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# Alternative Single And Two-Phase Factor-Type Estimators In Sample Survey

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## *Abstract*

***In this paper, alternative single and two-phase factor-type estimators for estimating finite population mean have been suggested. The suggested estimators were obtained by incorporating some known functions of auxiliary variables X in some existing factor-type estimators. Bias and Mean square error (MSE) of these suggested estimators have been obtained using Tailor's series expansion. The empirical simulation results obtained revealed that the suggested single and two-phase factor-type estimators ( $\bar{y}_{FTAA}$ and $\bar{y}_{FTAA}^{(d)}$ ) demonstrate high level of efficiency over the related existing single and two-phase factor-type estimators considered in this study, hence, they are improved version and therefore recommended for use but the single-phase factor-type estimator, $\bar{y}_{FTAA}$ is most preferred.***

**Keywords**: Factor-type Estimator, Auxiliary variable, Mean square error (MSE), Efficiency.

## 1.0 Introduction

Before 1940, estimation of population parameters like mean, total, proportion, variance, etc were done using corresponding sample statistics. Early 1940, in an attempt to improve the efficiency of sample mean of study variable, Cochran [1] developed ratio estimator by utilizing both the population and sample mean of auxiliary variable and it was found that the ratio estimator is more efficient than sample mean if there is strong and positive correlation between the study and auxiliary variables. In 1993, Singh and Shukla suggested a conventional factor-type estimator which is applicable when correlations between the study and auxiliary variables are either positive or negative. The work of Singh and Shukla [2] has been extended by other researchers like [3-6]. These works are presented below;

Let $A=(d-1)(d-2)$, $B=(d-1)(d-4)$, $C=(d-2)(d-3)(d-4)$,

$$\psi_1=\frac{A+C}{A+fB+C},\ \psi_2=\frac{fB}{A+fB+C},\ \psi_3=\frac{A+fB}{A+fB+C},\ \psi_4=\frac{C}{A+fB+C}\ ,\ P=\psi_3-\psi_1=\psi_2-\psi_4$$

$d$ is an unknown positive real number to be estimated i.e $d\in\Re^{+}$

Singh and Shukla [2] suggested a traditional factor-type estimator for population mean. This estimator is a class of traditional estimators for some values of $d$ and it is defined as:

$$\bar{y}_{FT}=\bar{y}\left[\frac{(A+C)\bar{X}+fB\bar{x}}{(A+fB)\bar{X}+C\bar{x}}\right] \tag{1}$$

for $d=1$, the estimator $\bar{y}_{FT}$ becomes $\bar{y}_r$, for $d=2$, $\bar{y}_{FT}$ becomes $\bar{y}_p$, for $d=3$, $\bar{y}_{FT}$ becomes $\bar{y}_{st}$ and for $d=4$, $\bar{y}_{FT}$ becomes $\bar{y}$

$$Bias(\bar{y}_{FT})=\bar{Y}\frac{1-f}{n}P\left(\psi_4C_x^2+\rho_{xy}C_xC_y\right) \tag{2}$$

$$MSE(\bar{y}_{FT})=\bar{Y}^2\frac{1-f}{n}\left[C_y^2+P^2C_x^2+2P\rho_{xy}C_xC_y\right] \tag{3}$$

In their work, it was observed that factor-type estimator $\bar{y}_{FT}$ was more efficient than classical ratio estimator $\bar{y}_r$ if $P>2C_{yx}-1$.

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Shukla [3] suggested a factor-type estimator for population mean under two-phase sampling as:

$$\bar{y}_{FTd} = \bar{y}_2 \frac{(A+C)\bar{x}_1 + fB\bar{x}_2}{(A+fB)\bar{x}_1 + C\bar{x}_2} \qquad (4)$$

$$Bias\left(\bar{y}_{FTd}\right)_I = \bar{Y}P\theta_3\left[\rho_{xy}C_xC_y - \psi_4 C_x^2\right] \qquad (5)$$

$$Bias\left(\bar{y}_{FTd}\right)_{II} = \bar{Y}P\left[\left(\theta_1\psi_3 - \theta_2\psi_4\right)C_x^2 + \theta_2\rho_{xy}C_xC_y\right] \qquad (6)$$

$$MSE\left(\bar{y}_{FTd}\right)_I = \bar{Y}^2\left[\theta_2 C_y^2 + \theta_3 P^2 C_x^2 + 2\theta_3 P\rho_{xy}C_xC_y\right] \qquad (7)$$

$$MSE\left(\bar{y}_{FTd}\right)_{II} = \bar{Y}^2\left[\theta_2 C_y^2 + \theta_4 P^2 C_x^2 + 2\theta_2 P\rho_{xy}C_xC_y\right] \qquad (8)$$

where $\theta_1 = \frac{1}{n_1} - \frac{1}{N}$, $\theta_2 = \frac{1}{n_2} - \frac{1}{N}$, $\theta_3 = \frac{1}{n_2} - \frac{1}{n_1}$, $\theta_4 = \theta_1 + \theta_2$

In his work, it was observed that factor-type estimator $\bar{y}_{FTd}$ was more efficient than classical ratio estimator $\bar{y}_r^d$, if $-2C_{yx} < P < 0$ under case I and if $-2C_{yx}\left(1+\delta\right)^{-1} < P < 0$ under case II where $\delta = \theta_1\theta_2^{-1}$.

Thakur and Gupta [6] suggested a linear combination based factor-type estimator $\bar{y}_{FTRP}$ for estimating population mean in sample surveys. It was discovered that when the correlation between the study and auxiliary variables is negative, the suggested estimator based on linear combination produced better estimate than some related existing estimators. The suggested estimator, its bias and MSE are given below;

$$\bar{y}_{FTRP} = f\bar{y}\frac{(A+C)\bar{X} + fB\bar{x}}{(A+fB)\bar{X} + C\bar{x}} + (1-f)\bar{y}\frac{(A+C)\bar{x} + fB\bar{X}}{(A+fB)\bar{x} + C\bar{X}} \qquad (9)$$

$$Bias\left(\bar{y}_{FTPR}\right) = \bar{Y}\frac{1-f}{n}P\left[\left(\psi_3 - f\right)C_x^2 + \left(2f-1\right)\rho_{xy}C_xC_y\right] \qquad (10)$$

$$MSE\left(\bar{y}_{FTRP}\right) = \bar{Y}^2\frac{1-f}{n}\left[C_y^2 + \left(2f-1\right)^2 P^2 C_x^2 + 2\left(2f-1\right)P\rho_{xy}C_xC_y\right] \qquad (11)$$

