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Vibration Analysis Of A Clamped-Clamped Rayleigh Beam Resting On Vlasov foundation And Under Partially Distributed Loads

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Abstract

The response of a prestressed Rayleigh beam with uniform cross-section and finite length supported by a two-parameter Vlasov foundation subjected to a partially distributed moving load is studied in this paper. The governing differential equation are obtained and are solved by paying attention on the boundary condition of the problem at the clamped-clamped end of the beam resting on a two-parameter foundation. The governing equation of the problem is evaluated using generalized Finite Integral Transform in conjunction with variation of parameters and Fresnel sine and Fresnel cosine identities. Numerical results are presented both in figures and graphs to demonstrate the behaviour of the beam-foundation system for various values of the foundation parameters of the problem of the moving load. Hence, this further confirms that certain parameters of the moving load must always be taken into consideration for accurate and safe assessment of the response to moving load which plays a vital role in structural design.

Keyword: Elastic beam, vibration, two-parameter foundation, moving load.

1. INTRODUTION

In recent years considerable attentions has been given to the response of elastic beams resting on elastic subgrade which is one of the structural engineering problems of theoretical and practical interest Celep et al [1]. A large number of studies have been devoted to the subject. Most of the early works in this area were directed at the dynamic of structures under the moving loads. Moving loads have been idealized as moving concentrated loads which acts at a certain point in the structure and along a single line segment. These include the work of Krylov [2], Gbadeyan and Oni [3] Belotserkovskiy [4]. The dynamic response of a simply supported transverse beam by a concentrated moving load was studied by Stanisic and Hardin [5]. They developed an interesting technique which, however, cannot easily be applied to various boundary conditions which are of practical interest. Akin and Mofid [6] presented an analytic numerical method that can be used to determine the dynamic behaviour of beams carrying concentrated moving mass. The problems of dynamic behaviour of an elastic beam subject to a moving concentrated mass were also studied by Sadiku and Leipholz [7]. Gbadeyan and Oni [8] presented a more versatile technique which can be used to determine the dynamic behaviour of beams having arbitrary end supports. It is remarked at this juncture that the elastic parameter of the beams in all the work discussed hitherto, are assumed constant. Although, the above completed works were impressive, only concentrated moving loads were considered. However, such loads do not represent the physical reality of the problem formulation. As a matter of fact, concentrated load do not exists physically. Moving loads are actually distributed over a small segment or over the entire length of the structure. To this end,Isede and J.A. Gbadeyan [9] carried out the analysis of a variable cross-section Timoshenko beam subjected to a moving partially distributed load. Finite element method with Lagrangian interpolation function was used to model the structure. Other recent works involving uniformly distributed moving mass model were carried out by Kargarnov in and Younesian [10], Wu [11], Bogacz and Czyczula [12]. More recently, Andi et al [13] investigate the dynamic behaviour of a finite simply supported uniform Rayleigh beam under travelling distributed loads resting on a one-parameter Winkler foundation. Generally speaking, Winkler foundation model assumed the foundation reaction to be proportional to the vertical displacement of the

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foundation at the same point. However, the Winkler model has various shortcomings due to the independence of the springs. Because the springs are assumed to be independent and unconnected to each other, no interaction exists between the springs. In order to eliminate the deficiency of Winkler model, improved theories have been introduced on refinement of Winkler's model by visualizing various types of interconnections such as shear layers and beams along the Winkler springs [14]. These theories have been attempted to find an applicable and simple model of representation of foundation medium.
In this paper, the response of a clamped-clamped Rayleigh beam will be obtained. The beam is subjected to partially distributed load and rested on a more reliable two-parameter elastic foundation known as Vlasov foundation.

2. Theoretical formulation

The problem of the flexural motion of a prestressed finite uniform Rayleigh beam resting on bi-Parametric Vlasov foundation to an arbitrary number of uniforms partially distributed moving masses moving at non-uniform velocities is considered. The Rayleigh beam has a constant cross sectional area and the mass M is assume to touch the beam at time $t = 0$ and travel across it with a non-uniform velocity such that the motion of the contact point of the moving load is given by

$$f(t) = x_0 + ct + \frac{1}{2}at^2 \tag{1}$$

where x_0 is the point of application of force $P(x,t)$ at the instance $t = 0$, c is the initial velocity and a is the constant acceleration of motion.
The equation of motion describing the lateral displacement of the Rayleigh beam is given by the fourth order partial differential equation

$$\begin{aligned} &EI\frac{\partial^4 V(x,t)}{\partial x^4} - N\frac{\partial^2 V(x,t)}{\partial x^2} + \mu\frac{\partial^2 V(x,t)}{\partial t^2} - \mu R^0\frac{\partial^4 V(x,t)}{\partial x^2 \partial t^2} - G\frac{\partial^2 V(x,t)}{\partial x^2} + KV(x,t) \\ &+ MH\left[x - \left(x_0 + ct + \tfrac{1}{2}at^2\right)\right]\left[\frac{\partial^2 V(x,t)}{\partial t^2} + 2(c+at)\frac{\partial^2 V(x,t)}{\partial x \partial t} + \left(c + at^2\right)\frac{\partial^2 V(x,t)}{\partial x^2}\right. \\ &\left. + a\frac{\partial V(x,t)}{\partial x}\right] = MgH\left[x - \left(x_0 + ct + \tfrac{1}{2}at^2\right)\right] \end{aligned} \tag{2}$$

where EI is the flexural rigidity of the structure, N is the axial force, R^0 is the rotatory inertia factor, μ is the mass per unit length of the beam, G is the shear modulus, K is the elastic foundation stiffness, $V(x,t)$ is the transverse displacement, M is the transverse distributed load, x is the spatial coordinate and t is the time, $H(\cdot)$ is the Heaviside function.

The boundary conditions of the structure under consideration is arbitrary and the initial conditions without any loss of generality is taken as

$$V(x,0) = 0 = \frac{\partial V(x,0)}{\partial t} \tag{3}$$

Since the load is assumed to be of mass M and the time t is assumed to be limited to that interval of time within the mass on the beam, that is

$$0 \le f(t) \le L \tag{4}$$

3. Analytical procedures

Equation (2) is a fourth order partial differential equation which has some coefficients which are not only variable but also singular.In this section, a general approach is developed in order to solve the initial value problem. The approach involves expressing the Heaviside function as a series form and then reducing the modified form of the equation above using the generalized finite integral transform. The resulting transformed differential equation having some variable coefficients is then simplified using modified Struble's asymptotic technique.

3.1 The generalized finite integral transform

For the dynamical systems, the governing equation is a fourth order partial differential equation with variable and singular coefficients. The Generalized Finite Integral Transform is employed to remove the singularities in the governing equations and to reduce it to a sequence of second order ordinary differential equations with variable coefficients. The generalized finite integral transform is defined by

$$\tilde{V}(m,t)=\int_0^L V(x,t)U_m(x)dx \tag{5}$$

with inverse

$$V(x,t)=\sum_{m=1}^{\infty}\frac{\mu}{V_m}\tilde{V}(m,t)U_m(x) \tag{6}$$

where

$$V_m=\int_0^L \mu U_m^2(x)dx \tag{7}$$

where $U_m(x)$ is any function chosen such that the pertinent boundary conditions are satisfied. An appropriate selection of functions for beam problems are beam mode shapes. Thus, for a uniform beam, the mth normal mode of vibration

$$U_m(x)=Sin\frac{\lambda_m x}{L}+A_m Cos\frac{\lambda_m x}{L}+B_m Sinh\frac{\lambda_m x}{L}+C_m Cosh\frac{\lambda_m x}{L} \tag{8}$$

is chosen as a suitable kernel of the integral transform (3.13), where A_m, B_m, C_m are constants and the mode frequencies λ_m can be determined using appropriate classical boundary conditions.

