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Equivalent Lagrangian and Lie Symmetry Analysis of a Class of Kuramoto Sivashinsky(KS)Equations

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Abstract

The equivalent Lagrangians of a class of Kuramoto Sivashinky equations are constructed via Noether approach. A remarkable feature of these Lagrangians, constructed through point transformations of the symmetries, is that they preserve the symmetry structures of the original Lagrangians

Keyword: Equivalent Lagrangian, Lie symmetries.

1. INTRODUTION

The Lagrangians of differential equations with the same algebra of Noether point symmetry generators may be mapped onto the other by a change of variables obtainable from the point symmetries of the equations. The method of equivalent Lagrangians can be used to generate solutions and conserved quantities of the differential equations via point transformations, thereby avoiding the complex integration procedures which are normally required. The method involves the construction of a regular point transformation which maps the Lagrangian of one differential equation to another Lagrangian [1]. Once this transformation is found, one can map the solutions and conserved quantities of the simpler equation to the corresponding solutions and conserved quantities of the given equation we want to analyse, respectively. The application of the concept of equivalent Lagrangians to construct Lagrangians for differential equations with a known Lie algebra of point symmetries has recently been a subject of extensive study. For example, Kara and Mahomed [2] applied the method to two cases of the equation of the form

$$\ddot{q} + p(t)\dot{q} + r(t)q = \mu\dot{q}^2 q^{-1} + f(t)q^n \qquad (1)$$

Kara[3] used the approach to derive equivalent Lagrangians for a unit second order wave equation and a system of second order ordinary differential equations. In this paper, we extend the application to a fourth order partial differential equation of the form

$$U_{tt} + \alpha U_{xxxx} - \gamma(U_x^n)_x = 0, \qquad (2)$$

where α,γ are constants and n > 0. This equation (2) is a modified nonlinear wave equation introduced by Yang and Chen [4]. It is associated with many equations. For example, in [5] a nonlinear wave equation.

$$U_{tt} + \alpha U_{xxxx} - \gamma(U_x^2)_x = 0, \qquad (3)$$

where u(x,t) is the longitudinal displacement, α > 0,γ 6= 0 are real numbers was presented. It is used to study some problems about vertical vibration of one dimensional elasticity pole and two dimensional anti-plane shear in the weak nonlinear analysis of micro-structure model in the elasticity and plasticity. Furthermore, the instability of its special solution and ordinary stain solution were studied [6]. Chen and Yang [7] and Zhang and Chen [8] considered the generalized equation of equation (3) and proved the existence and uniqueness of the global generalized solution and the global classical solution of several initial boundary value problems by the contraction mapping principle. The sufficient conditions of the nonexistence of the solution were also given. Yan [15] studied the equation (2) with the viscous damping term, by using the direct reduction method and obtained four new explicit solutions in the case of n = 2. The work of Yan [15] was extended by Wu and Fan [9] via the same method and presented the solutions for the equation for n ≥ 3. The outline of the paper is as follows. In the next section, we present some basic operators, definitions and concept of equivalent Lagrangians.

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In Section 3, the equivalent Lagrangians of the nonlinear wave equation (2) are constructed through point transformations of the symmetries. A brief discussion and conclusion is given in the last Section.

2 Preliminaries

Definition 2.1.

A $k^{th}-$ order (k $\geq$ 1) System E^{σ} of s partial differential equations of n independent variables x_i: $i = 1,2,\dots,n$ and m-dependent variables u^{α} : $\alpha = 1,2,\dots,m$ is defined by;

$$E^{\sigma}(x^i, U^{\alpha}, U_1, \cdots, U_k) = 0, \quad \sigma = 1, \cdots, s \qquad (4)$$

where u(1),...,u(k) denote the collection of all first, second, ..., kth-order partial derivatives.

Definition 2.2.