Jain and Shukla [5] suggested factor-type estimators $\bar{y}_{FT1}$ and $\bar{y}_{FT2}$ in multiprocessor environment for estimation of ready queue processing time. The suggested estimators were compared with ratio estimator in terms of total processing time and the suggested factor-type estimators were found to be more accurate, precise and efficient. The suggested estimators, their biases and MSEs are given below;

$$\bar{y}_{FT1} = \bar{y}\frac{9\bar{X} + 2f\bar{x}}{(6+2f)\bar{X} + 3\bar{x}} \qquad (12)$$

$$Bias\left(\bar{y}_{FT1}\right) = \frac{(3-2f)}{(9+2f)}\frac{1-f}{n}\bar{Y}\left[\frac{3}{9+2f}C_x^2 - \rho_{xy}C_xC_y\right] \qquad (13)$$

$$MSE\left(\bar{y}_{FT1}\right) = \bar{Y}^2\frac{1-f}{n}\left[C_y^2 + \frac{4f^2 - 12f + 9}{4f^2 + 36f + 81}C_x^2 + \frac{4f-6}{4f+18}\rho_{xy}C_xC_y\right] \qquad (14)$$

$$\bar{y}_{FT2} = \bar{y}\frac{22\bar{X} + 5f\bar{x}}{(10+5f)\bar{X} + 12\bar{x}} \qquad (15)$$

$$Bias\left(\bar{y}_{FT2}\right) = \frac{(12-5f)}{(22+5f)}\frac{1-f}{n}\bar{Y}\left[\frac{12}{22+5f}C_x^2 - \rho_{xy}C_xC_y\right] \qquad (16)$$

$$MSE\left(\bar{y}_{FT2}\right) = \bar{Y}^2\frac{1-f}{n}\left[C_y^2 + \frac{25f^2 - 120f + 144}{25f^2 + 220f + 448}C_x^2 + \frac{10f-24}{10f+44}\rho_{xy}C_xC_y\right] \qquad (17)$$

Shukla et al. [4] suggested a transformed factor-type estimator for population mean in multiprocessor environment for estimation of ready queue processing time. The suggested estimator was used to predict the remaining total processing time required to process completely the ready queue provided sources of auxiliary information are negatively correlated. The suggested estimator is given below as;

$$\bar{y}_{TFT} = \bar{y}\frac{(A+C)(d-1)\bar{X} + fB\left(d\bar{X} - \bar{x}\right)}{(A+fB)(d-1)\bar{X} + C\left(d\bar{X} - \bar{x}\right)} \qquad (18)$$

The function $\bar{y}_{TFT}$ equals zero at $d = 1$ and equals sample mean at $d = 4$, so the properties of the suggested estimator was studied at $d = 2$, $d = 3$, $d = 5$, and $d = 6$.

The bias and MSE of the suggested estimator at $d = 2$, $d = 3$, $d = 5$, and $d = 6$ respectively as follows;

$$Bias\left(\bar{y}_{TFT2}\right) = -\frac{1-f}{n}\bar{Y}\rho_{xy}C_xC_y \tag{19}$$

$$MSE\left(\bar{y}_{TFT2}\right) = \bar{Y}^2\frac{1-f}{n}\left[C_y^2 + C_x^2 - 2\rho_{xy}C_xC_y\right] \tag{20}$$

$$Bias\left(\bar{y}_{TFT3}\right) = -\frac{1}{2}\frac{1-f}{n}\bar{Y}\left[\frac{n}{N-n}\rho_{xy}C_xC_y\right] \tag{21}$$

$$MSE\left(\bar{y}_{TFT3}\right) = \bar{Y}^2\frac{1-f}{n}\left[C_y^2 + \left(\frac{f}{2-2f}\right)^2 C_x^2 - \left(\frac{f}{1-f}\right)\rho_{xy}C_xC_y\right] \tag{22}$$

$$Bias\left(\bar{y}_{TFT5}\right) = \frac{\left(3-2f\right)}{4\left(2f+9\right)}\frac{1-f}{n}\bar{Y}\left[\frac{3}{4\left(2f+9\right)}C_x^2 - \rho_{xy}C_xC_y\right] \tag{23}$$

$$MSE\left(\bar{y}_{TFT5}\right) = \bar{Y}^2\frac{1-f}{n}\left[C_y^2 + \left(\frac{3-2f}{36+8f}\right)^2 C_x^2 + 2\left(\frac{3-2f}{36+8f}\right)\rho_{xy}C_xC_y\right] \tag{24}$$

$$Bias\left(\bar{y}_{TFT6}\right) = \frac{\left(12-5f\right)}{5\left(22+5f\right)}\frac{1-f}{n}\bar{Y}\left[\frac{12}{5\left(22+5f\right)}C_x^2 - \rho_{xy}C_xC_y\right] \tag{25}$$

$$MSE\left(\bar{y}_{TFT6}\right) = \bar{Y}^2\frac{1-f}{n}\left[C_y^2 + \left(\frac{12-5f}{110+25f}\right)^2 C_x^2 + 2\left(\frac{12-5f}{110+25f}\right)\rho_{xy}C_xC_y\right] \tag{26}$$

The properties of the suggested estimator obtained at $d = 2$, $d = 3$, $d = 5$, and $d = 6$ were compared through empirical study and the result of the empirical analysis showed that the suggested estimator $\bar{y}_{TFT}$ performed better when $d = 6$.

The efficiency of factor-type estimators depends on the optimum estimate of the value of positive real number $d\left(d \in \Re^+\right)$ and they are expected to be applicable and robust irrespective of the direction of correlation $\left(\rho_{xy}\right)$ between the study and auxiliary variables. However, some existing factor-type estimators are found to be challenged either for some values of $d \in \Re^+$ or correlation $\left(\rho_{xy}\right)$. For example, Shukla et al. [4] factor estimator $\bar{y}_{TFT}$ is undefined at d=1, Jain and Shukla [5] factor estimators $\bar{y}_{FT1}$ and $\bar{y}_{FT2}$ are independent of d and Thakur and Gupta [6] factor estimator is less efficient when $\rho_{xy} > 0$ and also less robust for both $\rho_{xy} > 0$ and $\rho_{xy} < 0$ under super-population models. These limitations identified above prompt the present study.