Applying the generalized finite integral transform (5), equation (2) becomes

$$Z_1A(0,\ L,t)+Z_1T_A(t)-Z_2T_B(t)+\tilde{V}_{tt}(m,t)-R^0T_C(t)+Z_3T_B(t)+Z_4\tilde{V}(m,t)+T_D(t)$$
$$+T_E(t)+T_F(t)=\frac{Mg}{\mu}T_G(t) \tag{9}$$

where

$$Z_1=\frac{EI}{\mu},\ Z_2=\frac{N}{\mu},\ Z_3=\frac{G}{\mu},\ Z_4=\frac{K}{\mu} \tag{10}$$

$$A(0,L,t)=\frac{\partial^3V(x,t)}{\partial x^3}U_m(x)-\frac{\partial^2V(x,t)}{\partial x^2}U'_m(x)+\frac{\partial V(x,t)}{\partial x}U''_m(x)-V(x,t)U'''_m(x) \tag{11}$$

$$T_A(t)=\int_0^L V(x,t)\frac{\partial^4U_m(x)}{\partial x^4}dx$$

$$T_B(t)=\int_0^L \frac{\partial^2V(x,t)}{\partial x^2}U_m(x)dx$$

$$T_C(t)=\int_0^L \frac{\partial^4V(x,t)}{\partial x^2\partial t^2}U_m(x)dx$$

$$T_D(t)=\frac{M}{\mu}\int_0^L H\left[x-\left(x_0+ct+\tfrac{1}{2}at^2\right)\right]\frac{\partial^2V(x,t)}{\partial t^2}U_m(x)dx$$

$$T_E(t)=\frac{2M(c+at)}{\mu}\int_0^L H\left[x-\left(x_0+ct+\tfrac{1}{2}at^2\right)\right]\frac{\partial^2V(x,t)}{\partial x\partial t}U_m(x)dx$$

$$T_F(t)=\frac{M(c+at)^2}{\mu}\int_0^L H\left[x-\left(x_0+ct+\tfrac{1}{2}at^2\right)\right]\frac{\partial V(x,t)}{\partial x}U_m(x)dx$$

$$T_G(t)=\int_0^L H\left[x-\left(x_0+ct+\tfrac{1}{2}at^2\right)\right]U_m(x)dx$$

It is noted that integrals (12) are singular and in order to handle these singularities, use is made of the Fourier series representation for the Heaviside unit step function namely:

$$H\left[x-\left(x_0+ct+\frac{1}{2}at^2\right)\right]=\frac{1}{4}+\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{Sin(2n+1)\pi\left[x-\left(x_0+ct+\frac{1}{2}at^2\right)\right]}{2n+1},\ \ 0<x<L \tag{13}$$

Simplifying (12) in conjunction with (13), equation (9) after some simplifications and rearrangements yields

$$\tilde{V}_{tt}(m,t)+\left(\omega_m^2+\frac{K}{\mu}\right)\tilde{V}(m,t)-G\sum_{k=1}^{\infty}\tilde{V}(k,t)B_A(k,m)-R^0\sum_{k=1}^{\infty}\tilde{V}_{tt}(k,t)B_B(k,m)-\frac{N}{\mu}\sum_{k=1}^{\infty}\tilde{V}(k,t)B_A(k,m)$$
$$+\varepsilon_0\left\{\sum_{k=1}^{\infty}L\left[\tilde{V}_{tt}(k,t)B_c(k,m)+\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{Cos(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}\tilde{V}_{tt}(k,t)B_D(n,k,m)\right.\right.$$
$$-\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{Sin(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}\tilde{V}_{tt}(k,t)B_E(n,k,m)+\frac{(c+at)}{2}\tilde{V}_t(k,t)B_F(k,m)$$

$$+\frac{2(c+at)}{\pi}\sum_{n=0}^{\infty}\frac{Cos(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}\tilde{V}_t(k,t)B_G(n,k,m)$$

$$-\frac{2(c+at)}{\pi}\sum_{n=0}^{\infty}\frac{Sin(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}\tilde{V}_t(k,t)B_H(n,k,m)+\frac{(c+at)^2}{4}\tilde{V}(k,t)B_I(k,m)$$

$$+\frac{2(c+at)^2}{\pi}\sum_{n=0}^{\infty}\frac{Cos(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}\tilde{V}(k,t)B_J(n,k,m)$$

$$-\frac{2(c+at)^2}{\pi}\sum_{n=0}^{\infty}\frac{Sin(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}\tilde{V}(k,t)B_K(n,k,m)+\frac{a}{4}\tilde{V}(k,t)B_L(k,m)$$

$$+\frac{a}{\pi}\sum_{n=0}^{\infty}\frac{Cos(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}\tilde{V}(k,t)B_M(n,k,m)$$

$$\left.\left.-\frac{a}{\pi}\sum_{n=0}^{\infty}\frac{Sin(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}\tilde{V}(k,t)B_N(n,k,m)\right]\right\}=\frac{PL}{\lambda_m\mu}\Big[-Cos\lambda_m+A_mSin\lambda_m+B_mCosh\lambda_m$$

$$+C_mSinh\lambda_m+Cos\frac{\lambda_m}{L}\left(x_0+ct+\frac{1}{2}at^2\right)-A_mSin\frac{\lambda_m}{L}\left(x_0+ct+\frac{1}{2}at^2\right)-B_mCosh\frac{\lambda_m}{L}\left(x_0+ct+\frac{1}{2}at^2\right) \quad (14)$$

$$-C_mSinh\frac{\lambda_m}{L}\left(x_0+ct+\frac{1}{2}at^2\right)\Big]$$

where

$$\varepsilon_0=\frac{M}{\mu L} \quad (15)$$

$$B_A(k,m)=\int_0^L\frac{1}{\tau_k(x)}U_k''(x)U_m(x)dx \quad (16a)$$

$$B_B(k,m)=\int_0^L\frac{1}{\tau_k(x)}U_k''(x)U_m(x)dx \quad (16b)$$

$$B_C(k,m)=\int_0^L\frac{1}{\tau_k(x)}U_k(x)U_m(x)dx \quad (16c)$$

$$B_D(n,k,m)=\int_0^L\frac{1}{\tau_k(x)}Sin(2n+1)\pi xU_k(x)U_m(x)dx \quad (16d)$$

$$B_E(n,k,m)=\int_0^L\frac{1}{\tau_k(x)}Cos(2n+1)\pi xU_k(x)U_m(x)dx \quad (16e)$$

$$B_F(k,m)=\int_0^L\frac{1}{\tau_k(x)}U_k'(x)U_m(x)dx \quad (16f)$$

$$B_G(n,k,m)=\int_0^L\frac{1}{\tau_k(x)}Sin(2n+1)\pi xU_k'(x)U_m(x)dx \quad (16g)$$

$$B_H(n,k,m)=\int_0^L\frac{1}{\tau_k(x)}Cos(2n+1)\pi xU_k'(x)U_m(x)dx \quad (16h)$$

