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Axial Shear Wave Propagation In An Incompressible Cylindrical Solid of a Mooney Rivlin material of order one.

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Abstract

The problem of axial shear wave propagation in an incompressible cylindrical solid of a Mooney Rivlin material of order one was considered. The analysis of the model resulted into a linear second order partial differential equation for the determination of the stresses and axial displacement. Monge's method was used in solving the linear second order partial differential equation. The Monge's subsidiary equations reduced the linear second order partial differential equation into a linear ordinary separable equation. The method of solution adopted provided a closed form solution for the determination of the axial displacement and stresses at any cross section.

Keyword: Axial displacement, Displacement gradient, Deformed radius, Shear stress and shear strain.

1. INTRODUTION

The vibration and wave application characteristics of rubberlike solids are important for practical applications in sonar, ultrasound, engine mountings, seismic isolators and in modeling of some biological tissues. Haddow and Erbay [1] considered a condition which a strain energy function must satisfy for an axial shear waves to occur. They established that a necessary condition for the propagation of a pure axial shear wave to be admitted is that static pure shear is admitted and a sufficient condition for static pure axial shear to be admitted is that propagation of a pure axial shear wave is admitted. They also investigated the possibility of the simultaneous propagation, in the radius direction of a pure longitudinal wave and a finite amplitude pure axial transverse wave. Their results correspond with that of Haddow and Mioduchowski [2], Jiang and Beatty [3,4]. Jiang and Ogden [5] also obtained the same result in their work. In the work of Akbarov and Guliev [6], axisymmetric longitudinal wave propagation in a finite prestretched compound circular cylinder made of an incompressible material, they assumed that the inner and outer cylinder were made of incompressible neo-Hookean materials. Numerical result on the influence of the prestrains in the inner and outer cylinder on wave dispersion are presented and discussed. It is established that the pretension of the cylinder increases the wave velocity. Abo-el-nour and Fatimah [7] in their work investigated some aspect of dispersion relation of flexural waves propagation in a transversely isotropic hollow circular cylinder of infinite extent placed in a primary magnetic field. The result shows that the effect of the primary magnetic field is to increase the value of the material constants. According to Selim [8] damping of the medium has strong effect in the propagation of torsional waves and the velocity of such waves depend on the presence of initial stress. Dai and Wang [9] considered the stress wave propagation in piezoelectric reinforced by inextensible fibres. Rivlin [10] obtained an exact solution for an incompressible isotropic linearly elastic materials of the Mooney-Rivlin material, his result corresponds with that of Green and Zerna [11], and Ogden [12]. The same problem for incompressible materials was examined in [13]. Exact solution for some linear cases have been obtained in [14] for a class of neo-Hookean materials

The present investigation is to determine an analytic solution to the non linear case resulting from the axial displacement, caused by axial force in propagating a wave in a cylindrical incompressible solid material under axial shearing loading. In this work we use cylindrical material whose strain energy function is of the form

$$W = \frac{\mu_1}{2}(I_1 - 3) - \frac{\mu_2}{2}(I_2 - 3) \tag{1.1}$$

where μ_1 and μ_2 are material constants related to the distortional response and are determined from experimental data.

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2. Formulation of The Problem

Cylindrical polar coordinates of a material point in the spatial and material reference configurations are denoted by (r, θ, z) and (R, Θ, Z) , respectively where

$0 \leq R \leq a, 0 \leq \Theta \leq 2\pi, \ 0 \leq Z \leq h$.

The axial shear wave propagation that takes a point from the undefined to the defined configuration is given by

$$r = R \ , \theta = \Theta \ , \ z = Z + w(R, t) \qquad (2.1)$$

Where r is the radius of the cylinder in the deformed configuration and R is the undeformed radius and $w(R, t)$ is the axial displacement. The deformation equation (2.1) describes a superimposed pure wave propagating in the R direction and linearly polarized in the Z direction which is a time dependent deformation, so the superposed wave is to be transverse.