The Euler-Lagrangian operator is defined by

$$\frac{\delta}{\delta u^{\alpha}} = \frac{\partial}{\partial u^{\alpha}} + \sum_{s\geq 1}(-1)^s D_{i_1}\cdots D_{i_s}\frac{\partial}{\partial u^{\alpha}_{i_1\cdots i_s}}, \qquad \alpha = 1,2,\dots,m \qquad (5)$$

Where $D_i = \frac{\partial}{\partial x^i} + u_i^{\alpha} + u_{ij}^{\alpha}\frac{\partial}{\partial u_j^{\alpha}} + \cdots, \qquad \alpha = 1,2,\dots,m \qquad (6)$

is the total derivative operator with respect to xi.

Definition 2.3.

The Euler-Lagrangian equations, associated with (4) are the equations

$$\frac{\delta L}{\delta u^{\alpha}} = 0, \qquad \alpha = 1,2,\dots,m \qquad (7)$$

where L is referred to as a Lagrangian of (4).

Definition 2.4.

A Lie Backlund operator X is defined by

$$X = \varepsilon\frac{\partial}{\partial x^i} + \eta^{\alpha}\frac{\delta}{\delta u^{\alpha}} + \sum_{s\geq 1}\varsigma^{\alpha}_{i_1\cdots i_s}\frac{\partial}{\partial u^{\alpha}_{i_1\cdots i_s}}, \qquad \alpha = 1,2,\dots,m \qquad (8)$$

where $\varsigma^{\alpha}_{i_1\cdots i_s}$ are given as

$$\varsigma^{\alpha}_i = D_i(\eta^{\alpha}) - \varsigma^{\alpha}_{i_1\cdots i_s}D_i\varepsilon^j, \qquad \varsigma^{\alpha}_{i_1\cdots i_s} = D_{i_s}(\varsigma^{\alpha}_{i_1\cdots i_s}) - u^{\alpha}_{ji_1\cdots i_{s-1}}D_{i_s}(\varepsilon^j), \qquad s \geq 1 \qquad (9)$$

Definition 2.5. A Lie Backlund operator X of the form (8) is called a Noether symmetry generator associated with a Lagrangian L of (7) if there exists a vector $B = (B^1, B^2, \cdots, B^n)$ such that

$$XL + LD_i(\varepsilon^i) = D_i(B^i), \qquad (10)$$

where X is prolonged to the degree of L [3]. If the vector B is identically zero, then X is a strict Noether symmetry [10]. For each Noether symmetry generator X associated with a given Lagrangian L corresponding to the Euler-Lagrange differential equations, a conserved quantity is obtained [11] using the equation

$$T^i = B^i - N^i L, \qquad i = 1,2,\dots,n \qquad (11)$$

Definition 2.6. Two Lagrangians, $L = L(x, u, u_{(1)}, \cdots, u_{(r)})$ and $\bar{L} = \bar{L}(X, U, U_{(1)}, \cdots, U_{(r)})$, are said to be equivalent if and only if there exists a transformation, $X = X(x,u)$ *and* $U = U(x,u)$, such that

$$L(x, u, u_{(1)}, \cdots) = \bar{L}(X, U, U_{(1)}, \cdots)\det J \qquad (12)$$

where det J is the determinant of the Jacobian matrix of the base transformation $X = X(x,u)$ [12]. For ordinary differential equations in which $u = u(x)$, the definition of equivalence up to gauge function, $f = (x,u)$ is given as

Definition 2.7. Two Lagrangians, L and ¯ L, are said to be equivalent up to gauge function, $f = (x,u)$ if

$$L(x,u,u') = L\left(X, U, U^{-}\right)\frac{dX}{dx} + f_x + u' f_u \qquad (13)$$

where $X = X(x,u)$ *and* $U = U(x,u)$[1].

3. Equivalent Lagrangian for the class of KS Equations

Firstly, we present the Lie and Noether point symmetries of (2) which shall be used in this section to form the equivalent Lagrangian of the equation and in subsequent sections for further analysis. The symmetry structure splits into two different cases: case (i) $n \neq 1$ and case (ii) $n = 1$.