$$B_I(k,m)=\int_0^L\frac{1}{\tau_k(x)}U_k''(x)U_m(x)dx \quad (16i)$$

$$B_J(n,k,m)=\int_0^L\frac{1}{\tau_k(x)}Sin(2n+1)\pi xU_k''(x)U_m(x)dx \quad (16j)$$

$$B_K(n,k,m)=\int_0^L\frac{1}{\tau_k(x)}Cos(2n+1)\pi xU_k''(x)U_m(x)dx \quad (16k)$$

$$B_L(k,m)=\int_0^L\frac{1}{\tau_k(x)}U_k'(x)U_m(x)dx \quad (16l)$$

$$B_M(n,k,m)=\int_0^L\frac{1}{\tau_k(x)}Sin(2n+1)\pi xU_k'(x)U_m(x)dx \quad (16m)$$

$$B_N(n,k,m)=\int_0^L\frac{1}{\tau_k(x)}Cos(2n+1)\pi xU_k'(x)U_m(x)dx \quad (16n)$$

$$\tau_k(x) = \int_0^L U_k^2(x)dx \tag{17}$$

$$\varpi_m^2 = \frac{\phi^4}{L^4}\frac{EI}{\mu} \tag{18}$$

Equation (14) is the transformed equation describing the problem of transverse vibration of Rayleigh beam on Vlasovelastic foundation and traversed by uniform partially distributed masses moving at varying velocities. In what follows, two cases of equation (14) are considered.

3.2 Solution of the transformed governing equation

Case I: The Moving Force Uniform Rayleigh Beam Problem

The differential equation describing the behaviour of a uniform Rayleigh beam on a Vlasov elastic subgrade to a moving force moving at variable velocity may be obtained from equation (14) by setting $\varepsilon_0 = 0$. In this case, one obtains

$$\tilde{V}_{tt}(m,t) + \left(\omega_m^2 + \frac{K}{\mu}\right)\tilde{V}(m,t) - G\sum_{k=1}^{\infty}\tilde{V}_{tt}(k,t)B_A(k,m) - R^0\sum_{k=1}^{\infty}\tilde{V}_{tt}(k,t)B_B(k,m) - \frac{N}{\mu}\sum_{k=1}^{\infty}\tilde{V}_{tt}(k,t)B_A(k,m)$$

$$\tilde{V}_{tt}(m,t) + \left(\omega_m^2 + \frac{K}{\mu}\right)\tilde{V}(m,t) - G\sum_{k=1}^{\infty}\tilde{V}_{tt}(k,t)B_A(k,m) - R^0\sum_{k=1}^{\infty}\tilde{V}_{tt}(k,t)B_B(k,m)$$
$$-\frac{N}{\mu}\sum_{k=1}^{\infty}\tilde{V}_{tt}(k,t)B_A(k,m) = \frac{PL}{\lambda_m\mu}\Big[-Cos\lambda_m + A_m Sin\lambda_m + B_m Cosh\lambda_m + C_m Sinh\lambda_m$$
$$+ Cos\frac{\lambda_m}{L}\left(x_0 + ct + \frac{1}{2}at^2\right) - A_m Sin\frac{\lambda_m}{L}\left(x_0 + ct + \frac{1}{2}at^2\right) -$$
$$B_m Cosh\frac{\lambda_m}{L}\left(x_0 + ct + \frac{1}{2}at^2\right) - C_m Sinh\frac{\lambda_m}{L}\left(x_0 + ct + \frac{1}{2}at^2\right)\Big] \tag{19}$$

Equation (19) is a model, in which the inertia effects of the moving load is considered negligible and only its force effects are of interest. Evidently, an exact analytical solution to the equation is not feasible. Though the equation is amenable to numerical technique, an analytical approximate method is desirable as solutions so obtained often shed light on vital information about the dynamical system. To this end a modification of the asymptotic method due to Struble's often used in treating weakly homogeneous and non-homogeneous nonlinear oscillatory systems is resorted to. Hence equation (19) is rearranged to take the form

\
$$\tilde{V}_{tt}(m,t) + \frac{\gamma_{mf}^2\tilde{V}(m,t)}{\left[1-\varepsilon^* LB_B(m,m)\right]} - \frac{\varepsilon^*}{\left[1-\varepsilon^* LB_B(m,m)\right]}\left[\sum_{\substack{k=1\\k\neq m}}^{\infty} LB_A(k,m)\tilde{V}_{tt}(k,t) + F_0\sum_{\substack{k=1\\k\neq m}}^{\infty} LB_A(k,m)\tilde{V}_{tt}(k,t)\right]$$
$$= \frac{P}{\mu\lambda_m\left[1-\varepsilon^* LB_B(m,m)\right]}\Big[-Cos\lambda_m + A_m Sin\lambda_m + B_m Cosh\lambda_m + C_m Sinh\lambda_m + Cos\frac{\lambda_m}{L}\left(x_0 + ct + \frac{1}{2}at^2\right)$$
$$- A_m Sin\frac{\lambda_m}{L}\left(x_0 + ct + \frac{1}{2}at^2\right) - B_m Cosh\frac{\lambda_m}{L}\left(x_0 + ct + \frac{1}{2}at^2\right) - C_m Sinh\frac{\lambda_m}{L}\left(x_0 + ct + \frac{1}{2}at^2\right)\Big] \tag{20}$$

where

$$\varepsilon^* = \frac{R^0}{L},\ F_0 = \frac{(G+N)L}{R^0\mu},\ \omega_{mk}^2 = \omega_m^2 + \frac{K}{\mu},\ \gamma_{mf}^2 = \omega_{mk}^2 - \varepsilon^* F_0 B_A(m,m),\ B_A(m,m) = B_A(m,m)\big|_{k=m} \tag{21}$$

By this technique, one seeks the modified frequency corresponding to the frequency of the free system due to the presence of the effect of rotatory inertia. An equivalent free system operator defined by the modified frequency then replaces equation (20). Thus, we set the right hand side of (20) to zero and considered a parameter $\eta << 1$ for any arbitrary ratio ε^*, defined as

$$\eta = \frac{\varepsilon^*}{1+\varepsilon^*} \tag{22}$$

so that

$$\varepsilon^* = \eta + O(\eta^2) \tag{23}$$

Substituting equation (23) in the homogeneous part of equation (20) one obtains

$$\tilde{V}_{tt}(m,t)+\gamma_{mf}^{2}\left[1+\eta LB_{B}(m,m)\right]\tilde{V}(m,t)-\eta\left[1+\eta LB_{B}(m,m)\right]\times \left[\sum_{\substack{k=1\\k\neq m}}^{\infty}LB_{A}(k,m)\tilde{V}_{tt}(k,t)+F_{0}\sum_{\substack{k=1\\k\neq m}}^{\infty}LB_{A}(k,m)\tilde{V}_{tt}(k,t)\right]=0 \quad (24)$$

When η is set to zero in equation (20) a situation corresponding to the case in which the rotatory inertia effect is regarded as negligible is obtained, then the solution of (20) can be written as

$$\bar{V}_{mf}=\lambda(m,t)Cos\left[\omega_{mf}t-\psi_{mf}\right] \quad (25)$$

where $\lambda(m,t)$, ω_{mf}, ψ_{mf} are constants. Since $\eta<1$, Struble's technique requires that the asymptotic solutions of the homogeneous part of equation (20) be of the form

$$\bar{V}(m,t)=\lambda(m,t)Cos\left[\gamma_{mf}t-\phi(m,t)\right]+\eta\Phi_{1}+O(\eta^{2}) \quad (26)$$

where $\lambda(m,t)$ and $\phi(m,t)$ are slowly varying functions of time.

