Transactions of the Nigerian Association of Mathematical Physics
Volume 5,(September and November, 2017), pp27 – 32

Almost Sure Exponential Stochastic Stabilization Of First Order Neutral Delay Differential Equations With Positive And Negative Coefficients

A. O. Atonuje and J. S. Apanapudor

Department of Mathematics, Delta State University, Abraka, Nigeria.

Abstract

In this paper, we perturb an unstable deterministic first order neutral delay differential equation with positive and negative coefficients by an Ito-type Brownian white noise. It is then established that the presence of noise forces the new system of stochastic neutral delay differential equation (SNDDE) to become almost sure exponentially stable. This is achieved by replacing the noise scaling parameter with a sufficiently strong non-integral expression thereby forcing the SNDDE to be almost surely exponentially stable whereas the classical version still remains unstable.

Keywords: Brownian white noise, neutral delay differential equation, noise scaling parameter, almost sure exponential stability, stochastic neutral delay differential equation

1.0 Introduction

In recent decades, there has been much interest in studying stability of differential equations. It is much more robust to investigate the stability of stochastic differential equations and neutral delay differential equations than the stability of ordinary differential equations without delay because of the fact that these classes of equations arise in many areas of applied mathematics, engineering, population dynamics, etc. For recent results on stabilization of solutions of dynamical systems, we refer the reader to the important works found in [1 - 7].

In purely stochastic setting, stabilization of solutions of dynamical systems were modeled in [8 – 10] and some references therein. In [9], the authors investigated the stabilization of a system of deterministic differential equation using Levy white noise as follows: The authors stochastically perturbed a deterministic differential equation of the form

$$\frac{dx(t)}{dt} = p(x(t-)) \qquad (1)$$

By a Brownian motion and an independent compensated Poisson process into a stochastic differential equation

$$dx(t) = [p(x(t-)) - \lambda D(y)x(t-)]dt + \sum_{h-1}^{k} G_h x(t-)dB(t) + \int_{|y|\geq r} D(y)x(t-)\overline{N}(dt,dy),$$
$$+\int_{|y|\geq r} E(y)x(t-)N(dt,dy), \ t \geq t_0 \qquad (2)$$

On $x(t_0) = x_0 \in \Re^d$, *where* $G_h \in M_d(\Re)$, for $1 \leq h \leq k$. The integer $r > 0$ stands to separate compensated small jumps from large jumps so that N represents small jumps and it is an $F_t - adapted$ Poisson measure on $\Re^+ \times \Re^k \setminus \{0\} \to \Re^k$, $\overline{N}$ is the compensator of N defined by $\overline{N}(dt,dy) = N(dt,dy) - V(dy)dt$ for some Levy measure V. Where $B = \{B(t)\}_{t\geq 0} = B_1(t), B_2(t), \ldots, B_K(t)$ is an k – dimensional $F_t - adapted$ Brownian motion defined on the quadruplet $\{\Omega, F, \{F(t)\}_{t\geq 0}, P\}$, where (Ω, F, P) is a probability space and $\{F(t)\}_{t\geq 0}$ is its filtration. The suitable functions $D, E \in (\Re^k, M_d(\Re))$ are such that $y \to D(y), y \to E(y)$ are measurable maps from $\Re^k \to M_d(\Re)$ *and* $D \in M_d(\Re)$. The authors presented a result which runs as follows:

Correspondence Author: Atonuje A.O., E.mail: austino412@yahoo.com, Tel.:+2348035085758, +2348035390744 (JSA)

Suppose that the following conditions are satisfied for all
$x \in \Re^d$, *where* $\varepsilon > 0,\ \gamma \geq 0$

$$\left.\begin{aligned} &(i)\ \sum_{h-1}^{k} \left|G_h x\right|^2 \leq \varepsilon |x|^2 \\ &(ii)\ \sum_{h-1}^{k} \left|x^T G_h x\right|^2 \leq \gamma |x|^4 \end{aligned}\right\} \qquad (3)$$

