Transactions of the Nigerian Association of Mathematical Physics
Volume 5,(September and November, 2017), pp11 – 22

Some Properties of Certain New Classes of Analytic Functions Defined By a Generalized Differential Operator

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Abstract

In this work we investigate certain new classes of analytic functions defined by a generalized differential operator. The sufficiency conditions for a function to belong to these classes are investigated and some other properties of these classes of functions are obtained as well. Several interesting consequences of the main results are also mentioned.

Keywords: Coefficient inequalities, Subordination, Analytic functions, Extremepoints, Distortion inequalities

1.0 Introduction

Let A be the class of functions $f(z)$of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

and

$$A^+ = \left\{ f \in A: f(z) = z + \sum_{k=2}^{\infty} a_k z^k, (a_k \geq 0) \right\}, \tag{2}$$

which are analytic in the unit disk $U = \{z: z \in C: |z| < 1\}$.

We denote with S^* and K the classes of starlike ffunctions and convex functions respectively, i.e

$$S^* = \left\{ f(z) \in A: Re\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad z \in U \right\}$$

and

$$K = \left\{ f(z) \in A: Re\left(\frac{zf''(z)}{f'(z)} + 1\right) > 0, \quad z \in U \right\}$$

(see[1]).

Janowski [2] and Padmanaban*et al.* [3] respectively introduced the subclasses $S^*(A, B)$ and $K(A, B)(-1 \leq B < A \leq 1)$ as subclasses of starlike functions and convex functions. These subclasses are defined:

$$S^*(A, B) = \left\{ f(z) \in A: \frac{zf'(z)}{f(z)} \prec \frac{1 + Az}{1 + Bz}, \ z \in U \right\}$$

and

$$K = \left\{ f(z) \in A: \frac{zf''(z)}{f'(z)} + 1 \prec \frac{1 + Az}{1 + Bz}, \quad z \in U \right\}$$

Where "$\prec$" stands for surbodination and evidently

$f(z) \in K(A, B) \Leftrightarrow zf'(z) \in S^*(A, B)$.

$$US = \left\{ f(z) \in A: Re\left(\frac{zf'(z)}{f(z)}\right) > \left|\frac{zf'(z)}{f(z)} - 1\right|, \ z \in U \right\}$$

and

$$K = \left\{ f(z) \in A: Re\left(\frac{zf''(z)}{f'(z)} + 1\right) > \left|\frac{zf''(z)}{f'(z)}\right|, \quad z \in U \right\}$$

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(see [4]-[6]).

Further, we denote by $US(\alpha,\beta)\ and\ UK(\alpha,\beta)$ the classes of α-uniformly starlike functions andα-uniformly convex functions respectively which are defined:

$$US(\alpha,\beta)=\left\{f(z)\in A: Re\left(\frac{zf'(z)}{f(z)}\right)>\alpha\left|\frac{zf'(z)}{f(z)}-1\right|+\beta,\alpha\geq 0;0\leq\beta<1,z\in U\right\}$$

and

$$K(\alpha,\beta)=\left\{f(z)\in A: Re\left(\frac{zf''(z)}{f'(z)}+1\right)>\alpha\left|\frac{zf''(z)}{f'(z)}-1\right|+\beta,\alpha\geq 0;0\leq\beta<1,z\in U\right\}$$

Remark 1: In [7],

$f(z)=UK(\alpha,\beta)\Leftrightarrow zf'(z)\in US(\alpha,\beta)$.

We note that several interesting properties of various subclasses of functions $UK(\alpha,\beta)$ and $US(\alpha,\beta)$ can be found in [8].

In [9], Salagean introduced the differential operator $D^n: A\to A,\ n\in\aleph$ and defined it as:

$D^0f(z)=f(z)$

$D^1=Df(z)=zf'(z)$

$D^nf(z)=D\left(D^{n-1}f(z)\right)(n\in\aleph=\{1,2,\dots\})$,

which is known as the Salagean Differential Operator.

Following from the above, we now give the following definition:

Definition 1.1.[10] Let$1\leq\mu\leq\beta$ and $n\in\aleph_0=\{0,1,2,\dots\}$and $f(z)$ be defined by (1). We denote by $D^n_{\beta,\mu}f(z)$ the linear operator defined by

$$D^n_{\beta,\mu}f(z): A\to A, \qquad (3)$$

$$D^n_{\beta,\mu}f(z)=\sum_{k=2}^{\infty}[k(1+\beta-\mu)]^n a_k z^k \qquad (4)$$

Remark 2:

i. $D^n_{\beta,\mu}f(z)=(f*g)(z)f\in A$, where$g(z)=\sum_{k=2}^{\infty}[k(1+\beta-\mu)]^n a_k z^k$

ii. By letting $\beta=\mu=1$, we get the classical Sãlãgean operator:

$$D^nf(z)=z+\sum_{k=2}^{\infty}k^n a_k z^k.$$

Definition 1.2. Let $U_{m,n}(\alpha,A,B)$ denote the subclass of A consisting of functions $f(z)$ of the form (1) and satisfy the following surbodination:

$$\frac{D^mf(z)}{D^nf(z)}-\alpha\left|\frac{D^mf(z)}{D^nf(z)}-1\right|\prec\frac{1+Az}{1+Bz} \qquad (5)$$

$(-1\leq B<A\leq 1;\ \alpha\geq 0; m\in\mathbb{N}; n\in\mathbb{N}_0; m>n; z\in U)$.

Definition 1.3 [11]Let $\mathbb{V}^s_{m,n}(\alpha,A,B)(s\in\aleph_0)$ be the subclass of A consisting of functions $f(z)$ which satisfy the following condition:

$$f(z)\in\mathbb{V}^s_{m,n}(\alpha,A,B)\Leftrightarrow D^0f(z)\in U(\alpha,A,B) \qquad (6)$$

For $s=0$, it is easy to see that

$V^0_{m,n}(\alpha,A,B)=\ U(\alpha,A,B)$.

