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Real Hardy Spaces and local Hardy Spaces

Adewinbi H. S.[1*] and Peter O. J.[2]

Department of Mathematics, University of Ibadan, Ibadan, Oyo State, Nigeria.[1]
Department of Mathematics, University of Ilorin, llorin, Kwara State, Nigeria.[2]

Abstract

In this paper, we want to see the relationship between the local Hardy spaces (H^P) and real Hardy spaces (h^P)and the boundary behavior of functions in the Hardy spaces on the Disc, unit circle and on the Half-plane. We are concerned with Hardy spaces of vector-valued functions on the disk and on the unit circle. This paper will also gives a concrete answer to the following questions " Given a continuous function f on $\mathbb{T}$, does there exist a harmonic function U defined on $\mathbb{D}$ such that f is the boundary function of U? If yes, is U unique?

Keywords: Holomorphic functions, Lebesgue spaces, Orthogonal projection, Topological spaces, Dirichlet problem, Cauchy integral, Hardy spaces.

1.0 Introduction

The turn century brought forth a vast blooming in the field of analysis as functions spaces were equated with vector spaces, and given appropriate norms for their respective fields of study. The classical Lebesgue spaces, however, have very little regularity; at the extreme are complex analytic (holomorphic, regular, monogenic) functions [1]. Analytic functions are infinitely differentiable and locally converge to their Taylor series. The infinitely differentiable functions on a bounded set are known to form a Frechet space that is, a topological vector space which embeds into an infinite sequence of normed spaces, and whose topology is given as the limit of this sequence. But there is no norm which gives the same topology. A moment of insight can perhaps be had if we recall that an analytic function on a domain is harmonic on said domain, and a classical question in the study of Harmonic functions is the Dirichlet problem. In real analysis Hardy spaces are certain spaces of distributions on the real line, which are (in the sense of distributions) boundary values of the holomorphic functions of the complex Hardy spaces, and are related to the L^p spaces of functional analysis. For $1 \le p \le \infty$ these real Hardy spaces H^p are certain subsets of L^p, while for $p < 1$ the L^p spaces have some undesirable properties, and the Hardy spaces are better behaved. Today the H^p spaces and their local version h^p are important function spaces where it is possible to develop the analysis below the threshold $p = 1$ that bounds the L^p Lesbesgue spaces[2,3].

I. Behavior of Lebesgue spaces (L^p)

Theorem 1.1 (Holder's Inequality) : If p and q are real numbers greater than 1 such that $\frac{1}{p}+\frac{1}{q}=1$ and if $f \in L^p(\mu)$ and $g \in L^q(\mu)$ then $fg \in L^1(\mu)$ and $\| fg \|_1 \le \| f \|_p . \| g \|_q$

Theorem 1.2 (Minkowski's Inequality) : Let $1 \le p \le \infty$, then for $f, g \in L^p(X, d\mu)$

(a) $\| f \|_p \ge 0$ if $\| f \|_p = 0$ then $f = 0$.

(b) $\| \alpha f \|_p \le |\alpha| \| f \|_p$ for $\alpha \in \mathbb{C}$.

(c) $\| f + g \|_p \le \| f \|_p + \| g \|_p$.

Correspondence Author: Adewinbi H.S., E.mail: ayedunseun@gmail.com, Tel:+2347033221618, +2348033560280 (POJ)

Any L^p spaces must satistify both the Holder's and Minkowski's Inequality. The two theorems will be needed to prove some very important result. The proof of Theorem 1.1 and Theorem 1.2 can be found in [4].

II. Introduction to H^p spaces

Definition 2.1: For $1 \le p \le \infty$ the Hardy space H^p is defined as the space of all analytic functions f in

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$$

for which the norm

$$\| f \|_p = \sup_{0 \le r < 1} \left(\frac{1}{2\pi} \int_0^{2\pi} | f(re^{it}) |^p \, dt \right)^{\frac{1}{p}}$$

is finite[5,6]. The space H^∞ consists of all bounded analytic functions f in $\mathbb{D}$ and the norm is now [6]

$$\| f \|_\infty = \sup_{|z|<1} | f(z) |.$$

Notation

For $f \in H(\mathbb{D})$ and $r \in [0,1)$

$$M_0(f,r) = \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} log^+ | f(re^{i\theta}) | \, d\theta\right) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} log^+ | f(re^{i\theta}) | \, d\theta\right)$$