The deformation gradient tensor $\bar{F}$ associated with equation (2.1) is given by

$$\bar{F} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ w_R & 0 & 1 \end{pmatrix} \qquad (2.2)$$

The left Cauchy-Green tensor $\bar{B}$ is given by

$$\bar{B} = \bar{F}\bar{F}^T = \begin{pmatrix} 1 & 0 & w_R \\ 0 & 1 & 0 \\ w_R & 0 & w_R^2 + 1 \end{pmatrix} \qquad (2.3)$$

The Left Cauchy-Green tensor is a second order symmetric tensor . The principal strain invariants are

$$I_1 = 3 + w_R^2 = I_2; \ I_3 = 1 \qquad (2.4)$$

2.1 Equation of motion: The equations of motion in cylindrical polar coordinates (r,θ,z) is given by

$$\frac{\partial \tau_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{1}{r}(\tau_{rr} - \tau_{\theta\theta}) + \rho b_r = \rho a_r \qquad (2.5)$$

$$\frac{\partial \tau_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2}{r}\tau_{r\theta} + \rho b_\theta = \rho a_\theta \qquad (2.6)$$

$$\frac{\partial \tau_{zr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{z\theta}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z} + \frac{1}{r}\tau_{zr} + \rho b_z = \rho a_z \qquad (2.7)$$

Where b_r, b_θ *and* b_z *are the components of the body forces* a_r , a_θ *and* a_z are the components of the acceleration, the non –zero component of the equation of motion is the axial component and $\tau_{rr}.\tau_{r\theta}, \tau_{rz}, \tau_{\theta\theta}, \tau_{\theta z}, \tau_{zz}$ are components of the stress tensor which for this deformation are symmetric.

The stress tensor for incompressible material is given by

$$\bar{\tau} = -PI + 2W_1\bar{B} - 2W_2\bar{B}^{-1} \qquad (2.8)$$

where I is the unit tensor, P is the hydrostatic pressure in compression , $W(I_1, I_2, I_3)$ is the strain energy density function and

$$W_i = \frac{\partial W}{\partial I_i}, i = 1,2,3 \qquad (2.9)$$

Using equations (2.3) and (2.4) in (2.8) we obtain

$$\bar{\tau} = -P\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \mu_1\begin{pmatrix} 1 & 0 & w_R \\ 0 & 1 & 0 \\ w_R & 0 & w_R^2 + 1 \end{pmatrix} - \mu_2\begin{pmatrix} w_R^2 + 1 & 0 & -w_R \\ 0 & 1 & 0 \\ -w_R & 0 & w_R^2 + 1 \end{pmatrix} \qquad (2.10)$$

Consequently the stress tensor are

$$\tau_{rr} = -P + k_2 w_R^2 + k_1 \qquad (2.11)$$

$$\tau_{r\theta} = \tau_{\theta r} = 0 \qquad (2.12)$$

$$\tau_{rz} = \tau_{zr} = k_3 w_R \qquad (2.13)$$

$$\tau_{\theta\theta} = -P + k_1 \qquad (2.14)$$

$$\tau_{\theta z} = \tau_{z\theta} = 0 \qquad (2.15)$$

$$\tau_{zz} = -P + k_1 + k_1 w_R^2 \qquad (2.16)$$

Where $k_1 = \mu_1 - \mu_2$, $k_2 = -\mu_2$, $k_3 = \mu_1 + \mu_2$

Substituting equation (2.13), (2.15) and (2.16) in equation of motion , we find that the non-zero component of equation of motion is

$$k_3 w_{RR} + \frac{1}{R}(k_3 w_R) = \rho w_{tt} \qquad (2.17)$$

which reduces to

$$R k_3 w_{RR} + k_3 w_R = R\rho w_{tt} \qquad (2.18)$$

3. Monge method of solution

The equation for the determination of stresses and displacement in an incompressible cylindrical section of Mooney Rivlin material of order one undergoing an axial shear wave deformation is derived in (2.18)

This equation can be written as

$$R k_3 w_{RR} - R\rho w_{tt} = -k_3 w_R \qquad (3.1)$$

The standard form of the monge equation is given in [15]

$R^{\star}r + Ss + Tt^{\star} = V$ (3.2)
where $R^{\star}$,S,T and V are functions of R, t, w, p, and q.
$r = w_{RR}\ , s = w_{Rt},\ \ t^{\star} = w_{tt}\ $, p $= w_R$ and q $= w_t$ (3.3)
The monge subsidiary equations [15] are given by
$R^{\star}dpdt + \mathrm{T}dqdR - VdRdt = 0$ (3.4)
$R^{\star}(dt)^2 - SdRdt + T(dR)^2 = 0$ (3.5)
The relations which satisfy equation (3.4) and (3.5) are called intermediate integrals.
Comparing (3.1) and (3.2), we have
$R^{\star} = k_3R,\ \ T = -\rho R,\ \ S = 0\ ,\qquad V = -k_3w_R$ (3.6)
Consequently the subsidiary equations for our case become
$k_3\mathrm{R}w_{RR}\mathrm{dR}dt - \rho Rw_{tt}dtdR + k_3\mathrm{RdRdt} = 0$ (3.7)
$k_3\mathrm{R}(dt)^2 - \rho R(dR)^2 = 0$ (3.8)
From (3.8), we have