## 2.0 SUGGESTED FACTOR-TYPE ESTIMATORS

Motivated by several existing works [2,3, 7, 8, 9], the following suggested estimators are considered.

**Under single-phase sampling**

Let the sample of size $n$ be drawn by SRSWOR from population of size $N$, then the suggested factor-type estimator and its assumptions under single-phase sampling is;

$$\bar{y}_{FTAA} = \bar{y}\frac{\left[\left(A+C\right)\bar{X} + fB\bar{x}\right]a_x + \left[A+C+fB\right]b_x}{\left[\left(A+fB\right)\bar{X} + C\bar{x}\right]a_x + \left[A+C+fB\right]b_x} \tag{27}$$

where $a_x > 0$ and $b_x > 0$ are assumed to be known as either real numbers or (linear or non-linear) functions.

**Under two-phase sampling**

Consider a preliminary large sample $S_1$ of size $n_1$ drawn from population $\Omega$ of size $N$ by SRSWOR and secondary sample $S_2$ of size $n(n < n_1)$ drawn either of the following manners:

**Case I**: as a subset from the preliminary sample i.e. $S_2 \subset S_1$.

**Case II**: as an independent sample from population i.e. $S_2 \subset \Omega$.

The suggested factor-type estimator under two-phase sampling is;

$$\bar{y}_{FTAA}^{(d)}=\bar{y}_2\frac{\left[\left(A+C\right)\bar{x}_1+fB\bar{x}_2\right]a_x+\left[A+C+fB\right]b_x}{\left[\left(A+fB\right)\bar{x}_1+C\bar{x}_2\right]a_x+\left[A+C+fB\right]b_x} \qquad (28)$$

**3.0 Properties of The Suggested Single And Two-Phase Factor-Type Estimators**

In order to obtained the Bias and Mean Square Error (MSE) of the suggested single- phase factor-type estimator $\bar{y}_{FTAA}$, we defined the error terms $\Delta_{\bar{y}}=\left(\bar{y}-\bar{Y}\right)/\bar{Y}$ and $\Delta_{\bar{x}}=\left(\bar{x}-\bar{X}\right)/\bar{X}$ such that $\left|\Delta_{\bar{y}}\right|<1,\ \left|\Delta_{\bar{x}}\right|<1$

$$\left.\begin{aligned}&E\left(\Delta_{\bar{y}}\right)=E\left(\Delta_{\bar{x}}\right)=0,\ E\left(\Delta_{\bar{y}}^2\right)=\gamma C_y^2\\&E\left(\Delta_{\bar{x}}^2\right)=\gamma C_x^2,\ E\left(\Delta_{\bar{y}}\Delta_{\bar{x}}\right)=\gamma\rho_{xy}C_xC_y,\gamma=\frac{1}{n}-\frac{1}{N}\end{aligned}\right\} \qquad (29)$$

$$\bar{y}_{FTAA}=\bar{Y}\left(1+\Delta_{\bar{y}}\right)\left[1+\delta_x\psi_2\Delta_{\bar{x}}\right]\left[1+\delta_x\psi_4\Delta_{\bar{x}}\right]^{-1} \qquad (30)$$

where $\delta_x=\dfrac{a_x\bar{X}}{a_x\bar{X}+b_x}$

Here the assumption is that in (30), $\left|\delta_x\theta_2\Delta_{\bar{x}}\right|<1$ so that $\left(1+\delta_x\theta_2\Delta_{\bar{x}}\right)^{-1}$ is expandable.

Subtract $\bar{Y}$ for both side of equation (30) and using power series expansion; the simplification of (30) up to first order approximation $O(n^{-1})$ is given by

$$\bar{y}_{FTAA}-\bar{Y}=\bar{Y}\left[\Delta_{\bar{y}}+\delta_xP\Delta_{\bar{x}}+\delta_xP\Delta_{\bar{y}}\Delta_{\bar{x}}-\delta_x^2\psi_4P\Delta_{\bar{x}}^2\right] \qquad (31)$$

Taking expectation of $\bar{y}_{FTAAS}-\bar{Y}$ and using the results in (29), the bias of the suggested estimator is obtained as

$$Bias\left(\bar{y}_{FTAA}\right)=\bar{Y}P\gamma\left[\delta_x\rho_{xy}C_xC_y-\delta_x^2\psi_2C_x^2\right] \qquad (32)$$

Square both sides of $\bar{y}_{FTAA}-\bar{Y}$, taking expectation and using the results in equation (29), we obtain the MSE of the suggested estimators as:

$$MSE\left(\bar{y}_{FTAA}\right)=\bar{Y}^2\frac{1-f}{n}\left[C_y^2+\delta_x^2P^2C_x^2+2\delta_xP\rho_{xy}C_xC_y\right] \qquad (33)$$

Differentiate (33) partially with respect to $P$ and equate the result to zero as

$$\frac{\partial}{\partial P}MSE\left(\bar{y}_{FTAA}\right)=\bar{Y}^2\frac{1-f}{n}\left[2P\delta_xC_x^2+2\delta_x\rho_{xy}C_xC_y\right] \qquad (34)$$

$$\bar{Y}^2\frac{1-f}{n}\left[2P\delta_xC_x^2+2\delta_x\rho_{xy}C_xC_y\right]=0 \qquad (35)$$

$$P=-\rho_{xy}C_y/\delta_xC_x \qquad (36)$$

Substitute (36) in (33), the minimum $MSE\left(\bar{y}_{FTAA}\right)$ written as $MSE\left(\bar{y}_{FTAA}\right)_{\min}$ is obtained as

$$MSE\left(\bar{y}_{FTAA}\right)_{\min}=\bar{Y}^2\frac{1-f}{n}C_y^2\left[1-\rho_{xy}^2\right] \qquad (37)$$

**Remark 1:** The single-phase factor-type estimator $\bar{y}_{FTAA}$ under optimum condition is equally efficient as regression estimator, so therefore $\bar{y}_{FTAA}$ is an alternative to regression estimator when the slope is unknown.

