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Computing Square Root of Diagonalizable Matrix with Positive Eigenvalues and Iterative Solution to Nonlinear System of Equation: The Role of Lagrange Interpolation Formula

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Abstract

The iterative methods for solving system of nonlinear equation is presented and extension is made to include the so called SVD of a matrix for the positive eigenvalues of a matrix. By incorporating the use Lagrange interpolation formula, we are able to compute the square root of a matrix where the sign function of a matrix can be obtained cheaply. Particularly, we use the Schultz Hyper power method for approximating inverse of linear bounded operator in Hilbert space in speeding up Newton iteration for matrix square root. The discussed methods are amenable to Aerospace computation for polar decomposition for the direction matrix cosine. Details of above procedures form the peak of discussion.

***Keyword*:** Nonlinear systems, adaptive splitting, svd, matrix square roots, lagrange interpolation, Newton iteration.

1. INTRODUTION

The paper starts off with preliminary discussion on iterative solution of nonlinear systems of equations

$$F(x)=0 \quad . \qquad (1.1)$$

Using adaptive splitting, solution to system (1.1) may be found in exact arithmetic using any standard numerical iterative methods [1]. However, we often neglect the effects of square root of the resulting Jacobian matrix if such matrix has positive eigenvalues. Such computation is often facilitated by the aid of Singular value decomposition (SVD) in conjunction with Lagrange interpolation formula assuming no eigenvalues are on the left hand side of the real line. Such problems are often encounteredin diverse branches of dynamical systems and control theory [2].Computation of square root of a diagonalizable matrix with both positive and negative eigenvalues are major non trivial problems occurring in polar decomposition, an important aspect of Aeronautic space computation.

Before proceeding further, we let $F:R^n \to R^n$ be locally Lipschitz continuous. This means that F is Fretchet differentiable almost everywhere. Then, the generalized Jacobianof F at the vector $x \in R^n$ will be given by the set:

$$\partial F(x)=\alpha\left\{\lim_{x_j\to x, x_k\in D_F} F^{/}(x_k)\right\}, \qquad (1.2)$$

where, the term α , is the convex hull.

The Newton iteration is then defined by

$$x^{(k+1)}=x^{(k)}-A_k^{-1}F(x^{(k)}) \text{ ,(where, } A_k \in F^{/}(x^{(k)})) \qquad (1.3)$$

The corresponding ball radius r about a point $x \in R^n$ is represented by

$$B(x,r)=\{x_c \mid \|x-x_c\|<r\}. \qquad (1.4)$$

Additionally as is expected, the role ofLipschitz constant α gives a good measure on the rate of convergence in the fixed point iteration [3,4].This means that any decreasing sequence of such closed sets $x_1 \supset x_2 \supset x_3 \supset \ldots$, eventually stabilizes. The base of the topology it generates consists of complements of hyper-surfaces $x^{(k+1)}=x^{(k)}-N(x^{(k)})$ for which $x^{(k+1)}\bigcap x^{(k)}\neq\phi$ and, $N(x^{(k)})$ are the respective Newton corrections for each step which tends to zero vector as $k\to\infty$. Of special interest is that, the matrices A_k are of Baire's second category. That is, coefficients of $A_k \in F^{/}(x^{(k)})$ are non-meagre. If this is the case, then $A(x^k)$ has a completely regular topological space that is F -Suslinprovided that F is analytic at the point x_k .

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Thus $F: R^n \to R^n$ are both G_δ - closed and B_r complete in the sense of Cauchy sequence[5],e.g.,. Overall, the interest in this discussion precludes those of rare spaces although their occurrences are often met as well.
We reference further a categorical statement in our presentation that a Frechet space, a sequentially complete (Df) space to which adaptation of iterative methods succumb in line with Grothendieck, the strong or weak dual countable inductive limit of metrizable locally convex spaces are webbed.
We draw inspiration from [6] with the following theorems for verifications of initial conditions for the system of Equation (1.1) for any meaningful iterative Newton-Like processes. These are influenced by the Kantorovich,Borsuk's and Miranda theorems respectively.
Theorem 1.1, Standard Kantorovich Theorem [6]: Let $f: D \subset R^n \to R^n$ be differentiable in the open convex set D. Assume that for some point $x^{(0)} \in D$ the Jacobian $f^{/}(x^{(0)})$ is invertible with $\left\| f^{/}(x^{(0)}) \right\| \le \beta$, $\left\| f^{/}(x^{(0)})^{-1} f(x^{(0)}) \right\| \le \eta$.