Following from definitions 1.1, 1.2 and 1.3 we now define the classes $U(\alpha,\beta,\mu,A,B) and\ V^s_{m,n}(\alpha,\beta,\mu,A,B)$ as follows:

Definition 1.4. If $\alpha\geq 0, 1\leq\mu\leq\beta$, then

$$U_{m,n}(\alpha,\beta,\mu,A,B)=\{f\in A:\left[\frac{D^n_{\beta,\mu}f(z)}{D^n_{\beta,\mu}f(z)}-\alpha\left|\frac{D^n_{\beta,\mu}f(z)}{D^n_{\beta,\mu}f(z)}-1\right|\right]\prec\frac{1+Az}{1+Bz}$$
$$-1\leq B<A\leq 1; m\in\aleph; n\in\aleph_0; m>n; z\in U\} \qquad (7)$$

$$\mathbb{V}^s_{m,n}(\alpha,\beta,\mu,A,B)=\{f\in A: f(z)\in\mathbb{V}^s_{m,n}\left(U_{m,n}(0,1,1,A,B)\right)\Leftrightarrow D^0f(z)\in U(\alpha,\beta,\mu,A,B),$$
$$-1\leq B<A\leq 1; m\in\aleph; n\in\aleph_0; m>n; z\in U \qquad (8)$$

Remark 3:

*i.*when $\beta=\mu=1$ in (7) and (8), we get $U_{m,n}(\alpha,A,B)$ and $V^s_{m,n}(\alpha,A,B)$ defined by (2) and (4).

*ii.*when$m=1,n=0$ and $m=2,n=1$ in inequality (7), respectively we get the following two classes of functions:

$$US(\alpha,A,B)=\left\{f\in A:\frac{zf'(z)}{f(z)}-\alpha\left|\frac{zf'(z)}{f(z)}\right|\prec\frac{1+Az}{1+Bz}, z\geq 0;-1\leq B<A\leq 1\right\} \qquad (9)$$

And

$$UK(\alpha,A,B)=\left\{f\in A:1+\frac{zf''(z)}{f'(z)}-\alpha\left|\frac{zf''(z)}{f'(z)}-1\right|\prec\frac{1+Az}{1+Bz},\alpha\geq 0;-1\leq B<A\leq 1\right\} \quad (10)$$

From the above remark (ii) we note that
$f(z)\in UK(\alpha,A,B)\Leftrightarrow zf'(z)\in US(\alpha,A,B)$,
$US(1,1,-1)=US, UK(1,1,-1)=UK$.
By specializing the parameters A,B,α,β,μ,m, and n in definitions 1.4, we obtain the following subclasses studied by various authors:

(i) In Li and Tang [11],

$$U_{m,n}(\alpha,1,1,A,B)=U(\alpha,A,B)$$
$$=\{f\in A:\left[\frac{D^m f(z)}{D^n f(z)}-\alpha\left|\frac{D^m f(z)}{D^n f(z)}-1\right|\right]\prec\frac{1+Az}{1+Bz},$$
$$-1\leq B<A\leq 1; m\in\aleph; n\in\aleph_0;\alpha\leq 0; z\in U\};$$

(ii) In Shams et al. [7] and Shams and Kulkarni[8],

$$U_{1,0}(\alpha,1,1,1-2\lambda)=US(\alpha,\lambda)$$
$$=\{f\in A:Re\left(\frac{zf'(z)}{f(z)}-\lambda\right)>\alpha\left|\frac{zf'(z)}{f(z)}-1\right|$$
$$0\leq\lambda<1;\alpha\geq 0; z\in U\}$$
$$U_{2,1}(\alpha,1,1,1-2\lambda,-1)=US(\alpha,\lambda)$$
$$=\{f\in A:Re\left(1+\frac{zf''(z)}{f'(z)}-\lambda\right)>\alpha\left|\frac{zf''(z)}{f'(z)}\right|$$
$$0\leq\lambda<1;\alpha\geq 0; z\in U\};$$

(iii) In Eker and Owa [12], Strivastava and Eker [13],

$U_{m,n}(\alpha,1,1,1-2\lambda,-1)= U_{m,n}(\alpha,\lambda)$and
$V^s_{m,n}(\alpha,1,1,1-2\lambda,-1)=V^s_{m,n}(\alpha,\lambda)(0\leq\lambda<1;\alpha\geq 0; z\in U;$

(iv) In Rosy and Murugusundaramoorthy [14] and Aouf [15],

$$(iv)\quad U_{n+1,n}(\alpha,1,1,1-2\lambda,-1)$$
$$=S(n,\alpha,\lambda)=\left\{f\in A:Re\left(\frac{D^{n+1}f(z)}{D^n f(z)}-\lambda\right)>\alpha\left|\frac{D^{n+1}f(z)}{D^n f(z)}-1\right|\right\}$$
$$(0\leq\lambda<1;\alpha\geq 0; n\in\aleph_0; z\in U);$$

(v) In Janowski [2] and Padmanaban and Ganesan [3],

$$U_{1,0}(0,1,1,A,B)=S^*(A,B)=\left\{f\in A:\frac{zf'(z)}{f(z)}\prec\frac{1+Az}{1+Bz}\right\}$$
$$(-1\leq B<A\leq 1, z\in U)$$
$$U_{2,1}(0,1,1,A,B)=K^*(A,B)=\left\{f\in A:1+\frac{zf''(z)}{f'(z)}\prec\frac{1+Az}{1+Bz}\right\}$$
$$(-1\leq B<A\leq 1, z\in U).$$

Also we note that

$$U_{m,n}(0,1,1,A,B)=U(m,n,A,B)=f\in A:\frac{D^m f(z)}{D^n f(z)}\prec\frac{1+Az}{1+Bz}$$
$$(-1\leq B<A\leq 1; m\in\aleph; n\in\aleph_0; m>n; z\in U\}$$

Let
$\widetilde{US}(\alpha,A,B)= A^+\cap US(\alpha,A,B)$; $\widetilde{UK}(\alpha,A,B)=A^+\cap UK(\alpha,A,B)$;
$\widetilde{U}_{m,n}(\alpha,\beta,\mu,A,B)=A^+\cap U_{m,n}(\alpha,\beta,\mu,A,B)$;
$\tilde{V}^s_{m,n}(\alpha,\beta,\mu,A,B)=A^+\cap\tilde{V}^s_{m,n}(\alpha,\beta,\mu,A,B)$.
The object of this paper is to investigate the classes $U_{m,n}(\alpha,\beta,\mu,A,B)$and $\tilde{V}^s_{m,n}(\alpha,\beta,\mu,A,B)$. In particular, we provide coefficient inequalities, distortion theorem, extreme points, radius of close-to-convexity, starlikeness and convexity for the two classes.