To obtain the modified frequency, equations (26) and its derivatives are substituted into equation (24) and neglecting terms which do not contribute to the variational equations, one obtains

$$-2\dot{\lambda}(m,t)\gamma_{mf}Sin\left[\gamma_{mf}-\phi(m,t)\right]+2\lambda(m,t)\gamma_{mf}\dot{\phi}(m,t)Cos\left[\gamma_{mf}-\phi(m,t)\right] -\lambda(m,t)\gamma_{mf}^{2}Cos\left[\gamma_{mf}-\phi(m,t)\right]+\left[\gamma_{mf}^{2}-\eta L\gamma_{mf}^{2}B_{B}(m,m)\right]\lambda(m,t)Cos\left[\gamma_{mf}-\phi(m,t)\right]=0 \quad (27)$$

retaining terms to $O(\eta)$ only.

The variational equations are obtained by equating the coefficients of $Sin\left[\gamma_{mf}-\phi(m,t)\right]$ and $Cos\left[\gamma_{mf}-\phi(m,t)\right]$ on both sides of equation (27), we obtain

$$-2\dot{\lambda}(m,t)\gamma_{mf}=0 \quad (28)$$

and

$$2\lambda(m,t)\dot{\phi}(m,t)-\eta L\gamma_{mf}^{2}B_{B}(m,m)\lambda(m,t)=0 \quad (29)$$

Solving equations (28) and (29) respectively gives

$$\lambda(m,t)=D_{mf}^{*} \quad (30)$$

and

$$\phi(m,t)=\frac{\eta L\gamma_{mf}B_{B}(m,m)t}{2}+\psi_{mf} \quad (31)$$

where D_{mf}^{*} and ψ_{mf} are constants.

Therefore, when the inertia effect of the moving mass is considered, the first approximation to the homogeneous system is

$$\bar{V}(m,t)=D_{mf}^{*}Cos\left[\gamma_{bj}t-\psi_{mf}\right] \quad (32)$$

where

$$\gamma_{bj}=\gamma_{mf}\left[1-\frac{\eta LB_{B}(m,m)}{2}\right] \quad (33)$$

represents the modified natural frequency due to the effect of the rotatory inertia. It is observed that when $\eta=0$, we recover the frequency of the moving force problem when the rotatory inertia effect of the beam is considered negligible. Thus, to solve the non-homogeneous equation (20), the differential operator which act on $\bar{V}(m,t)$ and $\bar{V}(k,t)$ are replaced by the equivalent free system operator defined by the modified frequency γ_{bj}, thus, using (32), equation (20) can be written as

$$\frac{d^{2}}{dt^{2}}\bar{V}(m,t)+\gamma_{bj}^{2}\bar{V}(m,t)=P_{bj}^{0}\Big[-Cos\lambda_{m}+A_{m}Sin\lambda_{m}+B_{m}Cosh\lambda_{m}+C_{m}Sinh\lambda_{m} +Cos\frac{\lambda_{m}}{L}\left(x_{0}+ct+\frac{1}{2}at^{2}\right)-A_{m}Sin\frac{\lambda_{m}}{L}\left(x_{0}+ct+\frac{1}{2}at^{2}\right) -B_{m}Cosh\frac{\lambda_{m}}{L}\left(x_{0}+ct+\frac{1}{2}at^{2}\right)-C_{m}Sinh\frac{\lambda_{m}}{L}\left(x_{0}+ct+\frac{1}{2}at^{2}\right)\Big] \quad (34)$$

where

$$P_{bj}^0 = \frac{P}{\mu\lambda_m\left[1-\eta LB_B(m,m)\right]} \tag{35}$$

The general solution of equation (34) with the initial conditions gives expression for $\overline{V}(m,t)$ which on inversion yields

$$\begin{aligned}
\overline{V}(x,t) = \frac{1}{\rho_m(x)}\sum_{m=1}^{\infty}\Bigg\{&\frac{P_{bj}^o Sin\gamma_{bj}t}{\gamma_{bj}}\big[P_{13}S(d_{12}+d_{10}t)+P_{14}C(d_{12}+d_{10}t)+P_{11}S(d_{11}+d_{10}t)\\
&+P_{12}C(d_{11}+d_{10}t)+P_{21}C(d_{11}+d_{10}t)-P_{22}S(d_{11}+d_{10}t)+P_{23}C(d_{12}+d_{10}t)\\
&-P_{24}S(d_{12}+d_{10}t)-Q_{12}erf(d_{21}+d_{20}t)-Q_{11}erfi(d_{21}+d_{20}t)-Q_{13}erf(d_{22}+d_{20}t)\\
&-Q_{14}erfi(d_{22}+d_{20}t)+Q_{21}erf(d_{21}+d_{20}t)-Q_{22}erfi(d_{21}+d_{20}t)+Q_{23}erf(d_{22}+d_{20}t)\\
&-Q_{24}erfi(d_{22}+d_{20}t)+\frac{1}{2\gamma_{bj}}\big[Sin(\lambda_m-\gamma_{bj}t)-Sin(\lambda_m+\gamma_{bj}t)+A_m\big(Cos(\lambda_m-\gamma_{bj}t)\\
&-Cos(\lambda_m+\gamma_{bj}t)\big)-B_m\big(Sinh(\lambda_m+i\gamma_{bj}t)-Sinh(\lambda_m-i\gamma_{bj}t)\big)\\
&-iC_m\big(Cosh(\lambda_m+i\gamma_{bj}t)-Cosh(\lambda_m-i\gamma_{bj}t)\big)\big]-F_2^*\big]-\frac{P_{bj}^0 Cos\gamma_{bj}t}{\gamma_{bj}}\big[-P_{11}C(d_{11}+d_{10}t)\\
&+P_{12}S(d_{11}+d_{10}t)+P_{13}C(d_{12}+d_{10}t)-P_{14}S(d_{12}+d_{10}t)+P_{21}S(d_{11}+d_{10}t)\\
&+P_{22}C(d_{11}+d_{10}t)-P_{23}S(d_{12}+d_{10}t)-P_{24}C(d_{12}+d_{10}t)+iQ_{11}erfi(d_{21}+d_{20}t)\\
&-iQ_{12}erf(d_{21}+d_{20}t)-iQ_{13}erfi(d_{22}+d_{20}t)+iQ_{14}erf(d_{22}+d_{20}t)+iQ_{21}erf(d_{21}+d_{20}t)\\
&+iQ_{22}erfi(d_{21}+d_{20}t)-iQ_{23}erf(d_{22}+d_{20}t)-iQ_{24}erfi(d_{22}+d_{20}t)\\
&+\frac{1}{2\gamma_{bj}}\big[Cos(\lambda_m-\gamma_{bj}t)+\cos(\lambda_m+\gamma_{bj}t)-A_m\big(Sin(\lambda_m-\gamma_{bj}t)+Sin(\lambda_m+\gamma_{bj}t)\big)\\
&-B_m\big(Cosh(\lambda_m+i\gamma_{bj}t)+Cosh(\lambda_m-i\gamma_{bj}t)\big)-C_m\big(Sinh\lambda_m+i\gamma_{bj}t)+Sinh(\lambda_m-i\gamma_{bj}t)\big)\big]\\
&+F_1^*\big]\Bigg\}\left(Sin\frac{\lambda_m x}{L}+A_m Cos\frac{\lambda_m x}{L}+B_m Sinh\frac{\lambda_m x}{L}+C_m Cosh\frac{\lambda_m x}{L}\right)
\end{aligned} \tag{36}$$