In order to estimate unknown constant $d\in\Re^+$, $P=-C_{yx}/\delta_x$ and $P=\psi_3-\psi_1$ are equated as

$$\psi_3-\psi_1=-C_{yx}/\delta_x \qquad (38)$$

$$\frac{fB-C}{A+C+fB}=-C_{yx}/\delta_x \qquad (39)$$

$$\begin{aligned}\left(w-1\right)d^3+\left(fw+f-8w+9\right)d^2-\left(5fw+5f-23w+26\right)d&\\+\left(4fw+4f-22w+24\right)=0&\end{aligned} \qquad (40)$$

where $w=-C_{yx}/\delta_x$

By solving (40), at most 3 zeros $d_1$ , $d_2$ and $d_3$ of the polynomials for which (27) is optimal will be obtained.

In order to study the properties of the suggested two-phase factor-type estimator $\bar{y}_{FTAA}^{(d)}$, we define the following error terms $\Delta_{\bar{y}_2}=\left(\bar{y}_2-\bar{Y}\right)/\bar{Y}$, $\Delta_{\bar{x}_2}=\left(\bar{x}_2-\bar{X}\right)/\bar{X}$ and $\Delta_{\bar{x}_1}=\left(\bar{x}_1-\bar{X}\right)/\bar{X}$ such that $\left|\Delta_{\bar{y}_2}\right|<1,\ \left|\Delta_{\bar{x}_2}\right|<1,\ \left|\Delta_{\bar{x}_2}\right|$

**Under Case I**
In case I, we have

$$\left.\begin{array}{c} E\left(\Delta_{\bar{y}}\right)=E(\Delta_{\bar{x}})=E\left(\Delta_{\bar{x}_1}\right)=0, E\left(\Delta_{\bar{x}_1}\right)^2=\theta_1 C_x^2 \\ E\left(\Delta_{\bar{y}}\right)^2=\theta_2 C_y^2, E(\Delta_{\bar{x}})^2=\theta_2 C_x^2, E\left(\Delta_{\bar{y}}\Delta_{\bar{x}}\right)=\theta_2\rho_{xy}C_yC_x \\ E\left(\Delta_{\bar{y}}\Delta_{\bar{x}_1}\right)=\theta_1\rho_{xy}C_yC_x, E\left(\Delta_{\bar{x}}\Delta_{\bar{x}_1}\right)=\theta_1 C_x^2\ , \\ \theta_1=\frac{1}{n_1}-\frac{1}{N},\quad \theta_2=\frac{1}{n_2}-\frac{1}{N},\quad \theta_3=\theta_2-\theta_1,\ \theta_4=\theta_2+\theta_1 \end{array}\right\} \tag{41}$$

**Under Case II**
In case II, we have

$$\left.\begin{array}{c} E\left(\Delta_{\bar{y}}\right)=E(\Delta_{\bar{x}})=E\left(\Delta_{\bar{x}_1}\right)=0, E\left(\Delta_{\bar{x}_1}\right)^2=\theta_1 C_x^2 \\ E\left(\Delta_{\bar{y}}\right)^2=\theta_2 C_y^2, E(\Delta_{\bar{x}})^2=\theta_2 C_x^2, E\left(\Delta_{\bar{y}}\Delta_{\bar{x}}\right)=\theta_2\rho_{xy}C_yC_x \\ E\left(\Delta_{\bar{y}}\Delta_{\bar{x}_1}\right)=0, E\left(\Delta_{\bar{x}}\Delta_{\bar{x}_1}\right)=0\ , \\ \theta_1=\frac{1}{n_1}-\frac{1}{N},\quad \theta_2=\frac{1}{n_2}-\frac{1}{N},\quad \theta_3=\theta_2-\theta_1,\ \theta_4=\theta_2+\theta_1 \end{array}\right\} \tag{42}$$

Equation (28) can be expressed in terms of $\Delta_{\bar{x}}$ and $\Delta_{\bar{y}}$ as

$$\bar{y}_{FTAA}^{(d)}=\bar{Y}\left(1+\Delta_{\bar{y}_2}\right)\left[1+\delta_x\psi_1\Delta_{\bar{x}_1}+\delta_x\psi_2\Delta_{\bar{x}_2}\right]\left[1+\delta_x\psi_3\Delta_{\bar{x}_1}+\delta_x\psi_4\Delta_{\bar{x}_2}\right]^{-1} \tag{43}$$

Here the assumption is that in (43) $\left|\delta_x\psi_3\Delta_{\bar{x}_1}+\delta_x\psi_4\Delta_{\bar{x}_2}\right|<1$ so that $\left(1+\delta_x\psi_3\Delta_{\bar{x}_1}+\delta_x\psi_4\Delta_{\bar{x}_2}\right)^{-1}$ is expandable.

Subtract $\bar{Y}$ for both side of (43) and using power series expansion, the simplification of equation (43) up to first order approximation $O(n^{-1})$ is given by

$$\begin{aligned}\bar{y}_{FTAA}^{(d)}-\bar{Y}=\bar{Y}\Big[&\Delta_{\bar{y}_2}-\delta_xP\Delta_{\bar{x}_1}+\delta_xP\Delta_{\bar{x}_2}+\delta_x^2\psi_3P\Delta_{\bar{x}_1}^2-\delta_x^2\psi_4P\Delta_{\bar{x}_1}^2\\&+\delta_x^2\left(\psi_4-\psi_3\right)P\Delta_{\bar{x}_1}\Delta_{\bar{x}_2}-\delta_xP\Delta_{\bar{y}_2}\Delta_{\bar{x}_1}+\delta_xP\Delta_{\bar{y}_2}\Delta_{\bar{x}_2}\Big]\end{aligned} \tag{44}$$

Taking expectation of (44) and using results of (41), the bias of the suggested estimator to terms of order $n^{-1}$ is when $S_2\subset S_1$ is obtained as:

$$Bias\left(\bar{y}_{FTAA}^{(d)}\right)_I=\bar{Y}P\theta_3\left[\delta_x\rho_{xy}C_xC_y-\delta_x^2\psi_4C_x^2\right] \tag{45}$$