Then the sample Lyapunov exponent of the solution of Eq. (2) exists and satisfies

$$\underset{t\to\infty}{Lim}\, Sup \frac{1}{t} Log|x(t)| \leq -\left(\gamma + \lambda\mu_{\min} - k - \frac{\varepsilon}{2} - \lambda Log(1+\mu_{\max})\right)$$

almost surely (a.s) for any $x_0 \neq 0$, *where* $\mu_{\min}$ *and* $\mu_{\max}$ are the minimum and maximum eigen values of the $d \times d$ symmetric positive definite matrix D respectively. In particular, the trivial solution of Eq. (2) is almost surely exponentially stable provided that the following relationship is satisfied: $\gamma + \lambda\mu_{\min} - k - \frac{\varepsilon}{2} - \lambda Log(1+\mu_{\max}) > 0.$

Other results on stochastic stabilization to which the reader is referred can be found in [11 – 13]. Although, stabilization of dynamical systems (functional and non-linear differential equations), has been well developed during the last few decades, one can easily observe from existing literature that not much effort has been devoted to the study of stochastic stabilization of neutral differential equations to the best of the authors' knowledge.
The main aim of the present paper is to perturb a deterministic neutral delay differential equation (NDE) by a Brownian white noise of Ito-type into a stochastic neutral delay differential equation (SNDDE) so that with the choice of a sufficiently large non-integral noise scaling parameter, the presence of the Brownian motion is able to stochastically stabilize the system in an almost sure exponential sense. The unstable behavior which might have been exhibited by the original deterministic neutral delay differential equation is exterminated by the presence of multiplicative Brownian white noise.

2. NOTATIONS AND PRELIMINARIES:

We consider and propose to investigate the stability conditions for a deterministic neutral delay differential system of the form

$$\left.\begin{aligned} &\frac{d}{dt}\left[(x(t) - R(t)x(t-r))\right] = -P(t)x(t-\Gamma) + Q(t)x(t-\sigma) \\ &x(t) = \psi(t),\ t \in [-\Gamma_0, 0] \end{aligned}\right\} \qquad (4)$$

with $\Gamma_0 = \max(r, \Gamma, \sigma)$ *where* $R,\ P,\ Q \in C([t_0, \infty), \Re^+),\ r \in (0, \infty),\ \Gamma, \sigma \in \Re,\ \Gamma > \sigma$

$$\bar{P}(t) = P(t) - Q(t - r + \sigma) \geq 0 \qquad (5)$$

where Eq. (5) is not identically zero. By solution of Eq. (4), we refer to a function $x \in C([t_0, -\Gamma_0], \Re)$ *for some* $t_1 \geq t_0$ *such that* $x(t) - R(t)x(t-r)$ is continuously differentiable on $[t_0, \infty)$ and such that Eq. (4) is satisfied for $t \geq t_1$. In the present effort, Eq.(4) is perturbed by a Brownian white noise into a stochastic neutral delay differential equation of the form:

$$d\left[(X(t) - R(t)X(t-r))\right] = \left[-P(t)X(t-\Gamma) + Q(t)X(t-\sigma)\right]dt + uh(X(t), t)dB(t) \qquad (6)$$

On $t \in [t_0, T]$. Eq. (6) has its initial data

$$X(t_0) = \xi = \{\xi(\theta) : -\Gamma_0 \leq \theta \leq 0\} \in L^2_{F_{t_0}}([-T, 0], \Re^d),\ \ for\ t \geq t_0 \qquad (7)$$

By Eq. (7), we mean that ξ is an F_{t_0} −measurable $C([-\Gamma, 0], \Re^d)$−*valued* random variable such that $E\|\xi\|^2 < \infty$, where
$R : \Re^d \to \Re^d;\ P : \Re^d \times \Re^d \times \Re_+ \to \Re^d;$
$Q : \Re^d \times \Re^d \times \Re_+ \to \Re^d$ *and* $h : \Re^d \times \Re^d \times \Re_+ \to \Re^{d\times m}$ are smooth functions so that Eq. (6) has a unique solution for any given initial datum and $B(t)$ is a k- dimensional Brownian motion which represents an Ito-type white noise. Moreover, we let the functions P, Q, h satisfy the local Lipchitz condition and the linear growth condition and also request that R is Lipchitz continuous with Lipchitz coefficient less than 1, that is, there exists $L \in (0,1)$ such that
$|R(x) - R(y)| \leq L|x - y|,$ *for all* $x, y \in \Re^d$, then there exists a unique solution for the SNDDE (6).