where $C(x)$ and $S(x)$ are the well-known time-dependent Fresnel integrals defined by

$$S(x)=\int Sin(az^2)dz=\frac{1}{\sqrt{a}}\sqrt{\frac{\pi}{2}}S\left(\sqrt{a}\sqrt{\frac{2}{\pi}}z\right) \tag{37a}$$

$$C(x)=\int Cos(az^2)dz=\frac{1}{\sqrt{a}}\sqrt{\frac{\pi}{2}}C\left(\sqrt{a}\sqrt{\frac{2}{\pi}}z\right) \tag{37b}$$

and

$$P_{11}=\frac{1}{2\sqrt{a}}\sqrt{\frac{\pi}{2}}Cos\left(\frac{b_1^2}{4a}-c_0\right),\qquad P_{12}=\frac{1}{2\sqrt{a}}\sqrt{\frac{\pi}{2}}Sin\left(\frac{b_1^2}{4a}-c_0\right),$$

$$P_{14}=\frac{1}{2\sqrt{a}}\sqrt{\frac{\pi}{2}}Sin\left(\frac{b_2^2}{4a}-c_0\right),\qquad P_{13}=\frac{1}{2\sqrt{a}}\sqrt{\frac{\pi}{2}}Cos\left(\frac{b_2^2}{4a}-c_0\right),$$

$$P_{21}=\frac{A_m}{2\sqrt{a}}\sqrt{\frac{\pi}{2}}Cos\left(\frac{b_1^2}{4a}-c_0\right),\qquad P_{22}=\frac{A_m}{2\sqrt{a}}\sqrt{\frac{\pi}{2}}Sin\left(\frac{b_1^2}{4a}-c_0\right)$$

$$P_{23}=\frac{A_m}{2\sqrt{a}}\sqrt{\frac{\pi}{2}}Cos\left(\frac{b_2^2}{4a}-c_0\right),\qquad P_{24}=\frac{A_m}{2\sqrt{a}}\sqrt{\frac{\pi}{2}}Sin\left(\frac{b_2^2}{4a}-c_0\right),$$

$$Q_{11}=\frac{B_m\sqrt{\pi}}{8\sqrt{a}}e^{-\frac{b_3^2}{4a}-c_0}e^{2c_o},\quad Q_{12}=\frac{B_m\sqrt{\pi}}{8\sqrt{a}}e^{-\frac{b_3^2}{4a}-c_0}e^{\frac{b_3^2}{2a}},\quad Q_{13}=\frac{B_m\sqrt{\pi}}{8\sqrt{a}}e^{-\frac{b_4^2}{4a}-c_0}e^{2c_0}$$

$$Q_{14}=\frac{B_m\sqrt{\pi}}{8\sqrt{a}}e^{-\frac{b_4^2}{4a}-c_0}e^{\frac{b_4^2}{2a}},\quad Q_{21}=\frac{C_m\sqrt{\pi}}{8\sqrt{a}}e^{-\frac{b_3^2}{4a}-c_o}e^{\frac{b_3^2}{2a}},\quad Q_{22}=\frac{C_m\sqrt{\pi}}{8\sqrt{a}}e^{-\frac{b_3^2}{4a}-c_0}e^{2c_0},$$

$$Q_{23}=\frac{C_m\sqrt{\pi}}{8\sqrt{a}}e^{-\frac{b_4^2}{4a}-c_0}e^{\frac{b_4^2}{2a}},\qquad Q_{24}=\frac{C_m\sqrt{\pi}}{8\sqrt{a}}e^{-\frac{b_4^2}{4a}-c_0}e^{2c_0}$$

$$d_{10}=\frac{2a}{\sqrt{2\pi a}},\quad d_{11}=\frac{b_1}{\sqrt{2\pi a}},\quad d_{12}=\frac{b_2}{\sqrt{2\pi a}},\quad d_{20}=\frac{2a}{2\sqrt{a}},\quad d_{21}=\frac{b_3}{2\sqrt{a}},\quad d_{22}=\frac{b_4}{2\sqrt{a}}$$

$$F_1^*=-P_{11}C(d_{11})+P_{12}S(d_{11})+P_{13}C(d_{12})-P_{14}S(d_{12})+P_{21}S(d_{11})+P_{22}C(d_{11})-P_{23}S(d_{12})-P_{24}C(d_{12})$$
$$+iQ_{11}erfi(d_{21})-iQ_{12}erf(d_{21})-iQ_{13}erfi(d_{22})+iQ_{14}erf(d_{22})+iQ_{21}erf(d_{21})+iQ_{22}erfi(d_{21})$$
$$-iQ_{23}erf(d_{22})-iQ_{24}erfi(d_{22})+\frac{1}{\gamma_{bj}}\left[Cos\lambda_m-A_mSin\lambda_m-B_mCosh\lambda_m-C_mSinh\lambda_m\right]$$

$$F_2^*=P_{13}S(d_{12})+P_{14}C(d_{12})+P_{11}S(d_{11})+P_{12}C(d_{11})+P_{21}C(d_{11})-P_{22}S(d_{11})+P_{23}C(d_{12})$$
$$-P_{24}S(d_{12})-Q_{12}erf(d_{21})-Q_{11}erfi(d_{21})-Q_{13}erfi(d_{22})-Q_{14}erfi(d_{22})+Q_{21}erf(d_{21})$$
$$-Q_{22}erfi(d_{21})+Q_{23}erf(d_{22})-Q_{24}erfi(d_{22})$$

$$i=\sqrt{-1}$$

and

$$\rho_m(x)=\int_0^L U_m^2(x)dx \tag{38}$$

The equation (36) represents the transverse displacement response to forces moving at variable velocities of a prestressed uniform Rayleigh beam resting on Vlasov elastic foundation and having arbitrary end support conditions.