Also, taking expectation of (44) and using results of (42), the bias of the suggested estimator to terms of order $n^{-1}$ when $S_2\subset\Omega_N$ is obtained is obtained as

$$Bias\left(\bar{y}_{FTAA}^{(d)}\right)_{II}=\bar{Y}P\left[\theta_2\delta_x\rho_{xy}C_xC_y+\left(\theta_1\psi_3-\theta_2\psi_4\right)\delta_x^2C_x^2\right] \tag{46}$$

Squaring both sides of (44), then taking expectation and using results in (41), we obtain the MSE of the suggested estimator to terms of order $n^{-1}$ when $S_2\subset S_1$ as

$$MSE\left(\bar{y}_{FTAA}^{(d)}\right)_I=\bar{Y}^2\left[\theta_2C_y^2+\theta_3\delta_x^2P^2C_x^2+2\theta_3\delta_xP\rho_{xy}C_xC_y\right] \tag{47}$$

Also, squaring both sides of (44), then taking expectation and using results in (42), we obtain the MSE of the suggested estimator to terms of order $n^{-1}$ when $S_2\subset\Omega_N$ as

$$MSE\left(\bar{y}_{FTAA}^{(d)}\right)_{II}=\bar{Y}^2\left[\theta_2C_y^2+\theta_4\delta_x^2P^2C_x^2+2\theta_2\delta_xP\rho_{xy}C_xC_y\right] \tag{48}$$

Differentiate (47) partially with respect to $P$ and equate the result to zero as

$$\frac{\partial}{\partial P}MSE\left(\bar{y}_{FTAA}^{(d)}\right)_I=\bar{Y}^2\theta_3\left[2P\delta_xC_x^2+2\delta_x\rho_{xy}C_xC_y\right] \tag{49}$$

$$\bar{Y}^2\theta_3\left[2P\delta_xC_x^2+2\delta_x\rho_{xy}C_xC_y\right]=0 \tag{50}$$

$$P=-C_{yx}/\delta_x \tag{51}$$

Substitute (51) in (47), the minimum $MSE\left(\bar{y}_{FTAA}^{(d)}\right)_I$ written as $MSE\left(\bar{y}_{FTAA}^{(d)}\right)_{I\min}$ is obtained as

$$MSE\left(\bar{y}_{FTAA}^{(d)}\right)_{I\min}=\bar{Y}^2C_y^2\left[\theta_2-\theta_3\rho_{xy}^2\right] \quad (52)$$

Also, differentiate (48) partially with respect to $P$ and equate the result to zero as

$$\frac{\partial}{\partial P}MSE\left(\bar{y}_{FTAA}^{(d)}\right)_{II}=\bar{Y}^2\left[2\theta_4P\delta_x^2C_x^2+2\theta_2\delta_x\rho_{xy}C_xC_y\right] \quad (53)$$

$$\bar{Y}^2\left[2\theta_4P\delta_x^2C_x^2+2\theta_2\delta_x\rho_{xy}C_xC_y\right]=0 \quad (54)$$

$$P=-\theta_2C_{yx}/\delta_x\theta_4 \quad (55)$$

Substitute (55) in (48), the minimum $MSE\left(\bar{y}_{FTAA}^{(d)}\right)_{II}$ written as $MSE\left(\bar{y}_{FTAA}^{(d)}\right)_{II\min}$ is obtained as

$$MSE\left(\bar{y}_{FTAA}^{(d)}\right)_{II\min}=\bar{Y}^2C_y^2\left[\theta_2-\frac{\theta_2^2}{\theta_4}\rho_{xy}^2\right] \quad (56)$$

In order to estimate unknown constant $d\in\Re^+$, $P=-\theta_2C_{yx}/\delta_x\theta_4$ and $P=\psi_3-\psi_1$ are equated as

$$\psi_3-\psi_1=-\theta_2C_{yx}/\delta_x\theta_4 \quad (57)$$

$$\frac{fB-C}{A+C+fB}=\frac{-\theta_2C_{yx}}{\delta_x\theta_4} \quad (58)$$

$$\begin{aligned}(v-1)d^3+(fv+f-8v+9)d^2-(5fv+5f-23v+26)d\\+(4fv+4f-22v+24)=0\end{aligned} \quad (59)$$

where $v=\dfrac{-\theta_2C_{yx}}{\delta_x\theta_4}$

By solving (59), at most 3 zeros $d_1$ , $d_2$ and $d_3$ of the polynomials for which (48) is optimal will be obtained.

## 4.0 Empirical Study Using Simulated Data

In order to justify the efficiency of these suggested single and two-phase factor-type estimators over some related existing factor-type estimators considered, a numerical simulation study is conducted. The correlation coefficients between the study and auxiliary variables are assumed to be $\rho_{xy}=\pm0.2,\ \rho_{xy}=\pm0.5,\ \rho_{xy}=\pm0.8$ and $\rho_{xy}=\pm0.99$ using multivariate normal with the following parameters:

$$N=200,\ \bar{Y}=1.51,\ \bar{X}=2.02,\ \bar{Z}=3.05,\ S_y=0.241, S_x=0.387,\ S_z=0.239$$

Table 1: Bias, MSE and PRE of $\bar{y}_{FTAA}$ and some related existing single-phase factor-type ratio estimators when $\rho_{xy}=0.2, \rho_{xy}=0.5$