Definition 1:
Let $X(t,\ \xi):[-\Gamma_0,T]\times\Omega\to\Re$ be an $\{F(t)\}_{0\le t\le T}-adapted$ stochastic process which satisfies Eq. (6) together with its initial data $X(t_0)=\xi=\{\xi(\theta):-\Gamma_0\le\theta\le 0\}$, then $X(t,\ \xi)$ is called a strong solution of Eq. (6) if we request for any other solution $Y(t,\ \xi)$ such that
$P(X(t,\ \xi)=Y(t,\ \xi):t\in[-\Gamma_0,T])=1$, then $X(t,\ \xi)$ is called a path-wise unique solution of Eq. (6).
Assume that $P(0,t)\equiv 0,\ Q(0,t)\equiv 0\ and\ h(0,t)$ such that the SNDDE (6) has the solution $X(t)\equiv 0$ corresponding to the initial datum $X(t_0)=0$, then this solution $X(t)$ is called a trivial solution or equilibrium position of the SNDDE (6).

Definition 2:
The trivial solution $X(t:t_0,\ x_0)$ of the SNDDE (6) is said to be almost surely exponentially stable if the sample Lyapunov exponent is negative. More specifically, the trivial solution is almost surely exponentially stable if

$$\underset{t\to\infty}{Lim\,Sup}\frac{1}{t}Log\left|X(t:t_0,x_0)\right|<0 \qquad (8)$$

where the left hand side of Eq. (8) is called the sample Lyapunov exponent and as such if Eq.(8) holds one says that the state of the SNDDE (6) is insensitive to significantly small changes in the initial state or parameters of the system in an almost sure exponential sense.
The following is a special case of the result found in Mao [15] Theorem 4.3.3

Proposition 1:
Suppose that there exists a positive-definite function $V\in C^{2,1}(\Re^2\times[t_0,\infty),\Re_+)\ and$ constants $p>0,\ \alpha_1>0,\ \alpha_2\in\Re,\ \alpha_3\ge 0$ such that for all $x\neq 0,\ t\ge t_0$

$(a)\ \alpha_1|x|^p\le V(x,t)$

$(b)\ LV(x,t)\le\alpha_2 V(x,t)$

$(c)\ |V_x(x,t)h(x,t)|^2\ge\alpha_3 V^2(x,t)$

Then, $\underset{t\to\infty}{Lim\,Sup}\frac{1}{t}Log\left|X(t:t_0,x_0)\right|\le\frac{\alpha_3-2\alpha_2}{2p}\quad a.s.\ for\ all\ x_0\in\Re^d$. In particular, if $\alpha_3<2\alpha_2$, then the trivial solution of Eq. (6) is almost surely exponentially stable.

ASSUMPTIONS:
The following assumptions are useful for the proof of the main result:
A_1: We request the existence of a symmetric positive – definite matrix M and constants $\phi,\ \delta,\ \eta\ with\ 2\eta>\delta\ such\ that$

$(1)\ \left|x^T MP(x,t)\right|\le\phi|x|^2$

$(2)\ trace\left(h^T(x,t)Mh(x,t)\right)\le\delta\ x^T Mx$

$(3)\ \left|x^T Mh(x,t)\right|^2\ge\eta\left|x^T Mx\right|^2,\ for\ all\ t\ge 0,\ x\in\Re^d.$