Assuming further there be a Lipschitz constant $\kappa > 0$, for $f^{/}$ such that $\left\| f^{/}(x^{(0)})^{-1}(f^{/}(u) - f^{/}(v)) \right\| \le k\|u - v\|$ for all $u, v \in D$ .If $h = \eta\beta\kappa \le \frac{1}{2}$ and $\bar{B}(x^{(0)}, \rho_-) \subseteq D$, where $\rho_- = \frac{1-\sqrt{1-2h}}{\beta\kappa}$, then f has a zero x^* in $\bar{B}(x^{(0)}, \rho_-)$. Moreover, this zero is the unique zero of f in $\bar{B}(x^{(0)}, \rho_-) \cup B(x^{(0)}, \rho_+) \cap D$, where $\rho_+ = \frac{1+\sqrt{1-2h}}{\beta\kappa}$ and the Newton iterates x^* with

$$x^{(k+1)} = x^{(k)} - f^{/}(x^{(k)})^{-1} f(x^{(k)}) \qquad (1.5)$$

are well defined , remain in $\bar{B}(x^{(0)}, \rho_-)$ and converge to x^*.
Theorem 1.2, Borsuk's Theorem [6]: Assume that there is a symmetrisation of zeros of f around the disk D .Let $B \subseteq R^n$ be open, bounded, convex and symmetric with respect to $x^{(0)} \in B$ .Let $f: \bar{B} \to R^n$ be a continuous mapping, for which $f(x) \ne 0$ on ∂B and that $f(x^{(0)} + t) \ne \eta f(x^{(0)} - t)$, where $\eta > 0$ and all $x^{(0)} + t \in \partial B$. Then f has a zero in B.
Miranda Theorem [6]: Let $f: \bar{B}_\infty(x^{(0)}, \varepsilon) \subseteq R^n \to R^n$ be a continuous mapping. Assume that

$$f_i(x)\begin{cases} \ge 0 & \forall x \in B_\infty^{i,+}(x^{(0)}, r) \\ \le 0 & \forall x \in B_\infty^{i,+}(x^{(0)}, r) \end{cases} ; \text{ for } i = 1,2,\ldots,n$$

Then f has at least one zero x^* in $\bar{B}_\infty(x^{(0)}, r)$.
We situate the theorem of [7]to discriminate points of singularities in a system (1.1):
Lemma 1.2 [7]: Let A be a singular M - matrix, and let $A = M - N$ denote the weak regular splitting. Assuming there is $x > 0$ such that $Ax \ge O$,and $ind_0(B) = 1$ where $ind_i(B)$ is an index parameter. It holds that $B = I - M^{-1}N$ is a singular M - matrix .
Consequently following, we have that a weak regular splitting satisfies $\rho(M^{-1}N) = 1$ and $ind_1(M^{-1}N) = 1$.A graph compatible weak regular splitting [8] whose matrix A is a singular M -matrix, where $A = M - N$ is given by:

(a) $\rho(M^{-1}N) = 1$, (b) $mult_1(M^{-1}N) \ge mult_0(A)$, and (c) $ind_1(M^{-1}N) \le ind_0(A)$.

The rest of the paper is categorized in the form: Section2 gives commonly used iterative linear solvers - Jacobi, Gauss-Siedel and Successive over relaxation (SOR) methods. We relate that, SOR parameter can be estimated from ratio of the norm of errors of two successive Gauss-Siedel steps as iteration process approaches infinity and synchronized this with method due to Young of 1958 as reported in [8],aquite significant advantage in numerical computation. Thus, the number of steps to achieve numerical accuracy in the execution of SOR method is discussed. In section3, the SVD of a matrix is brought into discussion in collaboration with lagrange interpolation formula for computation of a matrix square root, an important step in the formation of matrix sign function. Numerical example is demonstrated in section4 and conclusion is given based on the strength of our findings.