Where

$$\log^+ | f(re^{i\theta}) | = \max\{\log | f(re^{i\theta}) |, 0\}$$

$$M_p(f,r) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} | f(re^{i\theta}) |^p \, d\theta\right)^{\frac{1}{p}} = \left(\frac{1}{2\pi}\int_0^{2\pi} | f(re^{i\theta}) |^p \, d\theta\right)^{\frac{1}{p}} \quad 0 < p < \infty$$

$$M_\infty(f,r) = \sup_{\theta \in [-\pi,\pi)} | f(re^{i\theta} |.$$

For $f \in H(\mathbb{D})$ and $p \in (0,\infty]$ set

$$\| f \|_p = \sup_{r \in [0,1)} M_p(f,r) = \lim_{r \to 1^-} M_p(f,r).$$

For $p \in (0,\infty]$ set

$$H^p = \{ f \in H(\mathbb{D}) : \| f \|_p < \infty)\}.$$

Further, set

$$N = \{ f \in H(\mathbb{D}) : \| f \|_0 < \infty\}.$$

$$H^\infty \subset H^p \subset H^s \subset N, \text{whenever } \; 0 < s < p \le \infty.$$

We were able to see the behaviour of Hardy spaces as p increases. The higher the value of p the bigger the space becomes[7].

Theorem 2.1: For functions in $H^p(\mathbb{D}), 1 \le p \le \infty$, the radial limit.

$$\overline{f}(e^{it}) = \lim_{r \to 1} f(re^{it})$$

exists almost everywhere in t (Fatou's theorem), and indeed $\overline{f} \in L^p(\mathbb{T})$, where

$$\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$$

Moreover

$$\| f \|_p = \left(\frac{1}{2\pi}\int_0^{2\pi} | \overline{f}(e^{it}) |^p \, dt\right)^{\frac{1}{p}} =: \| \overline{f} \|_{L^p(\mathbb{T})}$$

We normally identify f with $\overline{f}$, and can just regard H^p as the subspace of those $L^p(\mathbb{T})$ functions for which negative Fourier Coefficients vanish [7], that is

$$\frac{1}{2\pi}\int_0^{2\pi} \overline{f}(e^{it}) e^{-int} dt = 0 \, \forall n < 0$$

$$H^p(\mathbb{T}) := \{ f \in L^p(\mathbb{T}) : \overline{f}(n) = 0 \, \forall \, n < 0\}.$$

Then a function

$$\overline{f}:\sum_{n=0}^{\infty}a_n z^n$$

can be naturally identified with the power series

$$f(z)=\sum_{n=0}^{\infty}a_n z^n$$

defining an analytic function f on $\mathbb{D}$ [8,9].

Definition 2.2: Let $p\in[1,\infty]$ and $f\in H^p$. Then

$$\overline{f}(e^{it})=\lim_{r\to 1^-} f(re^{it})$$

exists for almost all $t\in[0,2\pi]$ and, moreover the following hold:

(1) $\overline{f}\in L^p(\mathbb{T})$

$$f(z)=f(re^{i\theta})=\frac{1}{2\pi}\int_0^{2\pi}P(re^{i\theta},e^{it})\overline{f}(e^{it})dt$$

$$P(z,e^{it})=\frac{1-|z|^2}{|e^{it}-z|^2}=\frac{1-r^2}{1-2rcos(\theta-t)+r^2},z=re^{i\theta}.$$

$$u(re^{it})=\int_{\mathbb{T}}^{\cdot} f(e^{it})\,\frac{1-r^2}{1-2rcos(\theta-t)+r^2}\,dt$$

Another way of writing the same identity is

$$u(z)=\int_{\mathbb{T}}^{\cdot} f(e^{it})P(e^{-it}z)dt \qquad (1)$$

Observe now that (1) makes sense for f continuous and defines a harmonic function u on D(we can also verify the mean value theorem using the fact that P is harmonic).