3.1 Case (i) $n \neq 1$

The Lie point symmetry generators of (2) for this case is a five-dimensional Lie algebra spanned with the following basis;

$$x_1 = \frac{\partial}{\partial x}, \quad x_2 = \frac{\partial}{\partial t}, \quad x_3 = \frac{\partial}{\partial u}, \quad x_4 = t\frac{\partial}{\partial u}, \quad x_5 = \frac{x}{2}\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + \frac{\mu u}{2}\frac{\partial}{\partial u}, \qquad (14)$$

where $\mu = \frac{n-3}{n-1}$. Obviously, when $n = 1$, the dilations in space and time are lost. This seems outstanding and distinguishes the symmetry structure of (2) for $n = 1$ from any other values of n. The Noether point symmetries of the Lagrangian

$$L = \frac{1}{2}u_t^2 - 1\ 2\alpha u_{xx}^2 - \frac{1}{n+1}\gamma u_x^{n+1} \qquad (15)$$

of equation (2), are the generators X_1 *to* X_4 above, all having zero gauge functions except X_4 which has a gauge function $(B_1, B_2) = (u, 0)$.

We want to construct a Lagrangian $\bar{L} = \bar{L}(r, s, v), v = v(r, s)$ equivalent to (15) using the transformation $x = x(r, s, v), t = t(r, s, v), u = u(r, s, v)$. Since any one parameter group G of a transformation can be reduced under a suitable change of variables to the translation group with the operator $G = \frac{\partial}{\partial t}$ [13], suitably equivalent quantities can be constructed using symmetry structures. Therefore, a point transformation that leaves the Lagrangians of (2) invariant under change of variables can be obtained through point transformations of its symmetry structures. Hence, by mapping a Noether symmetry generator

$$X_2 = \frac{\partial}{\partial t} \qquad (16)$$

to the dilation operator in (r, s, v) variables

$$X_2 = \frac{1}{2}r\frac{\partial}{\partial t} + s\frac{\partial}{\partial s} + \frac{1}{2}\mu v\frac{\partial}{\partial v} \qquad (17)$$

from the Lie point symmetries, we obtain the point transformation $x = f\left(\frac{s}{r^2},\ vr^{-\mu}\right)$, $t = log\, r + g\left(\frac{s}{r^2},\ vr^{-\mu}\right)$, $u = h(\frac{s}{r^2},\ vr^{-\mu})$As an example, we let

$$x = \frac{s}{r^2},\quad t = log\, r,\quad u = vr^{-\mu} \qquad (18)$$

Thus, it follows that $u_x = r^{2-\mu}v_s$, $u_t = 2sr^{-\mu}v_s + r^{1-\mu}v_r - \mu r^{-\mu}v$, $u_{xx} = r^{4-\mu}v_{ss}$ and the Jacobian of the transformation $J = -\frac{1}{r^3}$. By Definition 2.6, a Lagrangian equivalent to (15) is of the form

$$\bar{L} = \frac{1}{r^3}\left[\frac{1}{2}(\,2sr^{-\mu}v_s + r^{1-\mu}v_r - \mu r^{-\mu}v)^2 - \frac{1}{n+1}(r^{2-\mu}v_s)^{n+1}\right] \qquad (19)$$

The Euler Lagrangian equation associated with equation (19) is

$$\mu^2 vv_s - 4s(\mu-1)v_s^2 - n\gamma r^{n+5}v_s v_{ss} + v_s\big(4s^2 v_{ss} + r(r^7\alpha v_{sss} + (1-2\mu)v_r + 4sv_{rs} + rv_{rr})\big) = 0, \qquad (20)$$

To verify that this Lagrangian (19) is indeed equivalent to (15) under the point transformation (18), we calculate its Noether point symmetries in the new variables (r, s, v) given as

$$X_1 = r^2\frac{\partial}{\partial s},\quad B^1 = 0,\ B^2 = 0$$
$$X_2 = r\frac{\partial}{\partial r} + 2s\frac{\partial}{\partial s} + \mu v\frac{\partial}{\partial v},\quad B^1 = 0,\ B^2 = 0$$
$$X_3 = r^{\mu}\frac{\partial}{\partial v},\quad B^1 = 0,\ B^2 = 0 \qquad (21)$$
$$X_4 = r^{\mu}logr\frac{\partial}{\partial v},\quad B^1 = r^{-\mu}v,\ B^2 = 0$$