2.0 Commonly Used Iterative Solvers for Linear System.

Commonly used iterative solvers forEquation(1.3) are the Jacobi,Gausss-Siedel and Successive over relaxation methods and their various modifications [9,10] all based on fixed point theorem.
To this end, the matrix A is split in the following way
$A = D - L - U$, and $D = diag(a_{11}, a_{22}, \ldots, a_{nn})$, L = strictly Lower diagonal matrix, U strictly upper diagonal matrix respectively.
Basic stationary iterative method usually is in the form:

$$x^{(k+1)} = x^{(k)} + M^{-1}(b - Ax^{(k)}), \quad k = 0,1,2,\ldots,.) , \qquad (2.1)$$

based on matrix splitting $A = M - N$, it is set that:

$B = M^{-1}N = I - M^{-1}A$. For a preconditioned iteration to a linear system

$$M^{-1}Ax = M^{-1}b, \tag{2.2}$$

is described as follows: Firstly, convergence criterion for Jacobi iterative method is that, $\rho\left(D^{-1}(U+L)\right) < 1$.

Different iterative methods are obtained from system (2.1) in the form:

Setting as:

$M = D$, it is the Jacobi iterative method, $M = D - L$, it is Gauss-Siedel method, for $M = \frac{1}{\omega}D(I - \omega L)$, we have the Successive Over relaxation method.

Thus, Gauss-Siedel method also called the Single Step Method (SSM) is given by the equation:

$$(D-L)x^{(k+1)} = Ux^{(k)} + b \tag{2.3}$$

Equivalently, we rewrite Equation (2.3) in the form:

$$x^{(k+1)} = x^{(k)} + \left\{D^{-1}Lx^{(k+1)} + D^{-1}Ux^{(k)} + D^{-1}b - x^{(k)}\right\} = x^{(k)} + v^{(k)} \tag{2.4}$$

General standard form of Equation (2.4) is given by

$$x^{(k+1)} = x^{(k)} + \omega v^{(k)} \tag{2.5}$$

As is standard, we often write Equation (2.5) in the form:

$$Dx^{(k+1)} = Dx^{(k)} + \omega Lx^{(k+1)} + \omega Ux^{(k)} + \omega b - \omega Dx^{(k)}.$$

Thus, it follows that

$$x^{(k+1)} = (D - \omega L)^{-1}\left((1-\omega)D + \omega U\right)x^{(k)} + \omega(D - \omega L)^{-1}b \tag{2.6}$$

A simplified form of Equation (2.6) is setting $D = I$. In this case, we have the consequences:

$$x^{(k+1)} = (I - \omega L)^{-1}\left((1-\omega)I + \omega U\right)x^{(k)} + \omega(I - \omega L)^{-1}b \tag{2.7}$$

The iteration matrix for Equation (2.7) is that

$$L_\omega = (I - \omega L)^{-1}\left((1-\omega)I + \omega U\right).$$

The term $0 < \omega < 2$ is called Over-relaxation parameter. Various over relaxation methods are in force viz: for $\omega < 1$, it is under relaxation method; $\omega = 1$, it is single step method; $\omega > 1$, it is over relaxation method.

In case of Gauss-Siedel method (Single Step Method), the convergence is implied by

$|a_{ii}| > \sum_{j \neq i}|a_{ij}|$, $i = 1,2,\ldots,n$. Using the ideas due to Young of 1958 as reported in [8]it is that for a consistently ordered matrix and for real eigenvalues of Jacobi iteration matrix $B = D^{-1}(U+L)$ for which $\rho(B) < 1$, Successive Over relaxation method converges if

$\omega_b = \dfrac{2}{1 + \sqrt{1 - \rho(B)^2}}$, and, $\rho(H(\omega_b)) = \omega_b - 1 = \left(\dfrac{\rho(B)}{1 + \sqrt{1 - \rho(B)^2}}\right)^2$, where $H = (I - L)^{-1}U$.

