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Azimuthal Shear Wave Propagation In An Incompressible Hollow Circular Cylinder of a Mooney Rivlin material of order one.

Nwagwu I.O.

Department of Mathematics, Federal University of Technology Owerri, Nigeria.

Abstract

The problem of determining the stresses and angular displacement in a hollow cylindrical material under torsional shear wave propagation is considered. The cylinder under consideration is made up of Mooney Rivlin material of order one. The analysis of the model resulted into a linear second order partial differential equation. Monge's method was used in solving the linear second order partial differential equation. The method of solution adopted provided a closed form solution for the determination of the angular displacement and stresses at any cross section.

Keyword: Angular displacement, Displacement gradient, Shear stresses, Deformed radius and shear strain.

1. Introduction

The problem of azimuthal shear wave propagation in an elastic hollow circular cylinder is of considerable importance in engine mountings, seismic isolators sonar, ultrasound, and application of nonlinear elasticity to the modeling of some biological tissues. A reasonable amount of researches have be on done on torsional shear wave propagation with different strain energy function. Ertepinar and Erarslanoglu [1] considered the polynomial form of the strain-energy proposed by Levinson and Burgess. Haughton [2] obtained the same solution for $g(R)$ for class of strain energy function considered by Agarwal. Polignone and Horgan [3] worked on azimuthal shear wave using the generalized Blatz-ko material .Haddow and Jiang [4] considered on the condition which a strain energy function must satisfy for an azimuthal shear waves to occur in a hyperelastic material, they established that Azimuthal shear cannot be separated into two uncoupled systems, which govern the propagation of pure azimuthal shear waves and pure radial longitudinal waves, respectively. This means that the simultaneous propagation of a finite amplitude pure azimuthal shear wave and a pure radial longitudinal wave is not admitted in any compressible, isotropic, hyperelastics solids which admit the propagation of a finite amplitude pure azimuthal shear wave. Beatty and Jiang [5] and Jiang and Ogden [6] obtained the same results in their work. According to Selim [7] in his paper "Torsional wave propagation in an initially stressed, dissipative cylinder" the study reveals that the damping of a medium has strong effect in the propagation of torsional waves. The velocity of such waves depends on the presence of initial stress. Akbarov and Guliev [8] assumed the inner and outer cylinder to be made of incompressible neo-Hookean materials. They presented numerical result on the influence of the prestrains in the inner and outer cylinder on wave dispersion are presented and discussed. They also established that the pretension of the cylinder increases the wave velocity. Haddow and Erbay [9] investigated the possibility of the simultaneous propagation, in the R direction of a pure longitudinal wave and a finite amplitude pure axial transverse wave. Abo-el-nour and Fatimah [10] in their work investigated some aspect of dispersion relation of magnetic field. The result shows that the effect of the primary magnetic field is to increase the value of the material constants. Horgan and Saccomandi [11] investigated on Simple torsion of isotropic, hyperelastic, incompressible materials with limiting chain extensibility. Simmonds and Warne [12] worked on Azimuthal shear of compressible or incompressible, rubberlike, polar orthotropic tubes of infinite extent. Haddow [13] studied Nonlinear waves in hyperelastic solids. Erumaka [14] investigated on finite deformation of a class of Ogden solid under anti-plane shear. Ogden [15] worked on non-linear elastic deformation . The available literature reveals that it is has been difficult to obtain an analytic solution for the resulting non-linear second order partial differential equations. In this work we show how the application of Monge's method can ease this problem.

The present study is to determine the angular displacement and stresses caused by torsional force in propagating a wave in a hollow cylindrical material. In this work we consider moore rivlin material of order one whose strain energy function is of the form.

$$W = \frac{\mu_1}{2}(I_1 - 3) - \frac{\mu_2}{2}(I_2 - 3) \qquad (1)$$

where μ_1 and μ_2 are material constants related to the distortional response and are determined from experimental data.

Correspondence Author: Nwagwu I.O., E-mail: isaacogazie@yahoo.com, Tel: +2348038981875

FORMULATION OF THE PROBLEM:

Let the open region $D_0 = \{(\mathrm{R},\Theta,\mathrm{Z}), a \le \mathrm{R} \le \mathrm{b}, \quad 0 \le \Theta \le 2\pi, 0 \le \mathrm{Z} \le \mathrm{h}\}$ denote the cross section of a right circular hollow cylinder with inner radius a and outer radius b in its undeformed configuration. The hollow circular cylinder is subjected to azimuthal shear force of magnitude ρ. The resulting deformation is a one-to-one axisymmetric deformation which maps the point with cylindrical polar coordinate $(\mathrm{R},\Theta,\mathrm{Z})$ in the undeformed configuration D_0 to the point $(\mathrm{r},\theta,\mathrm{z})$in the deformed region D.