Case II: The Moving Mass Uniform Rayleigh Beam Problem

In this case, the mass of the moving load is considered commensurable with that of the structure and as such, the inertia effect of the moving load is not negligible. All the components of inertia terms are retained in the governing equation (14) and the solution of the entire equation is sought. This is termed the ***moving mass problem***. An exact closed form solution is not possible; hence an approximate analytical method due to Struble is resorted to. It is remarked at this junction that neglecting the terms representing the inertia term of the moving mass, we obtain equation (20). The homogeneous part of this equation can be replaced by a free system operator defined by the modified frequency γ_{bj} due to the presence of the effect of rotatory inertia. To this end, equation (14) can be written in the form

$$\tilde{V}_{tt}(m,t)+\gamma_{bj}^2\tilde{V}(m,t)+\varepsilon_0\left\{\sum_{k=1}^{\infty}L\left[\frac{1}{4}\tilde{V}_{tt}(k,t)B_c(k,m)+\frac{1}{\pi}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{Cos(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}V_{tt}(k,t)B_D(n,k,m)\right.\right.$$
$$-\frac{1}{\pi}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{Sin(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}V_{tt}(k,t)B_E(n,k,m)+\frac{(c+at)}{2}\sum_{k=1}^{\infty}\tilde{V}_t(k,t)B_F(k,m)$$
$$+\frac{2(c+at)}{\pi}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{Cos(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}V_t(k,t)B_G(n,k,m)$$
$$-\frac{2(c+at)}{\pi}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{Sin(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}V_t(k,t)B_H(n,k,m)+\frac{(c+at)^2}{4}\sum_{k=1}^{\infty}\tilde{V}(k,t)B_I(k,m)$$
$$+\frac{(c+at)^2}{\pi}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{Cos(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}V(k,t)B_J(n,k,m)$$
$$-\frac{(c+at)^2}{\pi}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{Sin(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}V(k,t)B_K(n,k,m)+\frac{a}{4}\sum_{k=1}^{\infty}\tilde{V}(k,t)B_L(k,m)$$
$$+\frac{a}{\pi}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{Cos(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}V(k,t)B_M(n,k,m)$$
$$\left.\left.-\frac{a}{\pi}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{Sin(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}V(k,t)B_N(n,k,m)\right]\right\}=\frac{P^*}{\mu B_0}\left[-Cos\lambda_m+A_mSin\lambda_m+B_mCosh\lambda_m\right.$$
$$+C_mSinh\lambda_m+Cos\frac{\lambda_m}{L}\left(x_0+ct+\frac{1}{2}at^2\right)-A_mSin\frac{\lambda_m}{L}\left(x_0+ct+\frac{1}{2}at^2\right)-B_mCosh\frac{\lambda_m}{L}\left(x_0+ct+\frac{1}{2}at^2\right)$$
$$\left.\left.\left.-C_mSinh\frac{\lambda_m}{L}\left(x_0+ct+\frac{1}{2}at^2\right)\right]\right]\right\} \tag{39}$$

As in the previous case, an exact analytical solution to the above equation is not possible. Thus, the same techniqueused in case I is employed to obtain the modified frequency due to the presence ofmoving mass, namely

$$\gamma_{mm} = \gamma_{bj}\left\{1-\frac{\eta_1}{8}\left[B_C(m,m)-\frac{\left(c^2 B_I(m,m)+aB_L(m,m)\right)}{\gamma_{bj}^2}\right]\right\} \tag{40}$$

where

$$\eta_1 = \frac{\varepsilon_0}{1+\varepsilon_0}, \quad \varepsilon_0 = \frac{N}{\mu L}, \quad B_c(m,m) = B_c(k,m)\Big|_{k=m}$$

$$B_I(m,m) = B_I(k,m)\Big|_{k=m}, \quad B_L(m,m) = B_L(k,m)\Big|_{k=m} \tag{41}$$

retaining $O(\eta)$ only.

Thus, equation (39) takes the form

$$\frac{d^2\bar{V}(m,t)}{dt^2} + \gamma_{mm}^2\bar{V}(m,t) = \frac{\eta_1 g L^2}{\lambda_m B_0}\Big[-Cos\lambda_m + A_m Sin\lambda_m + B_m Cosh\lambda_m + C_m Sinh\lambda_m + Cos\frac{\lambda_m}{L}\left(x_0+ct+\tfrac{1}{2}at^2\right)$$
$$- A_m Sin\frac{\lambda_m}{L}\left(x_0+ct+\tfrac{1}{2}at^2\right) - B_m Cosh\frac{\lambda_m}{L}\left(x_0+ct+\tfrac{1}{2}at^2\right) - C_m Sinh\frac{\lambda_m}{L}\left(x_0+ct+\tfrac{1}{2}at^2\right)\Big]\Big\} \tag{42}$$

This is analogous to equation (34). Thus, using similar argument as in moving force $\tilde{V}(m,t)$ can be obtained and on inversion yields

$$\bar{V}(x,t) = \sum_{m=1}^{\infty}\frac{\eta_1 g L^2}{\lambda_m\gamma_{mm}\rho_m(x)B_0}\Big\{Sin\gamma_{mm}t\big[P_{13}S(d_{12}+d_{10}t)+P_{14}C(d_{12}+d_{10}t)+P_{11}S(d_{11}+d_{10}t)+P_{12}C(d_{11}+d_{10}t)$$
$$+P_{21}C(d_{11}+d_{10}t)-P_{22}S(d_{11}+d_{10}t)+P_{23}C(d_{12}+d_{10}t)-P_{24}S(d_{12}+d_{10}t)-Q_{12}erf(d_{21}+d_{20}t)$$
$$-Q_{11}erfi(d_{21}+d_{20}t)-Q_{13}erf(d_{22}+d_{20}t)-Q_{14}erfi(d_{22}+d_{20}t)+Q_{21}erf(d_{21}+d_{20}t)$$
$$-Q_{22}erfi(d_{21}+d_{20}t)+Q_{23}erf(d_{22}+d_{20}t)-Q_{24}erfi(d_{22}+d_{20}t)+\frac{1}{2\gamma_{mm}}\big[Sin(\lambda_m-\gamma_{mm}t)$$
$$-Sin(\lambda_m+\gamma_{mm}t)+A_m\left(Cos(\lambda_m-\gamma_{mm}t)-Cos(\lambda_m+\gamma_{mm}t)\right)-iB_m\left(Sinh(\lambda_m+i\gamma_{mm}t)-Sinh(\lambda_m-i\gamma_{mm}t)\right)$$
$$-iC_m\left(Cosh(\lambda_m+i\gamma_{mm}t)-Cosh(\lambda_m-i\gamma_{mm}t)\right)\big]-F_2^*\big]-Cos\gamma_{mm}t\big[-P_{11}C(d_{11}+d_{10}t)$$
$$+P_{12}S(d_{11}+d_{10}t)+P_{13}C(d_{12}+d_{10}t)-P_{14}S(d_{12}+d_{10}t)+P_{21}S(d_{11}+d_{10}t)+P_{22}C(d_{11}+d_{10}t)$$
$$-P_{23}S(d_{12}+d_{10}t)-P_{24}C(d_{12}+d_{10}t)+iQ_{11}erfi(d_{21}+d_{20}t)-iQ_{12}erf(d_{21}+d_{20}t)-iQ_{13}erfi(d_{22}+d_{20}t)$$
$$+iQ_{14}erf(d_{22}+d_{20}t)+iQ_{21}erf(d_{21}+d_{20}t)+iQ_{22}erfi(d_{21}+d_{20}t)-iQ_{23}erf(d_{22}+d_{20}t)$$
$$-iQ_{24}erfi(d_{22}+d_{20}t)+\frac{1}{2\gamma_{mm}}\big[Cos(\lambda_m-\gamma_{mm}t)+\cos(\lambda_m+\gamma_{mm}t)-A_m\left(Sin(\lambda_m-\gamma_{mm}t)+Sin(\lambda_m+\gamma_{mm}t)\right)$$
$$-B_m\left(Cosh(\lambda_m+i\gamma_{mm}t)+Cosh(\lambda_m-i\gamma_{mm}t)\right)-C_m\left(Sinh\lambda_m+i\gamma_{mm}t)+Sinh(\lambda_m-i\gamma_{mm}t)\right)\big]$$
$$+F_1^*\big]\Big\}\left(Sin\frac{\lambda_m x}{L}+A_m Cos\frac{\lambda_m x}{L}+B_m Sinh\frac{\lambda_m x}{L}+C_m Cosh\frac{\lambda_m x}{L}\right) \tag{43}$$

Equation (43) represents the transverse displacement response to distributed masses, moving with non-uniform velocity of a highly prestressed Rayleigh beam resting onVlasovelastic subgrade. Equation (43) is valid for all variants of classical boundary conditions.