Clearly, this Noether algebra (21) is isomorphic to the Noether algebra of the Lagrangian L of (15). Hence, L and ¯ L are equivalent under the point transformation (18). Thus, using the equivalent Lagrangian approach, we derive new wave equation (20) from (2) which has some physical interpretations in physics but appears complex. However, its solutions can be obtained using the transformation (18) once the solutions of the wave equation (2) are known. Many Lagrangians equivalent to (15) can also be constructed through mappings of other symmetry generators using similar approach.

3.2 Case (ii) n = 1

This case gives rise to the linear wave equation of (2)

$$u_{tt} + \alpha u_{xxxx} - \gamma u_{xx} = 0, \qquad (22)$$

which admits the Lie point symmetries in addition to translations, X_1 *and* X_2 of Case (i), $X_3 = u\frac{\partial}{\partial u}$ *and* $X_4 = F_1(x,t)\ \frac{\partial}{\partial u}$. The strict Noether point symmetries of its Lagrangian

$$L = \frac{1}{2}u_t^2 - \frac{1}{2}\alpha u_{xx}^2 - \frac{1}{2}\gamma u_x^2 \qquad (23)$$

also include X_1 and X_2, and $X_3 = \frac{\partial}{\partial u}$, which is a special case of $F_1(x,t) = 1$.

Similarly, by mapping the operator $X_3 = \frac{\partial}{\partial u}$ to the dilation generator

$$X_4 = v\frac{\partial}{\partial v}, \qquad (24)$$

a point transformation $x = f(r,s),\quad t = g(r,s),\quad u = logv + h(r,s)$ is obtained. Choosing;

$$x = r,\quad t = s,\quad u = logv, \qquad (25)$$

results to $u_x = \frac{v_r}{v}$, $u_t = \frac{v_s}{v}$, $u_{xx} = \frac{v_{rr}}{v} - \frac{v_r^2}{v^2}$, $J = 1$. Hence by Definition 2.6, a Lagrangian equivalent to (23) under the point transformation (25) is

$$L = \frac{1}{2}\left[\left(\frac{v_s}{v}\right)^2 - \alpha\left(\frac{v_{rr}}{v} - \frac{v_r^2}{v^2}\right)^2 - \gamma\left(\frac{v_r}{v}\right)^2\right] \qquad (26)$$

which has a corresponding Euler Lagrangian equation

$$v^2(v_s^2 - \gamma v_r^2) + 6\alpha v_r^4 - 12\alpha v_r^2 v_{rr} + v^2(3\alpha v_{rr}^2 + 4\alpha v_r v_{rr} - v v_{ss} - \gamma v v_{rr} + \alpha v v_{rrrr}) = 0. \quad (27)$$

The strict Noether point symmetries of the above Lagrangian (26) are

$$X_1 = \frac{\partial}{\partial r}, \quad X_2 = \frac{\partial}{\partial s}, \quad X_3 = v\frac{\partial}{\partial v} \qquad (28)$$

Thus, using the similar reason in Case (i), the Lagrangians L and $\bar{L}$ of equations (23) and (26) respectively, are equivalent.

4 Discussion and Conclusion

Equivalent Lagrangians of a class of Kuramoto Sivashinsky (KS) equations, which admit natural Lagrangians were constructed via Noether point symmetries. These equivalent Lagrangians obtained through point transformations of the symmetries give rise to the Noether symmetry algebras which are isomorphic to the symmetry algebras of the original Lagrangians as expected.

Interestingly, the point transformations obtained here can be used to find the exact solutions and conserved quantities of the equations associated with the equivalent Lagrangians.

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