The deformation equations for the propagation of an Azimuthal shear wave in a cylindrical Mooney Rivlin hollow material is given by

$$\mathrm{r}=\mathrm{R} \quad ,\theta=\Theta+\mathrm{g}(\mathrm{R},\mathrm{t}) \quad , \ \mathrm{z}=\mathrm{Z} \tag{2}$$

where r is the radius of the cylinder in the deformed configuration and R is the undeformed radius and $\mathrm{g}(\mathrm{R},\mathrm{t})$ is the angular displacement. Equation (2) describes a superimposed pure wave propagation in the R direction and linearly polarized in the Θ direction results in a time dependent deformation. The deformation gradient tensor $\bar{\mathrm{F}}$ associated with equation (2) is given by

$$\bar{\mathrm{F}} = \begin{pmatrix} 1 & 0 & 0 \\ \mathrm{rg_R} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3}$$

The left Cauchy-Green deformation gradient tensor $\overline{\mathrm{B}}$ is given by

$$\overline{\mathrm{B}} = \bar{\mathrm{F}}\bar{\mathrm{F}}^{\mathrm{T}} = \begin{pmatrix} 1 & \mathrm{rg_R} & 0 \\ \mathrm{rg_R} & (\mathrm{rg_R})^2+1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4}$$

The Left Cauchy-Green tensor is a second order tensor and it is symmetric, then it has three principal strain invariants $\mathrm{I}_1,\mathrm{I}_2$ and I_3 given by

$$\mathrm{I}_1 = 3+(\mathrm{rg_R})^2 = \mathrm{I}_2 \ , \mathrm{I}_3 = 1 \tag{5}$$

Stress Tensor $\bar{\tau}$: The stress tensor for incompressible material is given by

$$\bar{\tau} = -\mathrm{PI} + 2\mathrm{W}_1\overline{\mathrm{B}} - 2\mathrm{W}_2\overline{\mathrm{B}}^{-1} \tag{6}$$

Where I is unit tensor, P is the hydrostatic pressure and $\mathrm{W}(\mathrm{I}_1,\mathrm{I}_2,\mathrm{I}_3)$ is the strain energy density function. $\mathrm{W_i} = \frac{\partial \mathrm{W}}{\partial \mathrm{I_i}}, \mathrm{i} = 1,2,3$

Using equations (4) and (5) in (6) we obtain

$$\bar{\tau} = -\mathrm{P}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \mu_1\begin{pmatrix} 1 & \mathrm{rg_R} & 0 \\ \mathrm{rg_R} & (\mathrm{rg_R})^2+1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \mu_2\begin{pmatrix} (\mathrm{rg_R})^2+1 & -\mathrm{rg_R} & 0 \\ -\mathrm{rg_R} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{7}$$

The components of the stress tensors are

$$\tau_{rr} = -P + k_1 + k_2(\mathrm{rg_R})^2 \tag{8a}$$
$$\tau_{r\theta} = \tau_{\theta r} = k_3\mathrm{rg_R} \tag{8b}$$
$$\tau_{rz} = \tau_{zr} = 0 \tag{8c}$$
$$\tau_{\theta\theta} = -P + k_4(\mathrm{rg_R})^2 \tag{8d}$$
$$\tau_{\theta z} = \tau_{z\theta} = 0 \tag{8e}$$
$$\tau_{zz} = -P + k_1 \tag{8f}$$

Where $k_{\mathrm{i}}; \mathrm{i} = 1,2,3,4$ are constants.