2.0 BASIC PROPERTIES

Unless otherwise mentioned, we assume in the remainder (i.e remaining part) of this paper that, $-1 \le B < A \le 1, \alpha \ge 0, 1 \le \mu \le \beta, n \in \aleph_0, m \in \aleph, (m > n)$and $z \in U$. First we derive the coefficient inequalities for the classes $U_{m,n}(\alpha,\beta,\mu,A,B)$and $\tilde{V}_{m,n}^{s}(\alpha,\beta,\mu,A,B)$.

2.1A SET OF COEFFICIENT INEQUALITIES

Theorem 2.1.A function $f(z)$ of the form (1) is in the class $U_{m,n}(\alpha,\beta,\mu,A,B)$ if

$$\sum_{k=2}^{\infty} \Omega(m,n,k,\alpha,\beta,\mu,A,B)\,|a_k| \le A-B \qquad (11)$$

Where

$$\Omega(m,n,k,\alpha,\beta,\mu,A,B) = \{[1+\alpha(1+|B|)][(k(1+\beta-\mu))^m-(k((1+\beta-\mu))^n]$$
$$+|B[(1+\beta-\mu)]^m - a[k(1+\beta-\mu)]^n|\} \qquad (12)$$

Proof. For function $f(z) \in A$, let us define the function $p(z)$ by

$$p(z) = \frac{D_{\beta,\mu}^m f(z)}{D_{\beta,\mu}^n f(z)} - \alpha\left|\frac{D_{\beta,\mu}^m f(z)}{D_{\beta,\mu}^n f(z)} - 1\right|.$$

It suffice to show that

$$\left|\frac{p(z)-1}{A-Bp(z)}\right| < 1, (z \in U).$$

We have

$$\left|\frac{p(z)-1}{A-Bp(z)}\right| = \left|\frac{D_{\beta,\mu}^m f(z) - \alpha e^{i\theta}\left|D_{\beta,\mu}^m f(z) - D_{\beta,\mu}^n f(z)\right| - D_{\beta,\mu}^n f(z)}{AD_{\beta,\mu}^n f(z) - B[D_{\beta,\mu}^m f(z) - \alpha e^{i\theta}\left|D_{\beta,\mu}^m f(z) - D_{\beta,\mu}^n f(z)\right|]}\right|$$

$$= \left|\frac{\begin{array}{c}\sum_{k=2}^{\infty}\{[(k(1+\beta-\mu)]^m - [(k(1+\beta-\mu)]^n\}a_k z^k \\ -\alpha e^{i\theta}|\sum_{k=2}^{\infty}\{[k(1+\beta-\mu)]^m - [k(1+\beta-\mu)]^n\}a_k z^k|\end{array}}{\begin{array}{c}(A-B)z - \sum_{k=2}^{\infty}\{B[k(1+\beta-\mu)]^m - A[k(1+\beta-\mu)]^n\}a_k z^k \\ -B\alpha e^{i\theta}|\sum_{k=2}^{\infty}\{[k(1+\beta-\mu)]^m - [k(1+\beta-\mu)]^n\}a_k z^k|\end{array}}\right|$$

$$\le \left|\frac{\begin{array}{c}\sum_{k=2}^{\infty}\{[(k(1+\beta-\mu)]^m - [(k(1+\beta-\mu)]^n\}|a_k||z|^k \\ +\alpha\sum_{k=2}^{\infty}\{[k(1+\beta-\mu)]^m - [k(1+\beta-\mu)]^n\}|a_k||z|^k\end{array}}{\begin{array}{c}(A-B)|z| - \{\sum_{k=2}^{\infty}|B[(k(1+\beta-\mu)]^m - A[(k(1+\beta-\mu)]^n|\}|a_k||z|^k \\ +|B|\alpha|\sum_{k=2}^{\infty}\{[k(1+\beta-\mu)]^m - [k(1+\beta-\mu)]^n\}||a_k||z|^k\end{array}}\right|$$

$$\le \left|\frac{\begin{array}{c}\sum_{k=2}^{\infty}\{[(k(1+\beta-\mu)]^m - [k(1+\beta-\mu)]^n\}|a_k| \\ +\alpha\sum_{k=2}^{\infty}\{[k(1+\beta-\mu)]^m - [k(1+\beta-\mu)]^n\}|a_k|\end{array}}{\begin{array}{c}(A-B) - \{\sum_{k=2}^{\infty}|B[k(1+\beta-\mu)]^m - A[k(1+\beta-\mu)]^n|\}|a_k| \\ -|B|\alpha\sum_{k=2}^{\infty}\{[k(1+\beta-\mu)]^m - [k(1+\beta-\mu)]^n|a_k|\end{array}}\right| \qquad (13)$$

The last expression is bounded above by 1, if

$$\sum_{k=2}^{\infty}\{[k(1+\beta-\mu)]^m - [k(1+\beta-\mu)]^n\}|a_k|$$
$$+\alpha\sum_{k=2}^{\infty}\{[k(1+\beta-\mu)]^m - [k(1+\beta-\mu)]^n\}|a_k|$$

$$\le \begin{array}{l}(A-B) - \sum_{k=2}^{\infty}\{|B[k(1+\beta-\mu)]^m - A[k(1+\beta-\mu)]^n|\}|a_k| \\ -|B|\alpha\sum_{k=2}^{\infty}\{[k(1+\beta-\mu)]^m - [k(1+\beta-\mu)]^n\}|a_k|\end{array} \qquad (14)$$

Thus by rearranging (14), we have

$$\sum_{k=2}^{\infty}\{[1+\alpha(1+|B|)]\,[k(1+\beta-\mu))^{m}-(k(1+\beta-\mu))^{n}$$
$$+|B(k(1+\beta-\mu))^{m}-A(k(1+\beta-\mu))^{n}|\}|a_k|\le A-B \qquad (15)$$

which completes the proof of Theorem 2.1

Corollary 2.2. If $f(z)\in A$ satisfies

$$\sum_{k=2}^{\infty}\Omega(1,0,k,\alpha,1,1,A,B)\,|a_k|\le A-B$$

where

$\Omega(1,0,k,\alpha,1,1,A,B)=[1+\alpha(1+|B|)](k^{m}-1)+|Bk-A|$,

then $f(z)\in US(\alpha,A,B)$.