| Estimators | $\rho_{xy}=0.2$ | | | $\rho_{xy}=0.5$ | | |
|---|---|---|---|---|---|---|
| | Bias | MSE | PRE | Bias | MSE | PRE |
| Sample mean $\bar{y}$ | 0 | 11.38X10$^{-4}$ | 100 | 0 | 8.54X10$^{-4}$ | 100 |
| Cochran [1] $\bar{y}_r$ | 9.62X10$^{-4}$ | 23.10X10$^{-4}$ | 49.273 | 5.1 X10$^{-4}$ | 11.1X10$^{-4}$ | 77.58 |
| Srivenkataramana [10] | 0.25X10$^{-4}$ | 11.06X10$^{-4}$ | 102.90 | -2.1X10$^{-5}$ | 6.47X10$^{-4}$ | 131.95 |
| Sisodia and Diwivedi [11] | 8.21X10$^{-4}$ | 21.15X10$^{-4}$ | 53.813 | 4.0X10$^{-4}$ | 9.74X10$^{-4}$ | 87.62 |
| Singh and Tailor [12] $t_3$ | 8.17X10$^{-4}$ | 21.11X10$^{-4}$ | 53.927 | 2.7X10$^{-4}$ | 8.45X10$^{-4}$ | 101.07 |
| Singh et. al. [13] $t_4$ | 10.2X10$^{-4}$ | 23.80X10$^{-4}$ | 47.826 | -1.7x10$^{-5}$ | 6.45X10$^{-4}$ | 132.45 |
| Upadhyaya and Singh [14] | 12.7X10$^{-4}$ | 27.34X10$^{-4}$ | 41.643 | -2.5X10$^{-5}$ | 7.66X10$^{-4}$ | 111.44 |
| Upadhyaya and Singh [14] | 18.4X10$^{-4}$ | 30.05X10$^{-4}$ | 37.887 | 4.8X10$^{-4}$ | 10.6X10$^{-4}$ | 80.32 |
| Singh and Shukla [2] | -1.28X10$^{-4}$ | 10.97X10$^{-4}$ | 103.74 | -1.48X10$^{-4}$ | 6.86X10$^{-4}$ | 142.86 |
| Jain and Shukla [5]) $\bar{y}_{FT1}$ | 0.49X10$^{-4}$ | 11.16X10$^{-4}$ | 102.04 | -20.2X10$^{-4}$ | 6.67X10$^{-4}$ | 147.07 |
| Jain and Shukla [5] $\bar{y}_{FT2}$ | 1.98 X10$^{-4}$ | 12.66X10$^{-4}$ | 89.904 | 44.1X10$^{-4}$ | 6.45X10$^{-4}$ | 152.07 |
| Thakur and Gupta [6]) | 0.12 X10$^{-4}$ | 11.84X10$^{-4}$ | 96.183 | 24.9X10$^{-4}$ | 9.36X10$^{-4}$ | 104.77 |
| Shukla et. al. [4] | -0.08X10$^{-4}$ | 11.03X10$^{-4}$ | 103.18 | -23.1X10$^{-4}$ | 7.75X10$^{-4}$ | 126.47 |
| $\bar{y}_{FTAA}$ | -0.01X10$^{-4}$ | 10.32X10$^{-4}$ | 110.31 | -13.5X10$^{-4}$ | 5.08X10$^{-4}$ | 167.95 |

Table 2: Bias, MSE and PRE of $\bar{y}_{FTAA}$ and some related existing single-phase factor-type ratio estimators when $\rho_{xy} = 0.8, \rho_{xy} = 0.99$

| Estimators | $\rho_{xy} = 0.8$ | | | $\rho_{xy} = 0.99$ | | |
|---|---|---|---|---|---|---|
| | Bias | MSE | PRE | Bias | MSE | PRE |
| Sample mean $\bar{y}$ | 0 | $6.64X10^{-4}$ | 100 | 0 | $6.64X10^{-4}$ | 100 |
| Cochran [1] $\bar{y}_r$ | $2.4X10^{-4}$ | $3.64X10^{-4}$ | 182.31 | $1.33X10^{-4}$ | $5.28X10^{-4}$ | 125.75 |
| Srivenkataramana [10] | $-6.3X10^{-5}$ | $2.88X10^{-4}$ | 230.69 | $-9.3X10^{-4}$ | $3.22X10^{-4}$ | 206.49 |
| Sisodia and Diwivedi [11] | $1.6X10^{-4}$ | $3.09 X10^{-4}$ | 214.61 | $7.9 X10^{-4}$ | $2.97X10^{-4}$ | 223.27 |
| Singh and Tailor [12] $t_3$ | $3.3X10^{-5}$ | $2.44X10^{-4}$ | 272.43 | $-3.8 X10^{-4}$ | $1.95X10^{-4}$ | 340.92 |
| Singh et. al. [13] $t_4$ | $-6.1X10^{-5}$ | $2.82X10^{-4}$ | 235.43 | $-1.1X10^{-4}$ | $1.65X10^{-4}$ | 401.74 |
| Upadhyaya and Singh [14] | $-4.6X10^{-5}$ | $5.07X10^{-4}$ | 131.05 | $-5.5 X10^{-4}$ | $4.82X10^{-4}$ | 137.72 |
| Upadhyaya and Singh [14] | $2.03X10^{-4}$ | $3.40X10^{-4}$ | 195.52 | $1.17X10^{-4}$ | $4.52X10^{-4}$ | 147.07 |
| Singh and Shukla [2] | $27.3X10^{-4}$ | $2.47 10^{-4}$ | 366.46 | $-7.15X10^{-4}$ | $59.4X10^{-4}$ | 67.63 |
| Jain and Shukla [5]) $\bar{y}_{FT1}$ | $-56.2X10^{-4}$ | $3.99X10^{-4}$ | 245.41 | $11.29X10^{-4}$ | $68.6X10^{-4}$ | 58.61 |
| Jain and Shukla [5] $\bar{y}_{FT2}$ | $-41.2X10^{-4}$ | $2.81 X10^{-4}$ | 349.99 | $23.77X10^{-4}$ | $96.3X10^{-4}$ | 41.74 |
| Thakur and Gupta [6]) | $30.7 X10^{-4}$ | $7.62 X10^{-4}$ | 128.70 | $23.9X10^{-4}$ | $29.6X10^{-4}$ | 135.8 |
| Shukla et. al. [4] | $-33.1X10^{-4}$ | $5.56 X10^{-4}$ | 176.24 | $1.79X10^{-4}$ | $45.4X10^{-4}$ | 88.46 |
| $\bar{y}_{FTAA}$ | $-40.4X10^{-4}$ | $1.72 X10^{-4}$ | 386.76 | $-4.16X10^{-4}$ | $1.32X10^{-4}$ | 502.51 |