More fundamentally is that for $\omega \in [1, \omega_b]$ in the SOR method, there follows:

$$\rho(\omega_b) = \left[\frac{\omega\mu}{2} + \frac{1}{2}\sqrt{\mu^2\omega^2 - 4(\omega - 1)}\right]^2 \tag{2.8}$$

Where from, $\mu = \dfrac{\rho(\Im_\omega) + \omega - 1}{\omega\sqrt{\rho(\Im_\omega)}}$ (2.9)

Recently, methods for estimating values of ω_b using Gauss-Siedel iterative sequence $\{x_k\}_{k=1}^n$ can be computed [4] and other references therein. For a considerable value of k, using notation $e^k = x^{(k)} - \bar{x}$ as the error for which $e^k = \Im_\omega^k e^{(0)}$, $d^k = x^{(k+1)} - x^{(k)}$, we then have

$$x^{(k+1)} - x^{(k)} = e^{(k+1)} - e^{(k)} = (\Im_\omega - I)e^{(0)} = \Im_\omega^k(\Im_\omega - 1)e^{(0)} = \Im_\omega^k d^{(0)}.$$

For a large $k \to \infty$, the estimate

$$c_k = \max_{1 \le i \le n}\frac{\left|x_i^{(k+1)} - x_i^{(k)}\right|}{\left|x_i^{(k)} - x_i^{(k-1)}\right|} \tag{2.30}$$

gives the approximate value of $\rho(\Im_\omega)$. We thus can estimate μ from Equation (2.5). The overall gain from estimate for c_k in Equation (2.6) is that the value for

$$\bar{\omega} = \frac{2}{\left(1 + \left[1 - (c_k - \omega - 1)^2/(\omega^2 c_k)\right]^{\frac{1}{2}}\right)}, \tag{2.31}$$

can be obtained cheaply without further calculation.
We see from the error $d^{(k)} = \Im_\omega^k e^{(0)} = (\Im_\omega - I)e^{(0)}$, that $e^{(k)} = (\Im_\omega - I)^{-1} d^{(k)}$. Using this connection, it holds that

$$\|e^{(k)}\|_\infty \le \frac{1}{1-\rho(\Im\ )}\|d^{(k)}\|_\infty .$$

Finally, the number of iterations required to achieve the above can be computed from

$$k = \frac{\ln \rho(\Im_{\omega_b})}{\ln \rho(B)} \qquad (2.32)$$

3.0 The SVD and Computation of a Matrix Sign Function, where lies The Lagrange Interpolation Formula?

The singular values decomposition (SVD) of a $m \times n$ matrix is an important tool for solving not only the rank of a matrix but alsoacts as a gateway providing many numerical solutions to linear systems. Its efficiency ranges from applications in least squares problems -such as image reconstruction from missing data [9,10] to polar decomposition, an important tool in numerical analysis and aerospacecomputations for the direction cosine [11].
In some cases, the nonlinear system of Equation (1.1) may have more number of equations than the unknowns which the system can accommodate. Then, for such a matrix $A \in R^{m\times n}$, $m > n$,has an SVD in the form:

$$U^H AV = \Sigma \ , \quad (3.1)$$

where, $U \in R^{m\times m}$, $V \in R^{n\times n}$ are unitary matrices for which $\Sigma = \begin{bmatrix} D & O \\ O & O \end{bmatrix}$,

$D = diag(\sigma_1, \sigma_2, ..., \sigma_k)$, $\sigma_1 > \sigma_2 > ... > \sigma_r > 0$ and, r is the rank of matrix A .
Via least squares approach, the equivalent problem leads to the form

$$(A^T A)x = A^T f(x) . \quad (3.2a)$$

On the other side, we may first reduce the matrix A to upper triangular form by QR decomposition. Thus the induced QR method is given by

$$A = Q\begin{pmatrix} R \\ O \end{pmatrix}, \quad m \ge n \qquad (3.2b)$$

By further introduction of SVD for R , would yield that $R = U_R \Sigma V^T$ implying that

$$A = U \Sigma V^T, U = Q\begin{pmatrix} U_R \\ O \end{pmatrix}. \qquad (3.3)$$

Supposing A has full rank, in view of Equation (3.2a) the generalized solution to system of equation

$$Ax = b \ , (3.4)$$

Where $A \in R^{m\times n}$, $b \in R^m$ will be written in the form:

$$X_{reg} = V\theta\Sigma^+ U^T \bar{b} = \sum_{i=1}^n f_i \frac{u_i^T b}{\sigma_i} v_i = \sum_{i=1}^n f_i \frac{u_i^T b}{\sigma_i} v_i + \varepsilon \sum_{i=1}^n f_i \frac{u_i^T s}{\sigma_i} v_i \quad (3.5)$$

The term $\theta \in R^{n\times n}$ is a diagonal matrix with well-known filter factors f_i on the diagonal while extra term $\frac{u_i^T s}{\sigma_i}$, corresponds to noise s which prevents the over determined system from blowing up [12,13].
We compute the square root of the successive Jacobian matrix from system of Equation(1.1) as follows:
Algorithm 3.1

(i) Define $F(\lambda) = \sqrt{\lambda}$; $F(\lambda)$ being the principal branch of $\lambda^{\frac{1}{2}}$ for square root function.