Corollary 2.3. If $f(z)\in A$ satisfies

$$\sum_{k=2}^{\infty}\Omega(2,1,k,\alpha,1,1,A,B)\,|a_k|\le A-B$$

where

$\Omega(2,1,k,\alpha,1,1,A,B)=[1+\alpha(1+|B|)]k(k-1)+|Bk-A|$,

then $f(z)\in UK(\alpha,A,B)$.

Remark 4:

(i) The result in Theorem 2.1 extend the result obtained by Aouf*et al* [16, Lemma 1] and also correct the one obtained by Li and Tang [10, theorem 1]

(ii) Putting then $A=1-2\lambda(0\le\lambda<1),\beta=\mu=1$, in Theorem 2.1 we correct the result obtained by Eker and Owa [12, Theorem 2.4]

(iii) Putting $=1-2\lambda(0\le\lambda<1),\beta=\mu=1,B=-1$ and $m=n+1(n\in\aleph_0)$ in Theorem 2.1 we obtain the result obtainedby Rosy and Murugusudaramoorthy [14, Theorem 2]

(iv) The result in Corollary 2.2, and Corollary 2.3 correct the results obtained by Li and Tang [11, Corollaries 1 and 2]

Next, we observe that by using Theorem 2.1, we have the following:

Theorem 2.4. A function $f(z)$ of the form (1) is in the class $\tilde{V}_{m,n}^{s}(\alpha,\beta,\mu,A,B)$ if

$$\sum_{k=2}^{\infty}k^{s}\Omega(m,n,k,\alpha,\beta,\mu,A,B)\,|a_k|\le A-B \qquad (16)$$

where$\Omega(m,n,k,\alpha,\beta,\mu,A,B)$is defined b (12)

Proof. From (4), replacing a_k by $k^{s}a_k$ in Theorem 2.1, we have Theorem 2.

We note that the classes $U_{m,n}(\alpha,\beta,\mu,A,B)$ and $V_{m,n}^{s}(\alpha,\beta,\mu,A,B)$ are non-empty since the functions given by

$$f(z)=z+\frac{(A-B)(2+\zeta)\xi_k}{(k+\zeta)(k+1+\zeta)\Omega(m,n,k,\alpha,\beta,\mu,A,B)}z^{k} \qquad (17)$$

and

$$f(z)=z+\frac{(A-B)(2+\zeta)\xi_k}{k^{s}(k+\zeta)(k+1+\zeta)\Omega(m,n,k,\alpha,\beta,\mu,A,B)}z^{k} \qquad (18)$$

belong to the classes $U_{m,n}(\alpha,\beta,\mu,A,B)$ and $\tilde{V}_{m,n}^{s}(\alpha,\beta,\mu,A,B)$ for $\zeta>-2$, $\xi_k\epsilon\, C$ and $|\xi_k|=1$.

Theorem 2.5. If $f(z)\in U_{m,n}(\alpha,\beta,\mu,A,B)$, then for $|z|=r<1$

$$\frac{1-(A-B)r-ABr^{2}}{1-B^{2}r^{2}}\le Re\left\{\frac{D_{\beta,\mu}^{m}f(z)}{D_{\beta,\mu}^{n}f(z)}-\alpha\left|\frac{D_{\beta,\mu}^{m}f(z)}{D_{\beta,\mu}^{n}f(z)}-1\right|\right\}\le\frac{1-(A-B)r-ABr^{2}}{1-B^{2}r^{2}},B\ \neq 0 \qquad (19)$$

and

$$1-Ar\le Re\left\{\frac{D_{\beta,\mu}^{m}f(z)}{D_{\beta,\mu}^{n}f(z)}-\alpha\left|\frac{D_{\beta,\mu}^{m}f(z)}{D_{\beta,\mu}^{n}f(z)}-1\right|\right\}\le 1+Ar,B=0 \qquad (20)$$

Proof. Janowski [2] proved that if

$$p(z)\prec\frac{1+Az}{1+Bz},|z|=r<1,$$

then

$$\left|p(z)-\frac{1-ABr^2}{1-B^2r^2}\right|<\frac{(A-B)r}{1-B^2r^2}, B\neq 0 \tag{21}$$

$$|p(z)-1|<Ar, B=0 \tag{22}$$

Using the definition of the class $U_{m,n}(\alpha,\beta,\mu,A,B)$, the inequalities (21) and(22) can be written in the form

$$\left|\frac{D^m_{\beta,\mu}f(z)}{D^n_{\beta,\mu}f(z)}-\alpha\left|\frac{D^m_{\beta,\mu}f(z)}{D^n_{\beta,\mu}f(z)}-1\right|-\frac{1-ABr^2}{1-B^2r^2}\right|<\frac{(A-B)r}{1-B^2r^2}, B\neq 0 \tag{23}$$

and

$$\left|\frac{D^m_{\beta,\mu}f(z)}{D^n_{\beta,\mu}f(z)}-\alpha\left|\frac{D^m_{\beta,\mu}f(z)}{D^n_{\beta,\mu}f(z)}-1\right|-1\right|<Ar, B=0 \tag{24}$$

From (23)and (24), we get (19) and (20) of Theorem 2.5.