Table 3: Bias, MSE and PRE of $\bar{y}_{FTAA}$ and some related existing single-phase factor-type product estimators $\rho_{xy} = -0.2, \rho_{xy} = -0.5$

| Estimators | $\rho_{xy} = -0.2$ | | | $\rho_{xy} = -0.5$ | | |
|---|---|---|---|---|---|---|
| | Bias | MSE | PRE | Bias | MSE | PRE |
| Sample mean $\bar{y}$ | 0 | $11.38X10^{-4}$ | 100 | 0 | $8.54X10^{-4}$ | 100 |
| Product | $9.63 X10^{-4}$ | $23.10X10^{-4}$ | 49.272 | $5.1X10^{-4}$ | $11.1X10^{-4}$ | 77.58 |
| Pandey and Dubey [15] $t_8$ | $8.21 X10^{-4}$ | $21.15X10^{-4}$ | 53.814 | $4.2X10^{-4}$ | $9.99X10^{-4}$ | 85.50 |
| Singh and Tailor [12] $t_9$ | $11.5X10^{-4}$ | $25.66X10^{-4}$ | 44.369 | $8.9X10^{-4}$ | $15.5X10^{-4}$ | 55.26 |
| Singh [16] $t_{10}$ | $7.06X10^{-4}$ | $19.61X10^{-4}$ | 58.079 | $3.5X10^{-4}$ | $9.23X10^{-4}$ | 92.54 |
| Singh et. al. [13] $t_{11}$ | $48.6 X10^{-4}$ | $78.80X10^{-4}$ | 14.446 | $10.1X10^{-4}$ | $16.9X10^{-4}$ | 50.46 |
| Upadhyaya and Singh [14] | $4.90 X10^{-4}$ | $20.41X10^{-4}$ | 55.788 | $24.3X10^{-4}$ | $39.8X10^{-4}$ | 24.42 |
| Upadhyaya and Singh [14] | $10.9X10^{-4}$ | $24.84X10^{-4}$ | 45.832 | $7.3X10^{-4}$ | $13.5X10^{-4}$ | 63.38 |
| Singh and Shukla [2] | $-1.28X10^{-4}$ | $12.22X10^{-4}$ | 93.139 | $-1.2X10^{-4}$ | $10.3X10^{-4}$ | 95.09 |
| Jain and Shukla [5] $\bar{y}_{FT1}$ | $1.53 X10^{-4}$ | $14.27X10^{-4}$ | 79.803 | $1.63X10^{-4}$ | $12.2X10^{-4}$ | 80.28 |
| Jain and Shukla [5] $\bar{y}_{FT2}$ | $3.76 X10^{-4}$ | $18.04X10^{-4}$ | 63.111 | $3.67X10^{-4}$ | $16.2X10^{-4}$ | 60.55 |
| Thakur and Gupta [6] | $-0.12X10^{-4}$ | $11.09X10^{-4}$ | 102.68 | $5.02 X10^{-4}$ | $6.41X10^{-4}$ | 152.9 |
| Shukla et. al. [4] | $0.73 X10^{-4}$ | $11.59X10^{-4}$ | 98.261 | $12.2X10^{-4}$ | $8.89 X10^{-4}$ | 110.2 |
| $\bar{y}_{FTAA}$ | $0.24 X10^{-4}$ | $10.34X10^{-4}$ | 109.34 | $-1.6X10^{-4}$ | $5.28X10^{-4}$ | 161.9 |

Table 4: Bias, MSE and PRE of $\bar{y}_{FTAA}$ and related existing single-phase factor-type product estimators $\rho_{xy} = -0.8, \rho_{xy} = -0.99$

| Estimators | $\rho_{xy} = -0.8$ | | | $\rho_{xy} = -0.99$ | | |
|---|---|---|---|---|---|---|
| | Bias | MSE | PRE | Bias | MSE | PRE |
| Sample mean $\bar{y}$ | 0 | $9.80X10^{-4}$ | 100 | 0 | $6.64 X10^{-4}$ | 100 |
| Product | $3.48X10^{-4}$ | $5.38X10^{-4}$ | 182.31 | $1.33 X10^{-4}$ | $5.28 X10^{-4}$ | 125.8 |
| Pandey and Dubey [15] $t_8$ | $2.6X10^{-4}$ | $4.7X10^{-4}$ | 207.93 | $7.96 X10^{-4}$ | $2.97 X10^{-4}$ | 223.3 |
| Singh and Tailor [12] $t_9$ | $11.4X10^{-4}$ | $13.05X10^{-4}$ | 75.09 | $8.85 X10^{-4}$ | $6.88 X10^{-4}$ | 96.56 |
| Singh [16] $t_{10}$ | $1.8X10^{-4}$ | $4.22X10^{4}$ | 232.18 | $3.88 X10^{-4}$ | $1.77 X10^{-4}$ | 374.8 |
| Singh et. al. [13] $t_{11}$ | $2.3X10^{-4}$ | $4.50X10^{-4}$ | 217.66 | $-1.04X10^{-4}$ | $1.88X10^{-4}$ | 352.6 |
| Upadhyaya and Singh [14] | $-10.1X10^{-4}$ | $3.53 X10^{-4}$ | 277.77 | $-6.56X10^{-4}$ | $4.42 X10^{-4}$ | 150.2 |
| Upadhyaya and Singh [14] | $1.07X10^{-4}$ | $3.81 X10^{-4}$ | 257.21 | $1.12 X10^{-4}$ | $4.31 X10^{-4}$ | 153.9 |
| Singh and Shukla [2] | $-1.25X10^{-4}$ | $12.50X10^{-4}$ | 78.44 | $-10.4X10^{-4}$ | $87.9X10^{-4}$ | 67.63 |
| Jain and Shukla [5] $\bar{y}_{FT1}$ | $2.58X10^{-4}$ | $16.11X10^{-4}$ | 60.87 | $16.7 X10^{-4}$ | $101.3X10^{-4}$ | 58.61 |
| Jain and Shukla [5] $\bar{y}_{FT2}$ | $5.43X10^{-4}$ | $22.21X10^{-4}$ | 44.14 | $35.1 X10^{-4}$ | $142.3X10^{-4}$ | 41.74 |
| Thakur and Gupta [6] | $5.54X10^{-4}$ | $3.78 X10^{-4}$ | 259.34 | $23.9X10^{-4}$ | $38.07X10^{-4}$ | 155.8 |
| Shukla et. al. [4] | $21.9 X10^{-4}$ | $10.45X10^{-4}$ | 93.825 | $1.09 X10^{-4}$ | $43.42X10^{-4}$ | 92.54 |
| $\bar{y}_{FTAA}$ | $3.65 X10^{-4}$ | $3.529 X10^{-4}$ | 277.78 | $0.01X10^{-4}$ | $1.32 X10^{-4}$ | 502.5 |