Moreover, the identity, $P(z)=2Re(1-z)^{-1}-1$, shows that P is harmonic in $\mathbb{D}$

(2) $\|\overline{f}\|_p=\|f\|_p$

(3) $f=P[f^*]$

(4) Let $\mathbb{C}$ be the positively oriented unit circle. Then

$$f(z)=\frac{1}{2\pi i}\int_{\mathbb{C}}\frac{\overline{f}(w)}{w-z}dw,\quad z\in\mathbb{D}$$

(5) If $p<\infty$, then

$$\lim_{r\to 1^-}\int_0^{2\pi}|f(re^{it})|^p\,dt=\int_0^{2\pi}|f(e^{it})|^p\,dt$$

$$\lim_{r\to 1^-}\frac{1}{2\pi}\int_{-\pi}^{\pi}|\overline{f}(e^{it})-f(e^{it})|^p\,dt=0.$$

(6) The limit $\overline{f}(e^{it})=\lim_{r\to 1^-} f(e^{it})$ exists for almost all $t\in[0,2\pi)$.

(7) If $p=2$, then $\|f\|^2_{H^2(\mathbb{D})}=\sum_{n=0}^{\infty}|a_n|^2=\|\overline{f}\|^2_{L^2(\mathbb{T})}$.

III. Relationship between H^p, h^p and L^p spaces

Here, we will like to see the connection between real Hardy space, local Hardy spaces and Lesbesgue spaces and how they can be use interchangeably.

Definition3.1: If $u\in h^p(\mathbb{D}),1<p<\infty$, then there exists $f\in L[0,2\pi]$ with

$$u(z)=\frac{1}{2\pi}\int_0^{2\pi}P(z,e^{it})f(t)dt\in h^p(\mathrm{D})$$

Where, the Poisson kernel is a function of the open disk given by

$$P(z,e^{it})=\frac{1-|z|^2}{|e^{it}-z|^2}=\frac{1-z\bar{z}}{(1-ze^{-it})(1-\bar{z}e^{it})}=\frac{1-r^2}{1-2rcos(\theta-t)+r^2}$$

Same holds if $p=\infty$. If $p=1$ there exists a finite signed measure on $[0,2\pi]$ such that

$$u(z)=\frac{1}{2\pi}\int_0^{2\pi}P(z,e^{it})d\mu(t)\in h^1(\mathrm{D})$$

Theorem 3.1: Let $f\in L[0,2\pi],\ p\geq 1$ and let

$$u(re^{i\theta})=u(z)=\frac{1}{2\pi}\int_0^{2\pi}P_r(\theta-t)f(t)dt$$

Then $u(re^{i\theta})\to f(\phi)$ almost everywhere in ϕ as $re^{i\theta}\to e^{i\phi}$

Proof

It is clear that if $u\in h^p,\ p>1$, then

$$u(re^{i\theta})\to f(\phi)\text{as}r\to 1$$

almost everywhere for some $f\in L[0,2\pi]$ and $u(z)=\frac{1}{2\pi}\int_0^{2\pi}P(z,e^{it})f(t)dt$

Theorem 3.2: Let $F\in H^p, 0<p\leq\infty$. Then

(1) For almost every t, the limit

$F(e^{it})=\lim\limits_{z\to e^{it}}F(z)$ exists. The function $f(t)=F(e^{it})$ is in L^p and if $p>1,\quad F=P(f)$.

(2) $\int_0^{2\pi}|F(re^{it})-f(t)|^p\ dt \underleftarrow{\ r\to 1\ }$ if $p<\infty$ if $p=\infty,\quad F(e^{it})\to f(t)$, in the weak*- topology of L^∞ when $r\to 1$. For each $0<p\leq\infty$, we have

$$\|F\|_{H^p}=\|f\|_{L^p}$$

(3) F is the Cauchy integral of its boundary function, i.e,

$$F(z)=\frac{1}{2\pi i}\int_{|\xi|=1}\frac{F(\xi)d\xi}{\xi-z}=\frac{1}{2\pi}\int_0^{2\pi}\frac{F(e^{it})e^{it}dt}{e^{it}-z}$$

Corollary 2.1: Every $F\in H^1$ is the Poisson integral and Cauchy integral of its boundary function.