Similarly by following the argument in Theorem 2.5, for the class $\widetilde{V}^s_{m,n}(\alpha,\beta,\mu,A,B)$ we have

Theorem 2.6. If $f(z)\in\tilde{V}^s_{m,n}(\alpha,\beta,\mu,A,B)$, then for $|z|=r<1$,

$$\frac{1-(A-B)r-ABr^2}{1-B^2r^2}\le Re\left\{\frac{D^m_{\beta,\mu}D^sf(z)}{D^n_{\beta,\mu}D^sf(z)}-\alpha\left|\frac{D^m_{\beta,\mu}D^sf(z)}{D^n_{\beta,\mu}D^sf(z)}-1\right|\right\}\le\frac{1-(A-B)r-ABr^2}{1-B^2r^2}, B\neq 0 \tag{25}$$

and

$$1-Ar\le Re\left\{\frac{D^m_{\beta,\mu}D^sf(z)}{D^n_{\beta,\mu}D^sf(z)}-\alpha\left|\frac{D^m_{\beta,\mu}D^sf(z)}{D^n_{\beta,\mu}D^sf(z)}-1\right|\right\}\le 1+Ar, B=0 \tag{26}$$

Corollary 2.7. If $f(z)\in US(\alpha,A,B)$, then for $|z|=r<1$

$$\frac{1-(A-B)r-ABr^2}{1-B^2r^2}\le Re\left\{\frac{zf'(z)}{f(z)}-\alpha\left|\frac{zf'(z)}{f(z)}-1\right|\right\}$$

$$\le\frac{1-(A-B)r-ABr^2}{1-B^2r^2}, B\neq 0 \tag{27}$$

and

$$1-Ar\le Re\left\{\frac{f'(z)}{f(z)}-\alpha\left|\frac{f'(z)}{f(z)}-1\right|\right\}\le 1+Ar, B=0 \tag{28}$$

Corollary 2.8. If $f(z)\in UK(\alpha,A,B)$, then for $|z|=r<1$

$$\frac{1-(A-B)r-ABr^2}{1-B^2r^2}\le Re\left\{1+\frac{zf''(z)}{f'(z)}-\alpha\left|\frac{f''(z)}{f'(z)}\right|\right\}$$

$$\le\frac{1-(A-B)r-ABr^2}{1-B^2r^2}, B\neq 0 \tag{29}$$

and

$$1-Ar\le Re\left\{1+\frac{f''(z)}{f'(z)}-\alpha\left|\frac{f''(z)}{f'(z)}\right|\right\}\le 1+Ar, B=0 \tag{30}$$

3.0 DISTORTION INEQUALITIES

Lemma 3.1. If $f(z)\in\widetilde{U}_{m,n}(\alpha,\beta,\mu,A,B)$, then we have

$$\sum_{k=p+1}^{\infty}a_k\le\frac{(\mathrm{A}-\mathrm{B})-\sum_{k=2}^{p}\Omega(m,n,k,\alpha,\beta,\mu,A,B)\,a_k}{\Omega(m,n,p+1,\alpha,\beta,\mu,A,B)} \tag{31}$$

where$\Omega(m,n,k,\alpha,\beta,\mu,A,B)$ is defined by (12).

Proof. In view of Theorem 2.1, we can write

$$\sum_{k=p+1}^{\infty}\Omega(m,n,k,\alpha,\beta,\mu,A,B)a_k\le(A-B)-\sum_{k=2}^{p}\Omega(m,n,k,\alpha,\beta,\mu,A,B)\,a_k \tag{32}$$

Clearly, $\Omega(m,n,k,\alpha,\beta,\mu,A,B)a_k$ is an increasing function for k. then from (12) and (32), we have

$$\Omega(m,n,P+1,\alpha,\beta,\mu,A,B)\sum_{k=p+1}^{\infty}a_k\le(A-B)-\sum_{k=2}^{p}\Omega(m,n,k,\alpha,\beta,\mu,A,B)a_k \tag{33}$$

such that

$$\sum_{k=p+1}^{\infty} a_k \leq \frac{(A-B)-\sum_{k=2}^{p}\Omega(m,n,k,\alpha,\beta,\mu,A,B)a_k}{\Omega(m,n,p+1,\alpha,\beta,\mu,A,B)} = C_k. \quad (34)$$

Lemma 3.2.$D_k$then we have

$$\sum_{k=p+1}^{\infty} a_k \leq \frac{(A-B)-\sum_{k=2}^{p}\Omega(m,n,k,\alpha,\beta,\mu,A,B)a_k}{\Omega(m-1,n-1,p+1,\alpha,\beta,\mu,A,B)} = D_k. \quad (35)$$

where$\Omega(m,n,k,\alpha,\beta,\mu,A,B)$ is defined by (12).

Corollary 3.3. If $f(z) \in \tilde{V}_{m,n}(\alpha,\beta,\mu,A,B)$, then

$$\sum_{k=p+1}^{\infty} a_k \leq \frac{(A-B)-\sum_{k=2}^{p}\Omega(m,n,k,\alpha,\beta,\mu,A,B)a_k}{(p+1)^s\Omega(m,n,p+1,\alpha,\beta,\mu,A,B)} = E_k. \quad (36)$$

and

$$\sum_{k=p+1}^{\infty} ka_k \leq \frac{(A-B)-\sum_{k=2}^{p}\Omega(m,n,k,\alpha,\beta,\mu,A,B)a_k}{(p+1)^s\Omega(m-1,n-1,p+1,\alpha,\beta,\mu,A,B)} = F_k. \quad (37)$$

Theorem 3.4. Let $f(z) \in \widetilde{U}_{m,n}(\alpha,\beta,\mu,A,B)$ then for $|z| = r < 1$

$$r - \sum_{k=2}^{p} a_k|z|^k - C_k r^{p+1} \leq |f(z)| \leq r + \sum_{k=2}^{p} a_k|z|^k + C_k r^{p+1} \quad (38)$$

and

$$1 - \sum_{k=2}^{p} ka_k|z|^{k-1} - D_k r^p \leq |f'(z)| \leq 1 + \sum_{k=2}^{p} ka_k|z|^{k-1} + Dr^p \quad (39)$$

whereC_k and D_k are given by Lemma 1 and Lemma 2.