The equations of motion in cylindrical polar coordinates (r,θ,z) is given by

$$\frac{\partial\tau_{rr}}{\partial r} + \frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta} + \frac{\partial\tau_{rz}}{\partial z} + \frac{1}{r}(\tau_{rr}-\tau_{\theta\theta}) + \rho b_r = \rho a_r$$
$$\frac{\partial\tau_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial\tau_{\theta\theta}}{\partial\theta} + \frac{\partial\tau_{\theta z}}{\partial z} + \frac{2}{r}\tau_{r\theta} + \rho b_\theta = \rho a_\theta$$
$$\frac{\partial\tau_{zr}}{\partial r} + \frac{1}{r}\frac{\partial\tau_{z\theta}}{\partial\theta} + \frac{\partial\tau_{zz}}{\partial z} + \frac{1}{r}\tau_{zr} + \rho b_z = \rho a_z$$

Where b_r, b_θ *and* b_z *are the components of the body force and* a_r , a_θ *and* a_z are the components of the acceleration.

The non- zero component of equation of motion is the azimuthal component, which reduces to

$$\frac{\partial\tau_{\theta r}}{\partial r} + \frac{2\tau_{r\theta}}{r} = \rho a_\theta \tag{9}$$

$$Now \quad \frac{\partial}{\partial r} = \frac{\partial}{\partial R}\frac{\partial R}{\partial r} = \frac{\partial}{\partial R}\left(\frac{1}{\frac{\partial r}{\partial R}}\right) = \frac{1}{\dot{r}}\frac{\partial}{\partial R}$$

Hence equation (14) becomes

$$\frac{\partial\tau_{r\theta}}{\partial R} + \frac{2\dot{r}}{r}\tau_{r\theta} = \dot{r}\rho a_\theta \tag{10}$$

From the deformation equation (2) and using the discussions above we have equation (10) as

$$k_3\mathrm{rg_{RR}} + 3k_3\mathrm{g_R} = \rho\mathrm{g}_{tt} \tag{11}$$

Monge method of solution

Equation (11) is the equation for the determination of stresses and displacement in an incompressible cylindrical section of Mooney Rivlin material of order one undergoing an azimuthal shear wave propagation

Equation (11) can be written as

$$k_3\mathrm{rg_{RR}} - \rho\mathrm{g}_{tt} = -3k_3\mathrm{g_R} \tag{12}$$

The standard form of the Monge equation is given see (Raisinghania 16) as

$$R^\star r + Ss + Tt^\star = V \tag{13}$$

where $R^\star$,S,T and V are functions of R, t, w, p, and q.

$r = g_{RR}\ , s = g_{Rt}, t^{\star} = g_{tt}$ (14)
$R^{\star} dpdt + \mathrm{T} dqdR - VdRdt = 0$ (15)
$R^{\star}(dt)^2 - SdRdt + T(dR)^2 = 0$ (16)
The equations (12) and (13) are called Monge's subsidiary equations and the relations which satisfy these equations are called intermediate integrals.
Comparing (12) with (13), we have the following
$R^{\star} = k_3 \mathrm{r},\ T = -\rho,\ S = 0\ ,\quad \mathrm{V} = -\,3k_3 \mathrm{g_R}$ (17)