A_2: There exists a pair of constants $W>0,\ \varsigma\ge 0$ such that

$$\|\pi(t)\|\le We^{\varsigma t},\quad for\ t\ge 0 \qquad (9)$$

Assumption A_1 guarantees that for any sufficiently large noise scaling parameter u, the trivial solution of the SNDDE (6) is almost surely exponentially stable and as such, the SNDDE (6) is the stochastically stabilized system of the deterministic neutral delay differential equation(4) which is generally unstable.
Our approach, here is to attempt to replace the noise scaling parameter u by a finite expression, (i.e. $u<\infty$), that satisfies A_2 thereby forcing the SNDDE (6) to be a. s. $L(\Re_+,\Re^d)$ self- stabilized in an almost sure exponential sense. $\pi(t)$ in assumption A_2 is called the convergence rate function.

3. **ALMOST SURE EXPONENTIAL STOCHASTIC SELF – STABILIZATION:**
In this section, we shall propose to stabilize the deterministic neutral delay differential equation. To be able to achieve this, the retarded arguments are assumed to be sufficiently small in which case the results of the system without delays may become similar to those with delays. Here, we choose to replace the noise scaling parameter u with a non-integral expression given by

$$u = \sup_{0 \le s \le t} |\pi(s)x(s)| \qquad (10)$$

Where $\pi(.)$ is a continuous $\Re^{n\times d} - valued$ function defined on $\Re_+$ which satisfies Eq.(9). We then establish that the stochastic neutral delay differential equation

$$d\,[(X(t) - R(t)X(t-r))] = [-P(t)X(t-\Gamma) + Q(t)X(t-\sigma)]dt + \left(\sup_{0 \le s \le t} |\pi(s)x(s)|\right) h(X(t),t)dB(t)$$
$$for \;\; t \ge 0,\;\; x(0) = x_0 \in \Re^d \qquad (11)$$

is a.s. exponentially stable. The choice of the parameter u, in the SNDDE (6) is appropriate, provided that the expression is made finite, that is, $\sup_{0 \le s \le t} |\pi(s)x(s)| < \infty$. Moreover, we shall apply the idea that if the retarded arguments r, Γ, σ are sufficiently small, then the stability behavior of differential equations with delays is similar to those of differential equations without delays, (see Driver et al [14]). The following which is called the exponential martingale inequality will be useful for the proof of the main result. It is a special case of the result found in Mao [15].

Proposition 1:

Assume that T, δ, η with $T \ge t$ and given a function $h = \{h_1,\ h_2,\ ,h_k\} \in L^2(\Re_+, \Re^{1\times k})$.Then

$$\mathrm{P}\left\{\sup_{0\le t\le T} \int_0^t h(s)dB(t) - \frac{\delta}{2}\int_0^t |h(s)|^2 ds > \eta \right\} \le e^{-\delta\eta} \qquad (12)$$

Proof:

We introduce the stopping time $\tau_d = \inf\left\{t \ge 0 : \left|\int_0^t h(s)B(s) + \int_0^t |h(s)|^2 ds \ge d\right|\right\}$ where $d \ge 1$ is some integer. Given an Ito process,

$x_d(t) = \delta\int_0^t h(s) I_{[[0,.\tau_d]]}(s)dB(s) - \frac{\delta^2}{2}\int_0^t |h(s)|^2 I_{[[0,.\tau_d]]}(s)ds$, we see that the process

$x_d(t) = \delta\int_0^t h(s) I_{[[0,.\tau_d]]}(s)dB(s) - \frac{\delta^2}{2}\int_0^t |h(s)|^2 I_{[[0,.\tau_d]]}(s)d \le L$, where L is its bound and as such, $\tau_d \to \infty$ with a highly positive probability. Applying Ito formula to $e^{\,x_d(t)}$, we get

$$e^{[x_d(t)]} = 1 + \int_0^t e^{[x_d(s)]}dx_d(s) + \frac{\delta^2}{2}\int_0^t e^{(x_d(s))}|h(s)|^2 I_{[[0,\tau_d]]}(s)ds$$
$$= 1 + \delta\int_0^t e^{[x_d(s)]}h(s) I_{[[0,\tau_d]]}(s)dB(s) \qquad (13)$$