Proof. Let $f(z)$ be given by (12). For $|z| = r < 1$, using Lemma 1, we have

$$|f(z)| \leq |z| + \sum_{k=2}^{p} a_k|z|^k + \sum_{k=p+1}^{\infty} a_k|z|^k \leq |z| + \sum_{k=2}^{p} a_k|z|^k + |z|^{p+1}\sum_{k=p+1}^{\infty} a_k$$

$$\leq r + \sum_{k=2}^{p} a_k|z|^k + C_k r^{p+1} \quad (40)$$

and

$$|f(z)| \geq |z| - \sum_{k=2}^{p} a_k|z|^k - \sum_{k=p+1}^{\infty} a_k|z|^k \geq |z| - \sum_{k=2}^{p} a_k|z|^k - |z|^{p+1}\sum_{k=p+1}^{\infty} a_k$$

$$\geq r - \sum_{k=2}^{p} a_k|z|^k - C_k r^{p+1} \quad (41)$$

Furthermore, for $|z| = r < 1$, using Lemma 2, we also obtain

$$|f'(z)| \leq 1 + \sum_{k=2}^{p} ka_k|z|^{k-1} + \sum_{k=p+1}^{\infty} ka_k|z|^{k-1} \leq 1 + \sum_{k=2}^{p} ka_k|z|^{k-1} + |z|^p\sum_{k=p+1}^{\infty} ka_k$$

$$\leq 1 + \sum_{k=2}^{p} ka_k|z|^{k-1} + D_k r^p \quad (42)$$

and

$$|f'(z)| \geq 1 - \sum_{k=2}^{p} ka_k|z|^{k-1} - \sum_{k=p+1}^{\infty} ka_k|z|^{k-1} \geq 1 - \sum_{k=2}^{p} ka_k|z|^{k-1} - |z|^p\sum_{k=p+1}^{\infty} ka_k$$

$$\geq 1 - \sum_{k=2}^{p} ka_k|z|^{k-1} - D_k r^p \quad (43)$$

where$C_k$and D_k are given by Lemma 1 and Lemma 2.

This completes the proof of Theorem 3.4

Theorem 3.5. Let $f(z) \in \tilde{V}_{m,n}^{s}(\alpha,\beta,\mu,A,B)$. Then for $|z| = r < 1$,

$$r - \sum_{k=2}^{p} a_k|z|^k - E_k r^{p+1} \le |f(z)| \le r + \sum_{k=2}^{p} a_k|z|^k + E_k r^{p+1} \tag{44}$$

and

$$1 - \sum_{k=2}^{p} k a_k|z|^{k-1} - F_k r^p \le |f'(z)| \le 1 + \sum_{k=2}^{p} k a_k|z|^{k-1} + F_k r^p \tag{45}$$

Where E_k and F_k are given by (36) and (37) respectively.

Proof. The proof of Theorem 3.6 is akin to the proof of Theorem 3.5. Hence, by making use of the Corollary 5 and method used in proving Theorem 3.5, we get the required result.

Taking $P = 1$ in Theorem 3.5 and Theorem 3.6, we have

Corollary 3.6. If $(z) \in U_{m,n}(\alpha,\beta,\mu,A,B)$. Then for $|z| = r < 1$

$$r - \frac{A-B}{\Omega(m,n,2,\alpha,\beta,\mu,A,B)}r^2 \le |f(z)| \le r + \frac{A-B}{\Omega(m,n,2,\alpha,\beta,\mu,A,B)}r^2 \tag{46}$$

and

$$1 - \frac{2(A-B)}{\Omega(m,n,2,\alpha,\beta,\mu,A,B)}r \quad \le |f'(z)| \le r + \frac{2(A-B)}{\Omega(m,n,2,\alpha,\beta,\mu,A,B)}r \tag{47}$$

Remark 5:

(i) By taking $\beta = \mu = 1$ in theorem 3.5 and theorem 3.6 we obtain the results in [11]

(ii) when $\beta = \mu = 1$ in Corollary 3.6 we obtain the results in [11].

Corollary 3.7. If $f(z) \in \tilde{V}_{m,n}(\alpha,\beta,\mu,A,B)$, then for $|z| = r < 1$

$$r - \frac{A-B}{2^s\Omega(m,n,2,\alpha,\beta,\mu,A,B)}r^2 \le |f(z)| \le r + \frac{A-B}{2^s\Omega(m,n,2,\alpha,\beta,\mu,A,B)}r^2 \tag{48}$$

and

$$1 - \frac{2(A-B)}{2^{s-1}\Omega(m,n,2,\alpha,\beta,\mu,A,B)}r \quad \le |f'(z)| \le r + \frac{2(A-B)}{2^{s-1}\Omega(m,n,2,\alpha,\beta,\mu,A,B)}r \quad . \tag{49}$$

Corollary 3.8. If $f(z) \in \tilde{U}S \quad (\alpha,A,B)$, then for $|z| = r < 1$

$$r - \frac{A-B}{\Omega(1,0,2,\alpha,1,1,A,B)}r^2 \le |f(z)| \le r + \frac{A-B}{\Omega(1,0,2,\alpha,1,1,A,B)}r^2 \tag{50}$$

and

$$1 - \frac{2(A-B)}{\Omega(1,0,2,\alpha,1,1,A,B)}r \quad \le |f'(z)| \le r + \frac{2(A-B)}{\Omega(1,0,2,\alpha,1,1,A,B)}r \quad . \tag{51}$$

Corollary 3.9. If $f(z) \in \tilde{U}K_{m,n}(\alpha,A,B)$, then for $|z| = r < 1$

$$r - \frac{A-B}{\Omega(2,1,2,\alpha,1,1,A,B)}r^2 \le |f(z)| \le r + \frac{A-B}{\Omega(2,1,2,\alpha,1,1,A,B)}r^2 \tag{52}$$

and

$$1 - \frac{2(A-B)}{\Omega(2,1,2,\alpha,1,1,A,B)}r \quad \le |f'(z)| \le r + \frac{2(A-B)}{\Omega(2,1,2,\alpha,1,1,A,B)}r \quad . \tag{53}$$

4.0 EXTREME POINTS

The extreme points of the classes $\tilde{U}_{m,n}(\alpha,\beta,\mu,A,B)$ and $\tilde{V}_{m,n}^{s}(\alpha,\beta,\mu,A,B)$are given by the following Theorems:

Theorem 4.1. Let $f_1(z) = z$ and

$$f_k(z) = z + \frac{A-B}{\Omega(m,n,k,\alpha,\beta,\mu)}z^k (k = 2,3,\dots),$$

where$\Omega(m,n,k,\alpha,\beta,\mu)$ is defined by (12). Then $f(z) \in \tilde{U}_{m,n}(\alpha,\beta,\mu,A,B)$ iff it can be expressed in the form

$$f(z) = \sum_{k=1}^{\infty} \Phi_k f_k(z) \tag{54}$$

where$\Phi_k > 0$ and $\sum_{k=1}^{\infty} \Phi_k = 1$.