Proof

Let $F\in H^1$ and fix $0<s<1$. For $z=re^{i\theta}$ in Δ we have,

$$F(sre^{i\theta})=\frac{1}{2\pi}\int_0^{2\pi}\frac{1-r^2}{1+r^2-2rcos(\theta-t)}F(se^{it})dt$$

When $s\to 1$, we see that the left hand side converges to $F(re^{i\theta})$ whereas, we have

$$|\int_0^{2\pi}\frac{1-r^2}{1+r^2-2rcos(\theta-t)}F(se^{it})[F(se^{i\theta})-F(e^{i\theta})dt|]\leq C_r\int_0^{2\pi}|F(se^{i\theta})-F(e^{i\theta})|\,dt \underleftarrow{\ s\to 1\ }$$

Then $F(re^{i\theta})=\frac{1}{2\pi}\int_0^{2\pi}\frac{1-r^2}{1+r^2-2rcos(\theta-t)}F(e^{it})dt$

Definition 3.2: An inner function is an H^{∞} function that has unit modulus almost everywhere on $\mathbb{T}$. An outer function is a function $f \in H^1$ which can be written in the form

$$f(re^{i\theta}) = \alpha \exp(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+re^{i\theta}}{e^{it}-re^{i\theta}} k(e^{it})dt)$$

for $re^{i\theta} \in \mathbb{D}$, where k is a real-valued integrable function and $|\alpha| = 1$.

Definition 3.4: The classical Hardy Spaces $H^p(D_+), 0 < p < +\infty$ are defined to consist of those functions f, holomorphic in the upper half plane $D_+ = \{x + iy : y > 0\}$ with the property that $M_p(f, y)$ is uniformly bounded for $y > 0$, where

$$M_p(f, y) = (\int_{-\infty}^{\infty} |f(x+iy)|^p\, dx)^{\frac{1}{p}}$$

Since $|f|^p$ is subharmonic for $f \in H^p(D_+)$, the function $M_p(f, y)$ decreases in $(0, \infty)$.

$$\| f \|_{H_+^p} = \sup\{M_p(f, y) : 0 < y < \infty\} = \lim_{y \to 0} M_p(f, y).$$

$$\| f \|_p = (\int_{-\infty}^{\infty} |f(x)|^p\, dx)^{\frac{1}{p}} = \| f \|_{H_+^p}.$$

Lemma 3.2: If $f \in h^p(\mathbb{D}_+)$, and $z = x + iy \in D_+$ then $|f(z)| \le C_p \frac{\| f \|_{h^p}}{y^{\frac{1}{p}}}$.

<u>Proof</u>

We assume that $p < \infty$, the other case being trivial. For every $r < y$

$$f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(z + re^{it})dt,$$

by the mean value property. Integrating in polar coordinates around z, we then have

$$\begin{aligned}
\frac{1}{|B(z,y)|}\int_{B(z,y)} f(w)dw &= \frac{1}{\pi y^2}\int_0^y \int_{-\pi}^{\pi} f(z+re^{it})dtrdr \\
&= \frac{2}{y^2}\int_0^y f(z)dzrdr \\
&= f(z)
\end{aligned}$$

Therefore, using Holders inequality and the inclusion $B(z, y) \subset \{u + iv, 0 < v < 2y\}$

$$\begin{aligned}
|f(z)| &\le \frac{1}{\pi y^2}\int_{B(z,y)} |f(w)|\, dw \\
&\le (\frac{1}{\pi y^2}\int_{B(z,y)} |f(w)|^p\, dw)^{\frac{1}{p}} \\
&\le (\frac{1}{\pi y^2}\int_{S_y} |f(w)|^p\, dw)^{\frac{1}{p}} \\
&\le (\frac{1}{\pi y^2}\int_{S_y} |f(u+iv)|^p\, dudv)^{\frac{1}{p}} \\
&\le (\frac{1}{\pi y^2}\int_0^{2y} M_p(f,v)^p\, dv)^{\frac{1}{p}} \\
&\le \| f \|_{h^p} (\frac{1}{\pi y^2}\int_0^{2y} dv)^{\frac{1}{p}} \\
&\le (\frac{2}{\pi y})^{\frac{1}{2}} \| f \|_{h^p}
\end{aligned}$$

This shows that the absolute value of f(z) is less than or equal to the product of a constant and the norm of f.