We note from the concept of multi-dimensional Ito integrals, that the integral in Eq. (13) is purely an $\Re^d - valued$ continuous martingale in respect to the filtration $\{F_t\}_{t\ge0}$ having the property that if $h \in M^2([0,T], \Re^{1\times K})$ *and* i, j are some stopping times such that $0 \le i \le j \le T$, *then*

$$E\left(\int_i^j h(s)dB(s) \mid F_i\right) = 0 \qquad (14)$$

$$E\left(\left|\int_i^j h(s)dB(s)\right|^2 \mid F_i\right) = E\left(\int_i^j |h(s)|^2 ds \mid F_i\right) \qquad (15)$$

Applying the idea of Eq. (14) and Eq. (15), we see that $e^{[x_d(t)]}$ is a non-negative martingale on $| \; t \ge 0$ *with* $E\left(e^{[x_d(t)]}\right) = 1$ and as such, by the Doob's sub -martingale inequalities, we get that

$$|\,\mathrm{P}\left\{\sup_{0 \le t \le T} e^{[x_d(t)]} \ge e^{-\delta\eta}\right\} \le e^{-\delta\eta} E\left(e^{(x_d(T))}\right) = e^{-\delta\eta}$$

which follows that

$$\mathrm{P}\left\{\sup_{0 \le t \le T}\left[\int_0^t h(s)\, I_{[0,\,\tau_d]}(s)dB(s) - \frac{\delta}{2}\int_0^t |h(s)|^2 I_{[0,\,\tau_d]}(s)ds\right] > \eta\right\} \le e^{-\delta\eta}$$

which gives Eq. (12) by letting $d \to \infty$ as required.

We now establish that if the noise scaling parameter is finite then the presence of Brownian white noise can stochastically self stabilize the SNDDE (6) in an almost sure exponential sense

Theorem 1:
Assume that assumptions A_1 and A_2 hold. Let the time delays be sufficiently small.

$$(i)\ if\ \lambda_{\min}\left(\pi^T(t)x(t)\right)\to\infty,\ as\ t\to\infty,\ then\ \lim_{t\to\infty}\left|x(t,x_0)\right|=0 \tag{16}$$

$$(ii)\ if\ \liminf_{t\to\infty}\log\left[\lambda_{\min}\left(\pi^T(t)\pi(t)\right)\right]/t\geq\lambda>0,\ then\ \limsup_{t\to\infty}Log\left(\left|x(t,x_0)\right|\right)\leq\frac{-\lambda}{2},\ a.s. \tag{17}$$

Moreover, for all $x_0\in\Re^d$, the solution of Eq. (11) must fulfill the condition

$$\sup_{0\leq s\leq t}\left|\pi(s)x(s)\right|<\infty \tag{18}$$

and hence the Eq. (11) is almost surely exponentially stable.

Proof:
On the contrary, let us assume that Eq. (14) does not hold, then there exist some $x_0\neq 0$ such that given $\Omega^*\subset\Omega$, where $\Omega^*=\left\{w\in\Omega:\sup_{0\leq s<\infty}\left|\pi(t)X(t,x_0)\right|=\infty\right\}$, $\mathrm{P}(\Omega^*)>0$, that is, $\mathrm{P}\left[w\in\Omega:\sup_{0\leq s<\infty}\left|\pi(t)X(t,x_0)\right|=\infty\right]>0$. We now write $X(t_0,x_0)=X(t)$ and apply Ito formula and assumption A_1 and get that for some $t\geq 0$, there exists a finite integer $\psi_1(w)$ such that

$$Log\left(X^T(t)MX(t)\right)\leq Log\left(x_0^TMx_0\right)+\frac{2\psi_1 t}{\lambda_{\min}(M)}+\delta\int_0^t\left(\sup_{0\leq s<\infty}\left|\pi(s)x(s)\right|\right)^2 ds$$