Proof. Suppose that

$$f(z)=\sum_{k=1}^{\infty}\Phi_k\,f_k(z)=z+\sum_{k=1}^{\infty}\Phi_k\frac{A-B}{\Omega(m,n,k,\alpha,\beta,\mu)}z^k$$

Then

$$\sum_{k=1}^{\infty}\Omega(m,n,k,\alpha,\beta,\mu)\frac{A-B}{\Omega(m,n,k,\alpha,\beta,\mu)}\Phi_k=\sum_{k=1}^{\infty}(\mathrm{A}-\mathrm{B})\Phi_k=(A-B)(1-\Phi_1)<A-B. \qquad (55)$$

Thus, $f(z)\in\widetilde{U}_{m,n}(\alpha,\beta,\mu,A,B)$ from the definition of the class $f(z)\in\widetilde{U}_{m,n}(\alpha,\beta,\mu,A,B)$.
Conversely, suppose that $f(z)\in\widetilde{U}_{m,n}(\alpha,\beta,\mu,A,B)$. Since

$$a_k\le\frac{A-B}{\Omega(m,n,k,\alpha,\beta,\mu)}\ (k=2,3,\dots) \qquad (56)$$

we may set

$$\Phi_k\le\frac{\Omega(m,n,k,\alpha,\beta,\mu)}{A-B}a_k\ (k=2,3,\dots) \qquad (57)$$

and

$$\Phi_k=1-\sum_{k=2}^{\infty}\Phi_k,$$

Then

$$f(z)=\sum_{k=1}^{\infty}\Phi_k f_k(z)$$

This completes the proof of Theorem 4.1

Corollary 4.2. Let $g_1\ (z)=z$ and

$$g_k\ (z)=z+\frac{A-B}{\Omega(m,n,k,\alpha,\beta,\mu,A,B)}z^k(k=2,3,\dots), \qquad (58)$$

Then $g(z)\in\tilde{V}_{m,n}^{s}(\alpha,\beta,\mu,A,B)$ if and only if it can be expressed in the form

$$g(z)=\sum_{k=1}^{\infty}\Phi_k g_k(z), \qquad (59)$$

where $\Phi_k>0$ and $\sum_{k=1}^{\infty}\Phi_k=1$.

Corollary 4.3. The extreme points of $\widetilde{U}_{m,n}(\alpha,\beta,\mu,A,B)$ are the functions $f_1\ (z)=z$ and

$$f_k\ (z)=z+\frac{A-B}{\Omega(m,n,k,\alpha,\beta,\mu,A,B)}z^k(k=2,3,\dots).$$

Corollary 4.4. The extreme points of $\tilde{V}_{m,n}^{s}(\alpha,\beta,\mu,A,B)$ are the functions $g_1\ (z)=z$ and

$$g_k\ (z)=z+\frac{A-B}{k^s\Omega(m,n,k,\alpha,\beta,\mu,A,B)}z^k(k=2,3,\dots)$$

Corollary 4.5. The extreme points of $\widetilde{U}S\ \ (\alpha,A,B)$ are the functions $f_1\ (z)=z$ and

$$(z)=z+\frac{A-B}{\Omega(1,0,k,1,1,\mu,A,B)}z^k(k=2,3,\dots).$$

Corollary 4.6. The extreme points of $\widetilde{U}K\ \ (\alpha,A,B)$ are the functions $f_1\ (z)=z$ and

$$f_k\ (z)=z+\frac{A-B}{\Omega(2,1,k,1,1,\mu,A,B)}z^k(k=2,3,\dots).$$

5.0 RADIUS OF CLOSE-TO-CONVEXITY, STAR-LIKENESS AND CONVEXITY

We determine the radius of close-to-convexity, starlikeness and convexity results for functions in the classes $\widetilde{U}_{m,n}(\alpha,\beta,\mu,A,B)$ and $\tilde{V}_{m,n}^{s}(\alpha,\beta,\mu,A,B)$ in the following theorems:

Theorem 5.1. Let $f\in\widetilde{U}_{m,n}(\alpha,\beta,\mu,A,B)$. Then $f(z)$ is close-to-convex of order $\delta(0\le\delta<1)$ in the disk $|z|=r_1$ where

$$r_1=\inf\left[\frac{(1-\delta)\Omega(m,n,k,\alpha,\beta,\mu,A,B)}{k(A-B)}\right]^{\frac{1}{k-1}}(k\ge 2) \qquad (60)$$

and $\Omega(m,n,k,\alpha,\beta,\mu,A,B)$ is defined by (12).

Proof. It is known in [17] that f is close-to-convexity of order δ, if it satisfies the condition:

$$|f'(z) - 1| < 1 - \delta \tag{61}$$

Hence, we must show that $|f'(z) - 1| < 1 - \delta$for $|z| < r_1(m, n, k, \alpha, \beta, \mu, A, B)$. From the left-hand side of (61), we have

$$|f'(z) - 1| \leq \sum_{k=2}^{\infty} k|a_k||z|^{k-1} \tag{62}$$

The last expression is less than $1 - \delta$ if

$$\sum_{k=2}^{\infty} \frac{k}{1-\delta}|a_k||z|^{k-1} \leq 1. \tag{63}$$

Using the fact that $f \in \widetilde{U}_{m,n}(\alpha, \beta, \mu, A, B)$ if

$$\sum_{k=2}^{\infty} \frac{\Omega(m, n, k, \alpha, \beta, \mu, A, B)}{A - B}|a_k| \leq 1 \tag{64}$$

We can say that (61) is true if

$$\frac{k}{1-\delta}|z|^{k-1} \leq \frac{\Omega(m, n, k, \alpha, \beta, \mu, A, B)}{A - B} \tag{65}$$

Or equivalently,

$$z \leq \left[\frac{(1-\delta)\Omega(m, n, k, \alpha, \beta, \mu, A, B)}{k(A - B)}\right]^{\frac{1}{k-1}} \tag{66}$$

Which completes the proof.