(ii) From the spectrum of A i.e. $\lambda_1, \lambda_2, ..., \lambda_n$, we form a polynomial interpolation $(\lambda_i \ge 0)$

$$P(\lambda) = f(\lambda_i)\prod_{\substack{j=1 \\ j\ne i}}^n \frac{(\lambda - \lambda_i)}{(\lambda_j - \lambda_i)} \qquad (3.6)$$

(iii) Form $F(A) = P(\lambda) = P(A)$

(iv) Thus $F(A)^2 = A$ as expected.

We should expect that $rank\, F(A) = rank\, F(A^{-1})$ and $F = F^+, F = F^-$ are self-dual operators for any non-singular matrix A .
The main idea in the above discussion is the principal square root of a matrix A whose eigenvalues lie in the open right half plane. This principal square root whenever it exists is a polynomial in the original matrix.
Definition 3.1[12]: The Jordan Canonical Form

Let $A \in C^{n\times n}$ matrix, there is a non singularmatrix $T^{-1}AT = J = diag\left(J_{m_1}(\lambda_i), ..., J_{m_i}(\lambda_i)\right)$,

Where

$$J_{m_i}(\lambda_i) = \begin{pmatrix} \lambda_i & 1 & & \\ & \lambda_i & 1 & \\ & & \ddots & \\ & & & 1 \\ & & & \lambda_i \end{pmatrix} = \lambda_i I + S \in C^{m_i \times m_i},$$

the Jordan blocks. The numbers $m_1 + m_2 + ... + m_p = n$ are unique and the matrix S is Nilpotent.

The form $T^{-1}AT = J = diag\left(J_{m_1}(\lambda_i), ..., J_{m_i}(\lambda_i)\right)$ is called the Jordan Canonical form of A.

Furthermore, we also compute that $T^{-1}(-A)T = -J$. In other words, $T^{-1}f(-A)T = f(-J) = diag(f(-J_i))$ so that, $f(-\lambda) = \pm f(\lambda)$ and this leads to $f^{(i)}(-\lambda) = \pm(-1)^i f^{(i)}(z)$. The same analysis goes for $f(-J_{m_i}) = \pm f(J_{m_i})$.

Using definition (3.1),we now state the following assertion analogous to[11,12].

Supposing F be defined on the spectrum of $A \in C^{n\times n}$ and assume further that A has the Jordan Canonical form as stated above, then

$f(A) = Tf(J)T^{-1} = Tdiag\left(f(J_i)T^{-1}\right)$, where, defined that

$$f(J_k) = \begin{pmatrix} f(\lambda_i) & f'(\lambda_i) & \cdots \dfrac{f^{(m_i-1)}(\lambda_i)}{(m_i-1)!} & \\ & f(\lambda_i) & \cdots\cdots & \\ & & f'(\lambda_i) & \\ & & & f(\lambda_i) \end{pmatrix} \quad (3.7)$$

Particularly, for the Jordan block J_i, we realize that the interpolating conditions are $P^{(k)}(\lambda_i) = f^{((k))}(\lambda_i); k = 0 : m_i - 1$ in the Hermit interpolating polynomial, and that:

$$P(t) = f(\lambda_i) + f'(\lambda_i)(t-\lambda_i)) + f''(\lambda_i)\frac{(t-\lambda_i)^2}{2!} + ... + f^{(m_i-1)}(\lambda_i)\frac{(t-\lambda_i)^{m_i-1}}{(m_i-1)!} \quad (3.8)$$

In passing, we reconcile the Lagrange interpolating polynomial with the Hermite formula in the form:

$$P(t) = \sum_{i=1}^{s}\left[\left(\sum_{k=0}^{n_i-1}\frac{1}{k!}\phi_i^{(k)}(\lambda_i)(t-\lambda_i)^k\right)\prod_{k\neq i}(t-\lambda_k)\right] \quad (3.9)$$

Where it is set that :

$$\phi_i(t) = \frac{f(t)}{\prod_{k\neq i}(t-\lambda_k)^{n_k}}.$$

We move to form an iterative Newton method for the square root of this matrix.

