variables of the point x.). They resulted from the direct influence of the initial prescribed factor (initial cause, which in our case is an initial inner source of heat) on the final desired result (in our case on the thermoelastic displacements).

5. CONCLUSION

To conclude,we note that both the introduced influence functions and suggested integral formulae can be realized in terms of elementary functions for a great number of thermoelastic bodies of a regular shape. Therefore, for the bodies with a regular shape there is a real possibility to generalize the known classical integral formulae from the theory of harmonic potentials onto uncoupled thermoelasticity.

References

[1] Seremet V.D. (2001) Some new influence functions and integral solutions in theory of thermal stresses. In: Proceedings of the IV-th international congress on thermal stresses, 8–11 June 2001, Osaka, Japan, pp 423–427Google Scholar

[2] Seremet,V. D. (1988). Fundamental Solutions of Problems in the theory of elasticity. Izv,Vuzov 85-88

[3]. Melnikov, Iu. A and Koshnarjova, V. A. (1994). Green's matrices and 2-D elasto-potentials for external boundary- value problems

[2] Seremet, V. D (1990). Constructing Green's Functions and Matrices for Mixed boundary values in the theory of Elasticity for the space and ½, ¼, 1/8 space.

[4] Udoh, P. J. (2011). On Generalized Green's Functions and Formulae in Uncoupled Thermoelastostatics. West African J. Sc and Tech. Vol1 (1)

[5] Seremet VD (2001) Some new influence functions and integral solutions in theory of thermal stresses. In: Proceedings of the IV-th international congress on thermal stresses, 8–11 June 2001, Osaka, Japan, pp 423–427Google Scholar

[6] Udoh, P. J. (2013). On Integral Representations For The First Basic Problem Of The Theory of Elasticity in Green's Function . World Journal of Applied Science and Technology, Vol. 5, No.2. pp218-223

[7] Greenberg, M. D. (1971). Application of Green's functions in Science and Engineering, Prentice-Hall,Inc., Englewood, N. J.

[8] Udoh, P. J. and Eyo, A. E. (2010). Application of Separation of Variables in Green's Function to Typical Half-Strip Problem for Elastic Material. Maejo International Journal of Science and Technology. University of Thailand.Vol.4(2). pp 88-92. ISSN 1905-7873

[9] Udoh, P. J., Abasiekwere, U. A.,Ekong, E. E.(2010). On Typical Elastic Problem for Rectangular Strip in Green's Function Using Separation of Variables. J.Math. Phy.Vol. 16

[10] Udoh, P. J. (2011). On Generalized Green's Functions and Formulae in Uncoupled Thermoelastostatics. West African J. Sc and Tech. Vol1 (1)

[11] Udoh, P. J **(2013).** On the separation of variables in Green's functions for elastic material. World Journal of Applied Science and Technology, Vol. 5, No.2. pp 184-190

[12] Erbe, H .H. (1974). Thermoelastic effects in incompressible elastic solids. Mech. Res. Comm.,1, 137 – 142.