$$-2\int_0^t\left(\sup_{0\leq s<\infty}\left|\pi(s)x(s)\right|\right)^2\frac{\left|x^T(s)h(x(s))\right|^2}{\left|x^T(s)Mx(s)\right|}ds+W(t) \tag{19}$$

where $W(t)=2\int_0^t\left(\sup_{0\leq s\leq t}\left|\pi(s)x(s)\right|\right)^2\frac{x^T(s)Mh(x(s),s)}{x^T(s)Mx(s)}$ represents a continuous Martingale which vanishes at $t=0$. Choose $\psi=1,2,3,\ldots$, then by the Exponential Martingale inequality, we see that

$$\mathrm{P}\left(\sup_{0\leq s<\psi}\left[W(t)-\frac{2\eta-\delta}{8\eta}\langle W(t),W(t)\rangle\right]>\frac{8\eta\log\psi}{2\eta-\delta}\right)\leq\frac{1}{\psi^2}$$

where $\langle W(t),W(t)\rangle=4\int_0^t\left(\sup_{0\leq s\leq\psi}\left|\pi(s)x(s)\right|\right)^2\frac{\left|x^TMh(x(s),s)\right|^2}{x^T(s)Mx(s)}ds$

By Borel Cantelli lemma, we see that for almost all $w\in\Omega^*$ there exists a random integer $\psi_2(w)$ such that for all $\sup_{0\leq t\leq\psi}\left[W(t)-\frac{2\eta-\delta}{8\eta}\langle W(t),W(t)\rangle\right]\leq\frac{8\eta\log\psi}{2\eta-\delta}$, that is, for $0\leq t\leq\psi,\ \psi\geq\psi_2$,

$$W(t)\leq\frac{8\eta\log\psi}{2\eta-\delta}+\frac{2\eta-\delta}{8\eta}\langle W(t),W(t)\rangle$$

$$\leq\frac{8\eta\log\psi}{2\eta-\delta}+\frac{2\eta-\delta}{2\eta}\int_0^t\left(\sup_{0\leq t\leq\psi}\left|\pi(s)x(s)\right|\right)^2\frac{\left|x^TMh(x(s),s)\right|^2}{\left(x^T(s)Mx(s)\right)}ds \tag{20}$$

By substituting Eq. (20) into Eq. (19) and applying A_1 we now have for the random integer ψ_2 that

$$Log\left(X^T(t)MX(t)\right)\leq Log\left(x_0^TMx_0\right)+\frac{2\psi t}{\lambda_{\min}(M)}+\frac{8\eta\log\psi}{2\eta-\delta}$$

$$\frac{-2\eta-\delta}{2}\int_0^t\left(\sup_{0\leq s<\infty}\left|\pi(s)x(s)\right|\right)^2 ds \tag{21}$$

$for\ all\ 0\leq t\leq\psi,\ \psi\geq\psi_2\ \ a..s.$

By definition, for every $w\in\Omega^*$ there exists a random number $\psi_3(w)$ such that

$$\sup_{0\leq s\leq t}\left|\pi(s)x(s)\right|\geq\sqrt{\frac{2\left(2\psi/\lambda_{\min}(M)+2(\xi+2)\right)}{2\eta-\delta}} \tag{22}$$

where $\lambda_{\min}(M)$ is the smallest Eigen value of the symmetric positive - definite matrix M. We have from the inequalities (21) and (22) and almost all $w \in \Omega^*$, that there exists a random number $\psi_4(w)$ such that

$$|x| \leq \frac{e^{-(\xi+1)t}}{\sqrt{\lambda_{\min}(M)}} \quad for \ all \ t \geq \psi_4$$

Applying A_2 for almost all $w \in \Omega^*$, we have

$\sup_{0\le s<\infty}|\pi(t)x(t)| \leq \sup_{0\le s<\psi_4} We^{\xi t}|x(t)| + \sup_{\psi_4 \le s<\infty}|\pi(s)x(s)|\frac{We^{-t}}{\sqrt{\lambda_{\min}(M)}} < \infty$ which is a contradiction, by the definition of Ω^* and as such,

$\sup_{0\le s\le t}|\pi(s)x(s)| < \infty$ in Eq. (18) must hold true and hence Eq.(16) and Eq.(17) follows which implies that

$\lim_{t\to\infty} \sup \frac{1}{t} Log|X(t:t_0,x_0)| < 0.$ and the result follows.