IV. Poisson Kernel and Poisson Integral

Let $h:\mathbb{D}\to\mathbb{C}$ be a harmonic function of the unit disc $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$. As h is harmonic, it can be expressed uniquely up to a constant as the sum of a holomorphic and antiholomorphic function.

$$h=f+g\,,\quad \partial_{\bar{z}}f=0=\partial_z g$$

Then as f and g are holomorphic and antiholomorphic respectively, they can be expressed as power series(convergent in the unit disk) in z and $\bar{z}$ respectively [10].

$$f=\sum_{i=0}^{\infty}a_n z^n\,,\quad g=\sum_{i=0}^{\infty}a_n \bar{z}^n$$

The solution to the Dirichlet lies with an integral kernel called the Poisson kernel. The Poisson kernel is a function of the open unit disk by

$$P(z)=\frac{1-z\bar{z}}{(1-z)(1-\bar{z})}=\frac{\bar{z}}{1-\bar{z}}+1+\frac{z}{1-z}$$

$$P(z)=\sum_{i=0}^{\infty}\bar{z}^n+1+\sum_{i=0}^{\infty}z^n, P_r:=P|_{|z|=r}$$

V. The Dirichlet Problem

Definition 5.1: The Dirichlet problem on $\mathbb{D}$ consists in assgning a continuous function f on $\mathbb{T}$ and seeking for a function u continuous on $\bar{\mathbb{D}}$ and harmonic in $\mathbb{D}$, which coincides with f on $\mathbb{T}$. In other words, we want to solve

$$\begin{cases}\Delta u=0 & \text{in}\,\mathbb{D}\\ u=f & \text{on}\,\mathbb{T}\end{cases}$$

in $u\in C(\bar{\mathbb{D}})$. The maximum modulus principle implies that a solution, if it exists is unique. In fact, one just has to observe that the difference of two solutions would be continuous on $\bar{\mathbb{D}}$, harmonic in $\mathbb{D}$, and identically zero on $\mathbb{T}$.

Given a continuous function f on $\mathbb{T}$, does there exist a harmonic function U defined on $\mathbb{D}$ such that f is the boundary function of U ? If yes, is U unique?

We will see that the answer to both questions is essentially yes (after we define what we mean by the limit of U). If $f(e^{it})=e^{int}$ with $n\in\mathbb{Z}$, the solution of Definition 5.1 easily found as

$$\begin{cases}u(z)=z^n & \text{if}\,n\ge 0\\ u(z)=\bar{z}^{|n|} & \text{if}\,n<0\end{cases}$$

Suppose that $f\in C^2(\mathbb{T})$, and let

$$f(e^{it})=\sum_{n\in\mathbb{Z}}\bar{f}(n)e^{int}=\lim_{N\to\infty}\sum_{n\in\mathbb{Z}}\bar{f}(n)e^{int}$$

be its Fourier series. It is natural to construct

$$u(z)=\sum_{n=0}^{\infty}\bar{f}(n)z^n+\sum_{n=1}^{\infty}\bar{f}(-n)\bar{z}^n$$

By the Fourier series

$$u_r(e^{it})=u(re^{it})\text{for}\,r<1$$

$$u_r(e^{it})=\sum_{n\in\mathbb{Z}}\bar{f}(n)r^{|n|}e^{int}$$

We set

$$P_r(e^{it})=\sum_{n\in\mathbb{Z}}r^{|n|}e^{int}$$

where the series converges uniformly on $\mathbb{T}$

$$\bar{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta} f(e^{i\theta})d\theta$$

$$\begin{aligned}
u(z) &= \sum_{n=0}^{\infty}\bar{f}(n)z^n + \sum_{n=1}^{\infty}\bar{f}(n)\bar{z}^{|n|} \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})[\frac{1}{1-ze^{-i\theta}} + \frac{\bar{z}e^{i\theta}}{1-\bar{z}e^{i\theta}}]d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})[\frac{1-\bar{z}e^{i\theta} + \bar{z}e^{i\theta}(1-ze^{-i\theta})}{(1-ze^{-i\theta})(1-\bar{z}e^{i\theta})}]d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})[\frac{1-\bar{z}e^{i\theta} + \bar{z}e^{i\theta} - \bar{z}z}{|1-ze^{-i\theta}|^2}]d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})[\frac{1-|z|^2}{|1-ze^{-i\theta}|^2}]d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\frac{1-|z|^2}{|e^{i\theta}-z|^2}d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\frac{1-|z|^2}{|z-e^{i\theta}|^2}d\theta \text{ for } |z|<1
\end{aligned}$$

For real-valued function f, the function u is also real-valued.