Taking Z_k to be approximation to $A^{\frac{1}{2}}$ and forming the perturbation to $Z = Z_k + E_k$, we have that

$$A = (Z_k + E_k)^2 = Z_k^2 + Z_k E_k + E_k X_k + E_k^2 \quad (3.10)$$

Ignoring the additional term E_k^2 , we obtain that

$$E_{k+1} = Z_k + E_K,\ Z_k E_k + E_k Z_k = A - Z_k^2 \quad (3.11)$$

Combining together the above ideas we see that iteration for obtaining square root of A is given by the equation

$$Z_{k+1} = \frac{1}{2}\left(Z_k + Z_k^{-1}A\right) \quad (3.12)$$

Equation (3.10) is the well-known Newton iteration for computing square root of a diagonalizable matrix.

To compute the square root of the matrix A we adopt the Newton iteration in the sense of [12, 14], a modified version of Equation (3.11) in the form

$$Z_{k+1} = \frac{1}{2}\left(Z_k + Y_k^{-1}\right); Y_{k+1} = \frac{1}{2}\left(Y_k + Z_k^{-1}\right) \quad (3.13)$$

where $Z_0 = A, Y_0 = I$, and, $\lim_{k\to\infty} = A^{\frac{1}{2}}$; $\lim_{k\to\infty} Y_k = A^{\frac{-1}{2}}$. We used Schultz Hyper Power method [15]to approximate the inverse of the matrix A^{-1} in Equation (3.13)in the form:

$$X_{i+1} = X_i(2I - AX_i) = (2I - X_iA)X_i, i = 0,1,... . \quad (3.14)$$

In method 3.14, the $X_0 = \beta A^H, \beta = \dfrac{2}{\sigma_1^2 + \sigma_p^2}$, where $\sigma_1 > \sigma_2 > ... > \sigma_p > 0$ are obtained fromSVD of A and $A^{-1} \to A^{+}$ as $i \to \infty$.

The matrix sign function is computed by the equation

Sign$(A) = A(A^2)^{-\frac{1}{2}}$ (3.15)

The spectral projectors corresponding to eigenvalues in the right and left hal-plane are

$P_{\pm} = \frac{1}{2}(I \pm sign(A))$ (3.16)

Finally, we state that given infinite power series $f(z) = \sum_{i=0}^{\infty} a_i z^i$ with radius of convergence r, the matrix $f(A) = \sum_{i=0}^{\infty} a_i z^i$ converges if $\rho < r$, and $\rho = \rho(A)$ is the spectral radius of A. If $\rho > r$, the matrix series diverges. The case $\rho = r$ requires further investigation.

Using additional information, we state the following properties concerning matrix A:

$K_{\log}(A) \geq \frac{K(A)}{\|\log(A)\|}$,and log$(I + A) = A - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} + ..., \rho(A) < 1$.

log$(A) = -2\sum_{i=0}^{\infty} \frac{1}{2i+1}\left((I-A)(I+A)^{-1}\right)^{2i+1}$, where $\min_k \operatorname{Re} \lambda_k(A) > 0$.

$K_{\cos}(A) \geq \frac{\|\sin(A)\|\|A\|}{\|\cos(A)\|}$, $K_{\sin}(A) \geq \frac{\|\cos(A)\|\|A\|}{\|\sin(A)\|}$.Thus A with no eigenvalues on R^- has

$\log(A) = 2^k \log(A^{\frac{1}{2}})^k$, where the value of k can be chosen such that $\log\left(A^{\frac{1}{2}}\right)^k$ can be easily computed.

4.0 Numerical Example.

PROBLEM 1.

$$A = \begin{pmatrix} 10 & 7 & 8 & 7 \\ 7 & 5 & 6 & 5 \\ 8 & 6 & 10 & 9 \\ 7 & 5 & 9 & 10 \end{pmatrix}$$

The tabulated results are presented in Table 1.