Theorem 5.2. Let the function $f(z)$ defined by (1) be in the class $\widetilde{U}_{m,n}(\alpha, \beta, \mu, A, B)$. Then the following are given

(i) f is starlike of order$\delta(0 \leq \delta < 1)$ in the disk $|z| < r_2$ where

$$\inf\left[\frac{(1-\delta)\Omega(m, n, k, \alpha, \beta, \mu, A, B)}{(k-\delta)(A - B)}\right]^{\frac{1}{k-1}} (k \geq 2) \tag{67}$$

(ii) f is convex of order $\delta(0 \leq \delta < 1)$ in the disk $|z| < r_3$where

$$r_3 = \inf\left[\frac{(1-\delta)\Omega(m, n, k, \alpha, \beta, \mu, A, B)}{k(k-\delta)(A - B)}\right]^{\frac{1}{k-1}} (k \geq 2) \tag{68}$$

Proof. We begin with the proof of (i)

It suffices to show that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 - \delta \tag{69}$$

For $|z| < r_2$ (see[18]).

Thus

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{\sum_{k=2}^{\infty}(k-1)|a_k||z|^{k-1}}{1 - \sum_{k=2}^{\infty}|a_k||z|^{k-1}} \tag{70}$$

The last expression is less than $1 - \delta$ if

$$\sum_{k=2}^{\infty} \frac{k-\delta}{1-\delta}|a_k||z|^{k-1} < 1. \tag{71}$$

Using the fact that $f \in \widetilde{U}_{m,n}(\alpha, \beta, \mu, A, B)$ if

$$\sum_{k=2}^{\infty} \frac{\Omega(m, n, k, \alpha, \beta, \mu, A, B)}{A - B}|a_k| \leq 1 \tag{72}$$

We can say that (69) is true if

$$\frac{k-\delta}{1-\delta}|z|^{k-1} \leq \frac{\Omega(m, n, k, \alpha, \beta, \mu, A, B)}{A - B} \tag{73}$$

Or equivalently,

$$|z|^{k-1} \leq \left[\frac{(1-\delta)\Omega(m, n, k, \alpha, \beta, \mu, A, B)}{(k-\delta)(A - B)}\right]^{\frac{1}{k-1}} \tag{74}$$

i.e.

$$|z| \quad \leq \left[\frac{(1-\delta)\Omega(m,n,k,\alpha,\beta,\mu,A,B)}{(k-\delta)(A-B)}\right]^{\frac{1}{k-1}} \tag{75}$$

which yields the starlkeness of the family.

Using the fact that f is convex iff $zf'(z)$ is starlike, we can prove (ii), following similar arguments to those in the proof of (i).

Proof. It suffices to show that

$$\left|\frac{zf''(z)}{f'(z)}\right| < 1-\delta \tag{76}$$

for $|z| \quad < r_3(m,n,k,\alpha,\beta,\mu,A,B)$.

We have

$$\left|\frac{zf'(z)}{f(z)}-1\right| \leq \frac{\sum_{k=2}^{\infty}(k-1)|a_k||z|^{k-1}}{1-\sum_{k=2}^{\infty}|a_k||z|^{k-1}} \leq \left|\frac{zf'(z)}{f(z)}-1\right| \leq \frac{\sum_{k=2}^{\infty}k(k-1)|a_k||z|^{k-1}}{1-\sum_{k=2}^{\infty}k|a_k||z|^{k-1}} \tag{77}$$

The last expression above is bounded by $(1-\delta)$ if

$$\sum_{k=2}^{\infty}\frac{(k-\delta)|a_k||z|^{k-1}}{1-\delta} \leq 1 \tag{78}$$

In view of (76) it follows that (78) is true if

$$\frac{k(k-\delta)|a_k||z|^{k-1}}{1-\delta} \leq \frac{\Omega(m,n,k,\alpha,\beta,\mu,A,B)}{A-B} \quad (k \geq 2) \tag{79}$$

Or equivalently,

$$|z| \quad \leq \left[\frac{(1-\delta)\Omega(m,n,k,\alpha,\beta,\mu,A,B)}{k(k-\delta)(A-B)}\right]^{\frac{1}{k-1}} \quad (k \geq 2 \tag{80}$$

which completes the proof.

Corollary 5.3. Let the function $f(z)$ defined by (1) be in the class $\tilde{V}_{m,n}^{s}(\alpha,\beta,\mu,A,B)$. Then $f(z)$ is close-to-convex of order $\rho(0 \leq \rho < 1)$ in $|z| \quad < r_{\rho,s}(m,n,k,\alpha,\beta,\mu,A,B)$ where

$$r_{\rho,s}(m,n,k,\alpha,\beta,\mu,A,B) = \inf\left[\frac{(1-\rho)k^s\Omega(m,n,k,\alpha,\beta,\mu,A,B)}{k(A-B)}\right]^{\frac{1}{k-1}} (k \geq 2)(81)$$

Corollary 5.4. Let the function $f(z)$ defined by (1) be in the class $\tilde{V}_{m,n}^{s}(\alpha,\beta,\mu,A,B)$. Then $f(z)$ is starlike of order $\eta(0 \leq \eta < 1)$ in $|z| \quad < r_{\eta,s}(m,n,k,\alpha,\beta,\mu,A,B)$ where

$$r_{\eta,s}(m,n,k,\alpha,\beta,\mu,A,B) = \inf\left[\frac{(1-\eta)k^s\Omega(m,n,k,\alpha,\beta,\mu,A,B)}{(k-\eta)(A-B)}\right]^{\frac{1}{k-1}} (k \geq 2)(82)$$

Corollary 5.5. Let the function $f(z)$ defined by (1) be in the class $\tilde{V}_{m,n}^{s}(\alpha,\beta,\mu,A,B)$. Then $f(z)$ is convex of order $\epsilon(0 \leq \epsilon < 1)$ in $|z| \quad < r_{\epsilon,s}(m,n,k,\alpha,\beta,\mu,A,B)$ where

$$r_{\epsilon,s}(m,n,k,\alpha,\beta,\mu,A,B) = \inf\left[\frac{(1-\epsilon)k^s\Omega(m,n,k,\alpha,\beta,\mu,A,B)}{k(k-\epsilon)(A-B)}\right]^{\frac{1}{k-1}} (k \geq 2)(83)$$

Concluding remarks: By suitably specializing the various parameters involved in the results presented in this paper, we can deduce numerous corresponding results, corollaries and their consequences for other relatively more familiar subclasses.

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