4. Applications

In this section, we shall illustrate the foregoing analysis by two practical examples. Particularly we shall consider classical boundary conditions such as clamped-clamped end conditions and clamped-free end conditions.

4.1 Clamped-Clamped End Conditions

In this case, both deflection and slope vanish. Thus

$$V(0,t)=0=V(L,t), \qquad \frac{\partial V(0,t)}{\partial x}=0=\frac{\partial V(L,t)}{\partial x} \tag{44}$$

and for normal modes

$$U_m(0)=0=U_m(L), \qquad \frac{\partial U_m(0)}{\partial x}=0=\frac{\partial U_m(L)}{\partial x} \tag{45}$$

which implies that

$$U_k(0)=0=U_k(L), \qquad \frac{\partial U_k(0)}{\partial x}=0=\frac{\partial U_k(L)}{\partial x} \tag{46}$$

Applying (45) to (8) yields

$$A_m = \frac{Sinh\lambda_m - Sin\lambda_m}{Cos\lambda_m - Cosh\lambda_m} = \frac{Cos\lambda_m - Cosh\lambda_m}{Sin\lambda_m + Sinh\lambda_m} = -C_m \text{ and } B_m = -1 \tag{47}$$

The frequency equation becomes

$$Cos\lambda_m Cosh\lambda_m = 1 \quad (48)$$

It follows from (48) that

$$\lambda_1 = 4.73004, \quad \lambda_2 = 7.85320, \quad \lambda_3 = 10.99561 \quad (49)$$

The expression for A_k, B_k, C_k and the corresponding frequency equation are obtained by a simple interchange of m with k in equation (47) and (48). Thus, the general solutions of the associated moving force and moving mass problems are obtained by substituting equations (47) and (48) into equations (36) and (43) respectively to obtain the displacement response to a moving force and moving mass respectively of a clamped-clamped uniform Rayleigh beam resting on a Vlasov elastic foundation.

5. Comments on closed form solutions

In an undamped system, it is pertinent to establish conditions under which resonance occurs. This phenomenon in structural and highway engineering is of great concern to researchers or in particular, design engineers because, it magnifies the amplitude of vibration in relatively undamped systems and can cause catastrophic failure in improperly constructed structures including bridges, buildings and airplanes – a phenomenon known as resonance disaster.

For the resonance of other classical boundary conditions, equation (36) clearly shows that the uniform elastic beam resting on Vlasov elastic foundation and traversed by partially distributed forces moving with variable speed reaches a state of resonance whenever

$$\gamma_{bj} = \frac{\lambda_m c_c}{L}, \quad \gamma_{bj} = \frac{\lambda_m c_c}{L} + \frac{2at_c}{L} \quad (50)$$

while equation (43) shows that the same beam under the action of moving mass experiences resonance effect whenever

$$\gamma_{mm} = \frac{\lambda_m c_c}{L}, \quad \gamma_{mm} = \frac{\lambda_m c_c}{L} + \frac{2at_c}{L} \quad (51)$$

From equation (40),

$$\gamma_{mm} = \gamma_{bj}\left\{1 - \frac{\eta_1}{8}\left[B_C(m,m) - \frac{\left(c^2 B_I(m,m) + aB_L(m,m)\right)}{\gamma_{bj}^2}\right]\right\} \quad (52)$$

which implies

$$\gamma_{bj} = \frac{\frac{\lambda_m c_c}{L}}{\left\{1 - \frac{\eta_1}{8}\left[B_C(m,m) - \frac{\left(c^2 B_I(m,m) + aB_L(m,m)\right)}{\gamma_{bj}^2}\right]\right\}} \quad (53)$$

Equations (51) and (53) show that for the same natural frequency, the critical speed for the same system consisting of a uniform Rayleigh beam resting on Vlasov elastic foundation and traversed by a moving partially distributed force is greater than that traversed by a moving partially distributed mass. Thus resonance is reached earlier in the moving partially distributed mass system than in the moving partially distributed force system.

6. Numerical results and discussion.

We shall illustrate the foregoing analysis in this paper by considering a uniform Rayleigh beam of length L=12.192m, modulus of elasticity $E = 3.1\times10^{10} N/m^2$, the moment of inertia $I = 2.87698\times10^{-3} m^4$ and the mass per unit length of the beam $\mu = 2758.291 kg/m$. The values of the foundation stiffness K varied between $0N/m^3$ and $4000000\, N/m^3$, axial force N is varied between $0N$ and $2\times10^6 N$ and shear modulus G is varied between $0N/m$ and $3\times10^5 N/m$. The transverse deflections of Rayleigh beam are calculated and plotted against time for various values of foundation stiffness K, axial force N, shear modulus G and rotatory inertia R^0.

Figure 1 displays the deflection profile of a clamped-clamped uniform Rayleigh beam under the action of partially distributed forces moving at variable velocity for various values of foundation stiffness K and fixed values of axial force N=20000, shear modulus G=10000 and rotatory inertia correction factor R^0=0.5. The figure shows that as the foundation stiffness K increases, the transverse displacement of the uniform Rayleigh beam decreases. Similar results are obtained when the clamped-clamped beam is subjected to partially distributed masses travelling at variable velocity as shown in figure 5. For various travelling time t, the transverse displacement of the beam for various values of axial force N and for fixed values of foundation stiffness K=40000, shear modulus G=10000 and rotatory inertia correction factor R^0=0.5 are shown in figure 2. It is observed that higher values of axial force N reduce the transverse displacement of the beam. The same behaviour characterizes the deflection profile of the clamped-clamped beam under the action of partially distributed masses moving at variable velocity for various values of axial force N as shown in figure 6.

In figure 3 the response amplitudes of the clamped-clamped uniform Rayleigh beam to partially distributed forces travelling at variable velocity for various values of shear modulus G and for fixed values of foundation stiffness K=40000, axial force N=20000 and rotatory inertia correction factor R^0=0.5 is displayed. It is seen from the figure that as the values of shear modulus increases, the response amplitude of the clamped-clamped uniform Rayleigh beam under the action of partially distributed forces travelling at variable velocity decreases. Similar results are obtained when the clamped-clamped beam is subjected to a partially distributed masses travelling at variable velocity as shown in figure 7. Also in figure 4, the deflection profile of clamped-clamped uniform Rayleigh beam under the action of uniform partially distributed forces is displayed. It is clearly shown that as we increase the values of rotatory inertia correction factor R^0, the deflection profile of the uniform beam reduces for fixed values of foundation stiffness K, axial force N and shear modulus G. Figure 8 displays the response amplitude of the clamped-clamped Rayleigh beam to partially distributed masses travelling at variable velocities for various values of rotatory inertia correction factor R^0 and for fixed values of foundation stiffness K=40000, axial force N=20000 and shear modulus G=10000. It is seen from the figure that as the values of rotatory inertia correction factor increases, the response amplitude of the clamped-clamped uniform beam under the action of partially distributed masses travelling at variable velocities decreases.