Dirichlet Problem on the Complex Plane $\mathbb{C}$

$$\begin{cases} -\Delta u = 0 & \text{in}\,\mathbb{D} \\ u = f & \text{on}\,\mathbb{T} \end{cases}$$

Where f is a given function and $\mathbb{T}$ is the boundary of the unit on the complex plane $\mathbb{C}$. The unknown is the function $u:\mathbb{D}\to\mathbb{C}$ It is ideal that $-\Delta u$ exists in the sense of the usual partial derivative. Given $f \in C(\mathbb{T})$, we can find a unique solution $u \in C^2(\mathbb{D}) \cap C(\bar{\mathbb{D}})$ of the above problem. To prove the existence, it turns out that we can write the solution out in full.

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\frac{1-|z|^2}{|z-e^{i\theta}|^2}d\theta \quad z\in\mathbb{D}$$

Observe that the function u inherits harmonicity from the kernel

$$z\in\mathbb{C}\mapsto\frac{1-|z|^2}{|z-e^{i\theta}|^2}\in\mathbb{R}$$

For a complex plne, if given any real-valued function f, there exist a real-valued function u, and the function is unique. That is, two different function f cannot have the same real-valued function u.

Definition 5.2: A C^2-function u defined on an open set $\Omega\subseteq\mathbb{R}^n$ is called harmonic on Ω if its Laplacian Δu defined as

$$\Delta u = \sum_{j=1}^{n}\frac{\partial^2 u}{\partial x_j^2}$$

is identically zero on Ω. A holomorphic function f is harmonic on its domain in $\mathbb{R}^2$. This follows from the fact that holomorphic functions are C^2 (infact analytic) and the Cauchy-Riemann equation.

$$\bar{\partial}_z f = \frac{1}{2}(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}) = 0$$

Differentiating with respect to x, we have

$$\frac{\partial^2 f}{\partial^2 x} = -\frac{\partial^2 f}{\partial x\partial y} = -\frac{\partial^2 f}{\partial^2 y}$$

Since Δ is real. Anti-holomorphic functions are also harmonic. Harmonic functions are characterized by mean value property.

VI. The Cauchy Projection

The Hardy space $H^p(\mathbb{D})$ is a closed subspace of $h^p(\mathbb{D})$. It is clear that the Poisson integral of a function(or measure) f on $\mathbb{T}$ is holomorphic in $\mathbb{D}$ if and only if $\hat{f}(n) = 0$ for $n < 0$. If we define

$$L^p_+(\mathbb{T}) = \{ f \in L^p(\mathbb{T}) : \hat{f}(n) = 0 \ \forall\, n < 0\}$$

and similarly

$$M_+(\mathbb{T}) = \{\mu \in M(\mathbb{T}) : \hat{\mu}(n) = 0 \ \forall\, n < 0\}$$

In the case $p = 2$ we are working with Hilbert Spaces, and we want to describe the orthogonal projection from $h^2(\mathbb{D})$ to $H^2(\mathbb{D})$, that we shall denote by C(for Cauchy) [8].

Lemma 6.1: For $u \in h^2(\mathbb{D})$, let $\hat{u}$ be its harmonic conjugate. Then

$$Cu = \frac{1}{2}(u + i\hat{u}) + \frac{1}{2}u(0)$$

Denote by $u^* \in L^2(\mathbb{T})$ the boundary function of u, i.e

$$u^* = \lim_{r\to 1} u_r$$

then

$$Cu(z) = \int_{\mathbb{T}} \frac{u^*(e^{it})}{1 - ze^{-it}}\, dt$$

<u>Proof</u>

Since $\{z^n\}_{n\geq 0} \cup \{\overline{z}^{n}\}_{n\geq 1}$ is an orthogonal basis of $h^2(\mathbb{D})$, and $\{z^n\}_{n\geq 0}$ spans $H^2(\mathbb{D})$, if we write

$$u(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} a_{-n} \overline{z}^{n}$$