Table 1 : Computing Square root of a Matrix and simultaneously its inverse with positive eigenvalues by Newton iteration method

No. of iteration k	$A_{Newt}^{\frac{1}{2}}$	$A_{Newt}^{-\frac{1}{2}}$
1	$\begin{pmatrix} 5.5000 & 3.5000 & 4.0000 & 3.5000 \\ 3.5000 & 3.0000 & 3.0000 & 2.5000 \\ 4.0000 & 3.0000 & 5.5000 & 4.5000 \\ 3.5000 & 2.5000 & 4.5000 & 5.5000 \end{pmatrix}$	$\begin{pmatrix} 13.000 & -20.5000 & 5.0000 & -3.0000 \\ -20.5000 & 34.5000 & -8.5000 & 5.0000 \\ 5.0000 & -8.5000 & 3.0000 & -1.5000 \\ -3.0000 & 5.0000 & -1.5000 & 1.5000 \end{pmatrix}$
2	$\begin{pmatrix} 3.3648 & 2.1210 & 2.0459 & 1.7889 \\ 2.1210 & 1.7756 & 1.6484 & 1.2217 \\ 2.0459 & 1.6484 & 3.3684 & 2.4655 \\ 1.7889 & 1.2217 & 2.4655 & 3.4708 \end{pmatrix}$	$\begin{pmatrix} 6.8852 & -10.6210 & 2.4541 & -1.5389 \\ -10.6210 & 17.9744 & -4.3964 & 2.5283 \\ 2.4541 & -4.3984 & 1.8816 & -0.9655 \\ -1.5389 & 2.5283 & -0.9655 & 1.0292 \end{pmatrix}$
3	$\begin{pmatrix} 2.5735 & 1.6198 & 1.2620 & 1.0769 \\ 1.6198 & 1.3051 & 1.1117 & 0.6918 \\ 1.2620 & 1.1117 & 2.5461 & 1.6926 \\ 1.0769 & 0.6918 & 1.6926 & 2.7268 \end{pmatrix}$	$\begin{pmatrix} 4.0811 & -6.0469 & 1.2763 & -0.8744 \\ -6.0469 & 10.3524 & -2.5052 & 1.3805 \\ 1.2763 & -2.5052 & 1.3741 & -0.7083 \\ -0.8744 & 1.3805 & -0.7083 & 0.8320 \end{pmatrix}$
4	$\begin{pmatrix} 2.3998 & 1.5198 & 1.0874 & 0.9190 \\ 1.5198 & 1.1919 & 0.9964 & 0.5720 \\ 1.0874 & 0.9964 & 2.3665 & 1.5258 \\ 0.9190 & 0.5720 & 1.5258 & 2.5676 \end{pmatrix}$	$\begin{pmatrix} 3.0443 & -4.3430 & 0.8419 & -0.6253 \\ -4.3430 & 7.5221 & -1.7991 & 0.9555 \\ 0.8419 & -1.7991 & 1.1899 & -0.6081 \\ -0.6253 & 0.9555 & -0.6081 & 0.7639 \end{pmatrix}$
5	$\begin{pmatrix} 2.3892 & 1.5169 & 1.0776 & 0.9110 \\ 1.5169 & 1.1820 & 0.9914 & 0.5651 \\ 1.0776 & 0.9914 & 2.3568 & 1.5172 \\ 0.9110 & 0.5651 & 1.5172 & 2.5591 \end{pmatrix}$	$\begin{pmatrix} 2.8471 & -4.0172 & 0.7597 & -0.5770 \\ -4.0044 & 6.9610 & -1.6585 & 0.8717 \\ 0.7565 & -1.6585 & 1.1542 & -0.5874 \\ -0.5751 & 0.8717 & -0.5874 & 0.7512 \end{pmatrix}$
6	$\begin{pmatrix} 2.3891 & 1.5170 & 1.0776 & 0.9110 \\ 1.5170 & 1.1818 & 0.9914 & 0.5651 \\ 1.0776 & 0.9914 & 2.3567 & 1.5172 \\ 0.9110 & 0.5651 & 1.5172 & 2.5591 \end{pmatrix}$	$\begin{pmatrix} 2.8394 & -4.0044 & 0.7565 & -0.5751 \\ -4.0044 & 6.9610 & -1.6585 & 0.8717 \\ 0.7565 & -1.6585 & 1.1542 & -0.5874 \\ -0.5751 & 0.8717 & -0.5874 & 0.7512 \end{pmatrix}$
7	$\begin{pmatrix} 2.3891 & 1.5170 & 1.0776 & 0.9110 \\ 1.5170 & 1.1818 & 0.9914 & 0.5651 \\ 1.0776 & 0.9914 & 2.3567 & 1.5172 \\ 0.9110 & 0.5651 & 1.5172 & 2.5591 \end{pmatrix}$	$\begin{pmatrix} 2.8393 & -4.0044 & 0.7565 & -0.5751 \\ -4.0044 & 6.9609 & -1.6585 & 0.8717 \\ 0.7565 & -1.6585 & 1.1542 & -0.5874 \\ -0.5751 & 0.8717 & -0.5874 & 0.7512 \end{pmatrix}$
8	$\begin{pmatrix} 2.3891 & 1.5170 & 1.0776 & 0.9110 \\ 1.5170 & 1.1818 & 0.9914 & 0.5651 \\ 1.0776 & 0.9914 & 2.3567 & 1.5172 \\ 0.9110 & 0.5651 & 1.5172 & 2.5591 \end{pmatrix}$	$\begin{pmatrix} 2.8393 & -4.0044 & 0.7565 & -0.5751 \\ -4.0044 & 6.9609 & -1.6585 & 0.8717 \\ 0.7565 & -1.6885 & 1.1542 & -0.5874 \\ -0.5751 & 0.8717 & -0.5874 & 0.7512 \end{pmatrix}$
9	$\begin{pmatrix} 2.3891 & 1.5170 & 1.0776 & 0.9110 \\ 1.5170 & 1.1818 & 0.9914 & 0.5651 \\ 1.0776 & 0.9914 & 2.3567 & 1.5172 \\ 0.9110 & 0.5651 & 1.5172 & 2.5591 \end{pmatrix}$	$\begin{pmatrix} 2.8393 & -4.0044 & 0.7565 & -0.5751 \\ -4.0044 & 6.9609 & -1.6585 & 0.8717 \\ 0.7565 & -1.6585 & 1.1542 & -0.5874 \\ -0.5751 & 0.8717 & -0.5874 & 0.7512 \end{pmatrix}$