Consequently the subsidiary equations for our case become
$k_3 \mathrm{r} g_{RR} \mathrm{dR}dt - \rho g_{tt} dtdR + 3k_3 \mathrm{g_R dRdt} = 0$ (18)
$k_3 \mathrm{r}(dt)^2 - \rho(dR)^2 = 0$ (19)
From (19), we have
$$dR = \pm\left(\sqrt{\frac{k_3 \mathrm{r}}{\rho}}\right)dt \qquad (20)$$
By using the two values of dR in equation (20) in (18), we have the two equations
$$k_3 \mathrm{r}\ g_{RR}\left(\sqrt{\frac{k_3 \mathrm{r}}{\rho}}\right) - \rho g_{tt} dt\left(\sqrt{\frac{k_3 \mathrm{r}}{\rho}}\right) + 3k_3 \mathrm{g_R}\ \left(\sqrt{\frac{k_3 \mathrm{r}}{\rho}}\right)=0 \qquad (21)$$
$$-k_3 \mathrm{r}\ g_{RR}\left(\sqrt{\frac{k_3 \mathrm{r}}{\rho}}\right) + \rho g_{tt} dt\left(\sqrt{\frac{k_3 \mathrm{r}}{\rho}}\right) - 3k_3 \mathrm{g_R}\ \left(\sqrt{\frac{k_3 \mathrm{r}}{\rho}}\right)=0 \qquad (22)$$
Solving equations (21) and (22) simultaneously by subtracting (21) from (22), we have the single equation
$\mathrm{r}\ g_{RR} \mathrm{dR} + 3\mathrm{g_R\ dR} = 0$
Which is equivalent to
$Rdp + 3pdR = 0$ (23)
Where $dp = g_{RR} \mathrm{dR},\ p = \frac{d\mathrm{g}}{d\mathrm{R}}$ and from equation 2 $\mathrm{r} = \mathrm{R}$
Equation (23) can be expressed as
$$\frac{\mathrm{dp}}{p} + \frac{3\mathrm{dR}}{R} = 0 \qquad (24)$$
integrating equation (24) we have
$\mathrm{pR}^3 = \mathrm{c}$ (25)
Equation (25) simplifies to
$p = \frac{\mathrm{c}}{R^3}$ (26)
where c is a constants
Equation (26) implies
$\frac{d\mathrm{g}}{d\mathrm{R}} = \frac{\mathrm{c}}{R^3}$
which integrate to
$$\int_0^{\mathrm{g}} d\mathrm{g} = \int_0^{\mathrm{R}} \frac{c d\mathrm{R}}{\mathrm{R}^3}$$
Let $R = c sec\theta$
$$\mathrm{g} = \int_0^{\theta} \frac{c^2 sec\theta tan\theta d\theta}{\mathrm{c}^3 \sec^2\theta sec\theta}$$
$$\mathrm{g} = \frac{1}{4\mathrm{c}}[1 - \cos 2\theta]$$
Using the boundary conditions $R(0) = 1\ and\ \mathrm{g}(0) = 0$
Therefore $\mathrm{c} = 1$
$R = sec\theta$ and $\mathrm{g} = \frac{1}{4}[1 - \cos 2\theta]$ (27)
$$\text{Now } \frac{d\mathrm{g}}{d\mathrm{R}} = \frac{d\mathrm{g}}{d\theta}\frac{d\theta}{d\mathrm{R}} = \frac{d\mathrm{g}}{d\theta} * \frac{1}{\frac{d\mathrm{R}}{d\theta}}$$
Which on simplification gives
$$Displacement\ gradient\ \ \frac{d\mathrm{g}}{d\mathrm{R}} = \cos^3\theta \qquad (28)$$
Azimuthal Shear Strain $\Upsilon = \mathrm{Rg_R}$
Therefore $\Upsilon = \mathrm{Rg_R} = \cos^2\theta$ (29)
From (8) the components of the stress tensors are
$\tau_{rr} = -P + k_1 + k_2(\mathrm{Rg_R})^2$ (30)
$\tau_{r\theta} = \tau_{\theta r} = k_3 \mathrm{Rg_R}$ (31)
$\tau_{rz} = \tau_{zr} = 0$ (32)
$\tau_{\theta\theta} = -P + k_4(\mathrm{Rg_R})^2$ (33)
$\tau_{\theta z} = \tau_{z\theta} = 0$ (34)

$\tau_{zz} = -P + k_1$ (35)

Where k_i; $i = 1,2,3,4$ are constants.

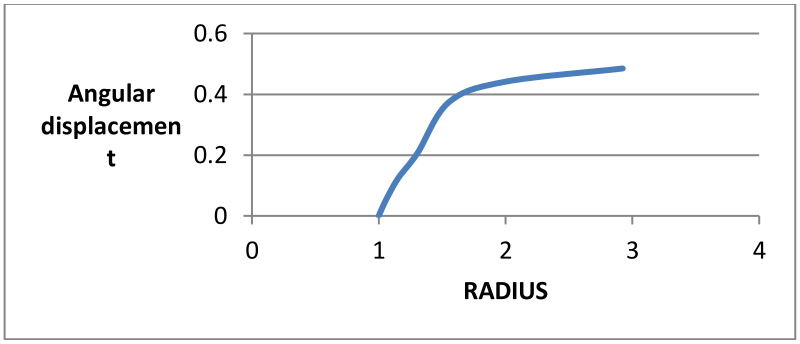

Fig 1: A graph of the equation (27) with angular displacement plotted against radius.

CONCLUSION: We were able to obtain a solution for the angular displacement given as equation (27) which is caused by Azimuthal force in propagating a wave in an incompressible hollow circular cylinder whose strain energy function is given as (1). We obtained the shear strain as the product of the radius of the cylinder and the angular displacement gradient given as (29). Finally, we obtained the components of the stress at any cross section of the hollow cylinder.

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