It follow that

$$Cu(z) = \sum_{n=0}^{\infty} a_n z^n$$

Consider now the Fourier series of u^*

$$u^*(e^{it}) = \sum_{n\in\mathbb{Z}} a_n (e^{it})^n = \sum_{n\in\mathbb{Z}} a_n e^{int}$$

with convergence in the $L^2(\mathbb{T})$ -norm. Since

$$(Cu)_r(e^{it}) = \sum_{n=0}^{\infty} a_n r^n e^{int}$$

one obtain $(Cu)_r$ from u^* by multiplying each Fourier coefficient

$$\hat{u}^*(n) = a_n \begin{cases} r^n & \text{if} n \geq 0 \\ 0 & \text{if} n < 0 \end{cases}$$

Consider, therefore the function

$$C_r(e^{it}) = \sum_{n=0}^{\infty} r^n e^{int} = \frac{1}{1 - re^{it}}$$

called the Cauchy Kernel. Then

$$\hat{(Cu)}_r(n) = \hat{u}^*(n)\hat{C}_r(n)$$

For every $n \in \mathbb{Z}$, so that

$$Cu(re^{i\theta}) = (Cu)_r(e^{i\theta}) = u^* * C_r(e^{i\theta}) = \int_{\mathbb{T}} \frac{u^*(e\ it)}{1-re^{i(\theta-t}})dt$$

$$Cu(z) = \int_{\mathbb{T}} \frac{u^*(e^{it})}{1-ze^{-it}}dt$$

Observe that the above equation can be rewritten as a contour integral

$$(Cu)(z) = \frac{1}{2\pi i}\oint_{\partial\mathbb{D}} \frac{u^*(w)}{w-z}dw$$

an expression resembling the Ordinary Cauchy Integral Formula. This is compatible with the fact that if u is already in $H^2(\mathbb{D})$ (i.e it is holomorphic), we then have the identity, for $|z| < r < 1$

$$u(z) = \frac{1}{2\pi i}\oint_{\Gamma} \frac{u(w)}{w-z}dw$$

$$u(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{u_r(e^{it})}{re^{it}-z}re^{it}dt$$

$$u(z) = \int_{\mathbb{T}} \frac{u_r(e^{it})}{1-\frac{z}{r}(e^{-it})}dt$$

where Γ is the circle of radius r oriented counter clockwise. Letting $r \to 1$, one obtains

$$u(z) = \int_{\mathbb{T}} \frac{u(e^{it})}{1-ze^{-it}}dt = Cu(z)$$

Since we are assuming that $Cu = u$. The orthogonal projection of $h^2(\mathbb{D})$ onto $H^2(\mathbb{D})$ corresponds, passing to boundary values, to the orthogonal projection $C*$ of $L^2(\mathbb{T})$ onto $L^2_+(\mathbb{T})$, via the following commutative diagram

$$\begin{array}{ccc} L^2(\mathbb{T}) & \underset{\longleftarrow}{C^*} & L^2_+(\mathbb{T}) \\ \downarrow \mathsf{P} & & \downarrow \mathsf{P} \\ h^2(\mathbb{D}) & \underset{\longleftarrow}{C} & H^2(\mathbb{D}) \end{array}$$

So

$$C^* f = \lim_{r\to 1} (C\mathsf{P}f)_r$$

where the limit is meant in the L^2-norm. We want to give an expression of C^* that does not involve the harmonic extension to the interior. The Lemma can be reduced to a more direct formula for the operator.

$$H : L^2(\mathbb{T}) \to L^2(\mathbb{T})$$

mapping f into

$$Hf = \lim_{r\to 1} \hat{P}_r * f$$

again in the L^2 -norm. The operator is bounded by conjugate Harmonic,

$$Hf(e^{it}) = \lim_{r\to 1} f * \hat{P}_r(e^{it}) = \lim_{r\to 1} \int_{\mathbb{T}} f(e^{i(t-t')}) \frac{2rsint'dt'}{1+r^2-2rcost'}$$

Conclusion

In this paper, we were able to see the relationship between $h^p(\mathbb{D})$ and $H^p(\mathbb{D})$, study the boundary behaviour of functions in the Hardy spaces on the Unit Disc and on the Half-Plane. We discussed also a crucial point in the theory of Hardy spaces, the fact that for $1 < p < \infty$ the conjugate harmonic function of an h^p-function is also in h^p.We were able to answer the question that given a continuous function f on $\mathbb{T}$, does there exist a harmonic function U defined on $\mathbb{D}$ such that f is the boundary function of U? If yes, is U unique? The Dirichlet problem has a solution for the unit disk. If f is a real-valued function, then U is unique.

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