We have also presented results for Lagrange method (3.6) in the form

$$f(A)=P(A)=\frac{1}{4281.60}\left(A^3-4.7114A^2-3.2049A+0.0332I\right)$$
$$-\frac{1}{156.1732}\left(A^3-31.142A^2+25.8539A-0.2605I\right)$$
$$+\frac{1}{80.5577}\left(A^3-34.158A^2+117.2334A-1.1922I\right)$$
$$-\frac{1}{961.0173}\left(A^3-34.9909A^2+145.6771A-98.5475I\right)$$

Thus

$$f_L(A)=p_L(A)=A_L^{\frac{1}{2}}=\begin{pmatrix}2.3739 & 1.5062 & 1.0652 & 0.9001\\ 1.5062 & 1.1743 & 0.9821 & 0.5573\\ 1.0652 & 0.9821 & 2.3414 & 1.5034\\ 0.9002 & 0.5573 & 1.5034 & 2.5439\end{pmatrix}$$

Therefore, we computed that $\left\|A_L^{\frac{1}{2}}-A_{Newt}^{\frac{1}{2}}\right\|=0.0467\cdot$

5.0 Conclusion.

The paper discussed commonly used iterative methods for solving system of nonlinear equation. Estimating a relaxation parameter for Successive Overrelaxation method (SOR) from the Gauss-Siedel iterative method was stressed. The described method may be found useful in the fast sweeping method as in the Eikonal equation, linearized steady compressible Euler equation, nonlinear hyperbolic PDE_s and, fast marching method for the ordered upward wind problem. Particularly in the paper, special emphasis was placedon computing square root of a positive diagonalizable matrix using two different approaches namely,

(i) Lagrange interpolationmethod, and (ii) Newton iterative formula.

In the case of Newton iterative method, we used the Schultz Hyper Power formula to approximate the inverse linear bounded operator in Hilbert space.

The norm bound for the $\left\|A_L^{\frac{1}{2}}-A_{Newt}^{\frac{1}{2}}\right\|=0.0467$ was obtained, which is quite encouraging. The gain in the described method is that they are amenable to Aerospace computation as in polar decomposition for the matrix direction cosine.

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