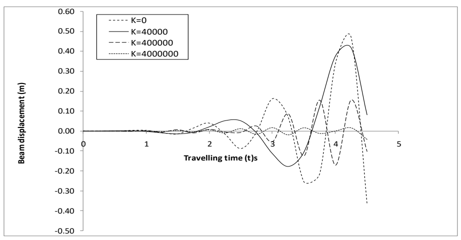

Figure 1: Transverse displacement of a clamped-clamped uniform Rayleigh beam under the actions of partially distributed forces travelling at variable velocity for various values of foundation stiffness K

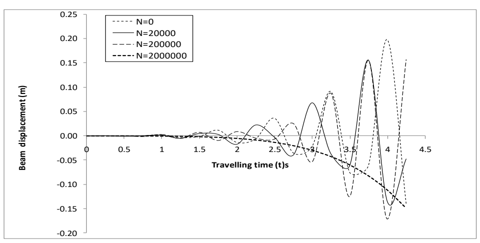

Figure 2: Deflection profile of a clamped-clamped uniform Rayleigh beam under the actions of partially distributed forces travelling at variable velocity for various values of axial force

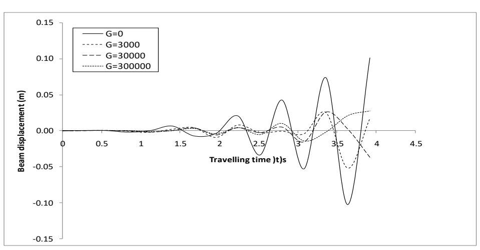

Figure 3: Transverse displacement of a clamped-clamped uniform Rayleigh beam under the actions of partially distributed forces travelling at variable velocity for various values of shear modulus G

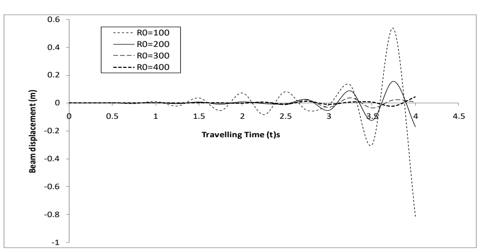

Figure 4: Response amplitude of a clamped-clamped uniform Rayleigh beam under the actions of partially distributed forces travelling at variable velocity for various values of rotatory inertia correction factor R0

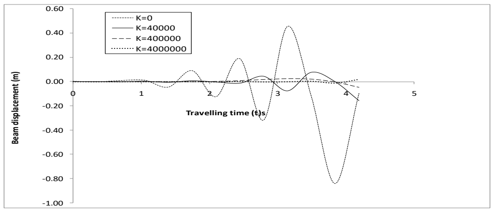

Figure 5: Displacement response of a clamped-clamped Rayleigh beam under the actions of partially distributed masses travelling at variable velocity for various values of foundation stiffness K

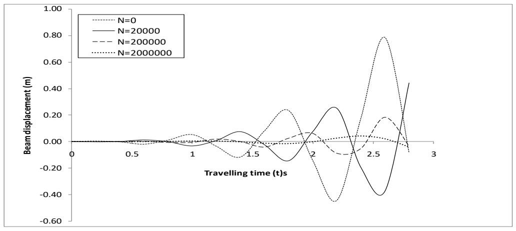

Figure 6: Deflection profile of a clamped-clamped uniform Rayleigh beam under the actions of partially distributed masses travelling at variable velocity for various values of axial force

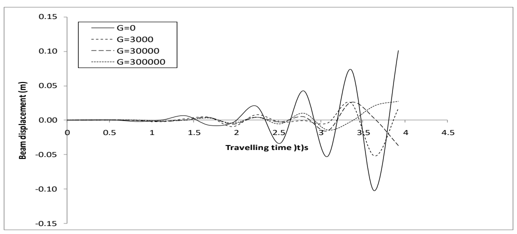

Figure 7: Transverse displacement of a clamped-clamped uniform Rayleigh beam under the actions of partially distributed masses travelling at variable velocity for various values of shear modulus G

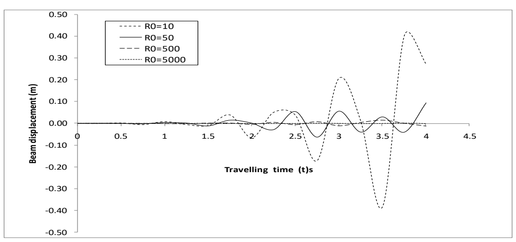

Figure 8: Response amplitude of a clamped-clamped uniform Rayleigh beam under the actions of partially distributed masses travelling at variable velocity for various values of rotatory inertia correction factor R0

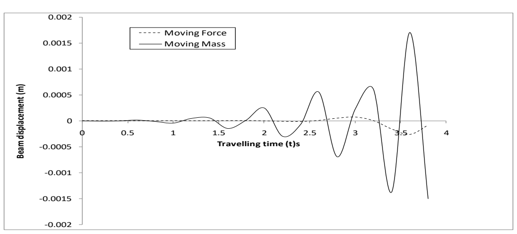

Figure 9: Comparison of the displacement response of moving force and moving mass cases for a uniform clamped-clamped Rayleigh beam for fixed values of $K = 400000$, $N = 200000$ $G = 100000$ $R^0 = 0.5$

Figure 9 displays the comparison of the transverse displacement response of moving force and moving mass cases of the clamped-clamped uniform Rayleigh beam traversed by a moving load travelling at variable velocity for fixed values of K=400000, N=200000, G=100000 and R^0=0.5.

7. CONCLUSION

The problem of the dynamic response of a uniform Rayleigh beam resting on bi-parametric Vlasov foundation and traversed by partially distributed masses travelling at variable velocity has been investigated. Closed forms solutions of the governing fourth order partial differential equations with variable and singular coefficients of uniform Rayleigh beam moving mass problems are presented. The solution techniques is based on generalized finite integral transformation, the expansion of the Heaviside function in series form, a modification of Struble's asymptotic method and Fresnel sine and Fresnel cosine integrals. Analytical solutions obtained are analysed and resonance conditions for the various beam problems are established. Results show that

(i) as the axial force N increases, the amplitude of the uniform Rayleigh beam under the action of load moving at non-uniform velocity decrease.

(ii) when the axial force N is fixed, the displacements of the uniform Rayleigh beam resting on elastic foundation and traversed by masses travelling with variable velocity decrease as the value of foundation moduli K increases for all variants of boundary conditions.

(iii) for fixed values of foundation modulus K , axial force N and shear modulus G , the response amplitude for the moving mass problem is greater than that of the moving force problem for the illustrative end conditions considered.

(iv) as the rotatory inertia correction factor R^0 increases, the transverse displacement response of the beam model decreases.

(v) it has been established for the illustrative examples considered, the moving force solution is not an upper bound for the accurate solution of the moving mass cases in uniform Rayleigh beam under partially distributed loads. Hence, the non-reliability of moving force as a safe approximation to the moving mass problem is confirmed and finally,

(vi) the critical speed for the same system consisting a uniform Rayleigh beam resting on bi-parametric Vlasov foundation and traversed by moving partially distributed mass is smaller than that traversed by a moving partially distributed force.

Resonance is reached earlier in the moving distributed mass problem. Hence, an increase in structural damage sensitivity is noticed under the effect of moving partially distributed load.

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