CONCLUSION:

Whereas the deterministic neutral delay differential equation (4) still remains unstable, the presence of the Brownian white noise in the SNDDE (6) introduces an almost sure exponential self stabilization. This is necessitated by the choice of a sufficiently strong noise scaling parameter which is established to be finite, provided that the stated assumptions hold. This a. s. exponential self stabilization was motivated by the presence of the Brownian noise which cannot occur in the deterministic equation, where noise is absent.

REFERENCES:

[1] Cao, K. C., Zhong, S. M. and Liu, B. S. (2003). BIBO Stabilization for systems with Time – Delay and non-linear Perturbations. Jour. Univ. Elect. Sci. Tech. China, 23, 287 – 289.

[2] Zhou, X. and Zhong, S. (2013). Mean Square Stability of Discrete – Time Stochastic Control Systems with Delays and Non-linear Perturbations. Jour. Fac. Sci. & Math. Univ. of Nis, Serbia.

[3] Dragan, V. and Morozan, T, (2010). Robust Stability and Robust Stabilization of Discrete –Time Linear Stochastic Systems. Anal of the Academy of Romanian Scientists Series on Mathematics and its Applications. Vol. 2, No. 2, 141 -171.

[4] Applebaum, D and Siakalli, M. (2009). Asymptotic Stability of Stochastic Differential Equations Driven by Levy Noise. Jour. Appl. Prob. Vol. 46, No. 4, 1116 – 11129.

[5] Li, C. W., Dong, Z. and Situ, R. (2002). Almost Sure Stability of Linear Stochastic Differential Equations with Jumps. Jour. Prob. Theo. And Related Fields, Vol.123, 121 – 155.

[6] Mao, X. (2000). Robustness of Stability of Stochastic Differential Equations with Markovian Switching. SACTA, Vol. 3, No.1, 48 – 61.

[7] Atonuje, A. O. and Ezenweani, U. L. (2011). Instability and Stochastic Stabilization in Solutions of Delay Differential Equations with Two Delays. Journal of Mathematical Sciences. Vol. 23, No. 1, 207 – 214.

[8] Ozbek, L., Babacan, E. K. and Efe, M. (2010). Stochastic Stability of the Discrete –Time Constrained Extended Kalman Filter. Turk. Jour. Elect. Eng. & Comp. Sci. Vol. 18, No. 2, 211 -223.

[9] Applebaum, D. and Siakalli, M. (2010).Stochastic Stabilization of Dynamical Systems Using Levy Noise. Stoch. Dyn. 10, No 4, 509 -527 .

[10] Atonuje, A. O. and Tsetimi, J. (2013). Stabilization of Non – Linear Delay Differential equations by Levy Process. Jour. Nig. Inst. Prod. Eng. Vol. 15, 184 – 194.

[11] Has'minskii, R.Z. (1980). Stochastic Stabilty of Differential Equations. Alphen: Stijtoff and Noordoff (Translation of the Russian Edition, Mosco, Nauka)

[12] Basak, G. K., Bisi, A. and Ghosh, M. K. (1996). Stability of Random Diffusion and Drift. Jour. Math. Anal. Appl. Vol. 202, 406 -422.

[13] Halaney, A. (1966). Differential Equations: Stabilty, Oscillations, Time Lag. Academic Press, New York.

[14] Driver, R. D., Sasser, D. W. and Slatter, M. l. (1973). The Equation $x^{|}(t) = ax(t) + bx(t-r)$ with small delay.American Mathematical Monthly, 80, 990 -995.

[15] Mao, X. (1997). Stochastic Differential Equations and their Applications. Horwood Publishing Limited, England.