$$dR = \pm\left(\sqrt{\frac{k_3}{\rho}}\right)dt \qquad (3.9)$$

By using the two values of dR in equation (3.9) in (3.7), we have the two equations

$$k_3Rw_{RR}\left(\sqrt{\frac{k_3}{\rho}}\right) - \rho Rw_{tt}\left(\sqrt{\frac{k_3}{\rho}}\right) + k_3w_R\left(\sqrt{\frac{k_3}{\rho}}\right)=0 \qquad (3.10)$$

$$-k_3Rw_{RR}\left(\sqrt{\frac{k_3}{\rho}}\right) + \rho Rw_{tt}\left(\sqrt{\frac{k_3}{\rho}}\right) - k_3w_R\left(\sqrt{\frac{k_3}{\rho}}\right)=0 \qquad (3.11)$$

Solving equations (3.10) and (3.11) simultaneously by subtracting (3.10) from (3.11), we have the single equation.

$$k_3R\mathrm{dp} + k_3\mathrm{pdR} = 0$$

$$\frac{\mathrm{dp}}{p} + \frac{\mathrm{dR}}{R} = 0$$

$$pR = c$$

$$w_R = \frac{c}{R} \qquad (3.12)$$

Equation (3.12) agreed with the result of [1]

$$\int_0^{\mathrm{w}} d\mathrm{w} = \int_0^{\mathrm{R}} \frac{cd\mathrm{R}}{\mathrm{R}}$$

Let $R = ccos\theta$

$$w = \int_0^{\theta} \frac{-c^2sin\theta d\theta}{ccos\theta}$$

$$w = Inccos\theta$$

Using the boundary condition $\mathrm{w}(0) = 0$ and $R\left(\frac{\pi}{2}\right) = 0$
Therefore $\mathrm{c} = 1$

$$R = cos\theta\ \ and\ w = Incos\theta \qquad (3.13)$$

Now $\frac{dw}{d\mathrm{R}} = \frac{dw}{d\theta}\frac{d\theta}{d\mathrm{R}} = \frac{dw}{d\theta} * \frac{1}{\frac{d\mathrm{R}}{d\theta}}$

Which on simplification gives

$$Displacement\ gradient\quad \frac{d\mathrm{w}}{d\mathrm{R}} = \sec\theta \qquad (3.14\ \ )$$

Therefore, shear strain $w_R = \sec\theta$
The components of the stress tensors are
$\tau_{rr} = -P + k_2w_R^2 + k_1$ (3.15)
$\tau_{r\theta} = \tau_{\theta r} = 0$ (3.16)
$\tau_{rz} = \tau_{zr} = k_3w_R$ (3.17)
$\tau_{\theta\theta} = -P + k_1$ (3.18)
$\tau_{\theta z} = \tau_{z\theta} = 0$ (3.19)
$\tau_{zz} = -P + k_1 + k_1w_R^2$ (3.20)
Where $k_1 = \mu_1 - \mu_2$, $k_2 = -\mu_2$, $k_3 = \mu_1 + \mu_2$ and $w_R = \sec\theta$

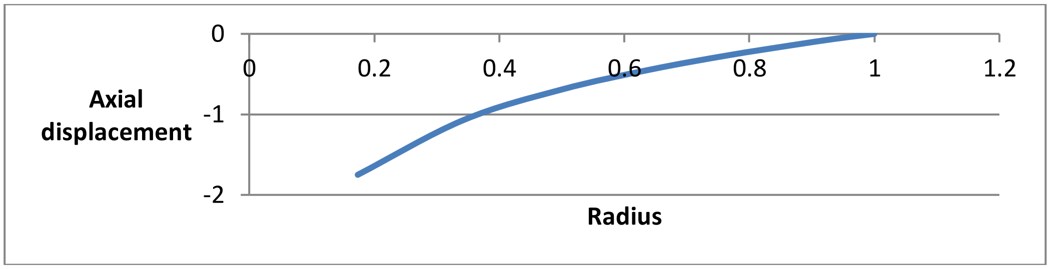

Fig 1: A graph of the equation (3.13) with Axial displacement plotted against radius.

Conclusion

In this work we considered an axial shear wave in an incompressible cylindrical solid of a Mooney Rivlin material of order one given in equation (2.1). The analysis of the model resulted into a linear second order partial differential equation. Solving the linear second order partial differential equation we obtained equation of an analytic result (3.13). We obtained the shear strain as (3.14) by differentiating displacement with respect to the radius of the deformed cylinder.

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