Table 5: Bias, MSE and PRE of $\bar{y}_{FTAA}^{(d)}$ and some related existing two-phase factor-type ratio estimators when $\rho_{xy} = 0.2$ and $\rho_{xy} = 0.5$

| Estimators | $\rho_{xy} = 0.2$ | | | $\rho_{xy} = 0.5$ | | |
|---|---|---|---|---|---|---|
| | Bias | MSE | PRE | Bias | MSE | PRE |
| **CASE I** | | | | | | |
| Sample mean $\bar{y}$ | 0 | $11.38X10^{-4}$ | 100 | 0 | $8.54 X10^{-4}$ | 100 |
| Sukhatme [17] $t_1$ | $7.22X10^{-4}$ | $20.17X10^{-4}$ | 56.429 | $3.98X10^{-4}$ | $10.46X10^{-4}$ | 81.65 |
| Srivenkataramana [10] | $11.4X10^{-4}$ | $28.60X10^{-4}$ | 39.799 | $12.1X10^{-4}$ | $26.85X10^{-4}$ | 31.80 |
| Malik and Tailor [9] | $6.13X10^{-4}$ | $18.68X10^{-4}$ | 60.947 | $2.13X10^{-4}$ | $8.47X10^{-4}$ | 100.83 |
| Shukla [3] | $0.05X10^{-4}$ | $11.36 X10^{-4}$ | 100.23 | $-23.6X10^{-4}$ | $7.66X10^{-4}$ | 111.44 |
| $\bar{y}_{FTAA}^{(d)}$ | $-29X10^{-4}$ | $11.04 X10^{-4}$ | 103.09 | $-23.9X10^{-4}$ | $6.88X10^{-4}$ | 124.14 |
| **CASE II** | | | | | | |
| Sample mean $\bar{y}$ | 0 | $11.38 X10^{-4}$ | 100 | 0 | $8.54X10^{-4}$ | 100 |
| Sukhatme [17] $t_1$ | $12.5X10^{-4}$ | $27.44 X10^{-4}$ | 41.488 | $7.04 X10^{-4}$ | $13.90 X10^{-4}$ | 61.444 |
| Srivenkataramana [10] | $16.2X10^{-4}$ | $38.68 X10^{-4}$ | 29.432 | $14.1X10^{-4}$ | $34.97X10^{-4}$ | 24.415 |
| Malik and Tailor [9] | $10.6X10^{-4}$ | $24.85 X10^{-4}$ | 45.819 | $3.97X10^{-4}$ | $10.31X10^{-4}$ | 82.856 |
| Shukla [3] | $0.32X10^{-4}$ | $11.31 X10^{-4}$ | 100.62 | $-25.5X10^{-4}$ | $6.98 X10^{-4}$ | 122.33 |
| $\bar{y}_{FTAA}^{(d)}$ | $-12X10^{-4}$ | $11.02 X10^{-4}$ | 103.31 | $-13.2X10^{-4}$ | $6.79X10^{-4}$ | 125.71 |

Table 6: Bias, MSE and PRE of $\bar{y}_{FTAA}^{(d)}$ and some related existing two-phase factor-type ratio estimators when $\rho_{xy} = 0.8$ and $\rho_{xy} = 0.99$

| Estimators | $\rho_{xy} = 0.8$ | | | $\rho_{xy} = 0.99$ | | |
|---|---|---|---|---|---|---|
| | Bias | MSE | PRE | Bias | MSE | PRE |
| **CASE I** | | | | | | |
| Sample mean $\bar{y}$ | 0 | $6.64 X10^{-4}$ | 100 | 0 | $6.64 X10^{-4}$ | 100 |
| Sukhatme [17] $t_1$ | $1.35X10^{-4}$ | $4.93 X10^{-4}$ | 134.77 | $9.47X10^{-4}$ | $2.27 X10^{-4}$ | 291.96 |
| Srivenkataramana [10] | $8.79X10^{-4}$ | $19.91X10^{-4}$ | 33.35 | $12.5X10^{-4}$ | $25.46X10^{-4}$ | 26.09 |
| Malik and Tailor [9] | $18.7X10^{-4}$ | $4.24X10^{-4}$ | 156.66 | $-2.7X10^{-4}$ | $2.034X10^{-4}$ | 326.09 |
| Shukla [3] | $1.28X10^{-4}$ | $7.52X10^{-4}$ | 130.42 | $-1.85X10^{-4}$ | $9.83X10^{-4}$ | 67.56 |
| $\bar{y}_{FTAA}^{(d)}$ | $-24X10^{-4}$ | $4.21X10^{-4}$ | 157.66 | $-17X10^{-4}$ | $1.99 X10^{-4}$ | 333.41 |
| **CASE II** | | | | | | |
| Sample mean $\bar{y}$ | 0 | $6.64 X10^{-4}$ | 100 | 0 | $6.64 X10^{-4}$ | 100 |
| Sukhatme [17] $t_1$ | $5.23X10^{-4}$ | $7.98X10^{-4}$ | 83.243 | $3.24X10^{-4}$ | $3.42 X10^{-4}$ | 194.28 |
| Srivenkataramana [10] | $13.9X10^{-4}$ | $34.20X10^{-4}$ | 19.415 | $13.9X10^{-4}$ | $35.87X10^{-4}$ | 18.511 |
| Malik and Tailor [9] | $1.80X10^{-4}$ | $4.66X10^{-4}$ | 142.45 | $6.20X10^{-4}$ | $1.71 X10^{-4}$ | 388.93 |
| Shukla [3] | $-67X10^{-4}$ | $6.58X10^{-4}$ | 148.90 | $-0.61X10^{-4}$ | $5.49 X10^{-4}$ | 120.88 |
| $\bar{y}_{FTAA}^{(d)}$ | $-17X10^{-4}$ | $3.67X10^{-4}$ | 181.16 | $-11X10^{-4}$ | $1.58 X10^{-4}$ | 420.70 |

Table 7: Bias, MSE and PRE of $\bar{y}_{FTAA}^{(d)}$ and some related existing two-phase factor-type product estimators when $\rho_{xy} = -0.2$ and $\rho_{xy} = -0.5$

| Estimators | $\rho_{xy} = -0.2$ | | | $\rho_{xy} = -0.5$ | | |
|---|---|---|---|---|---|---|
| | Bias | MSE | PRE | Bias | MSE | PRE |
| **CASE I** | | | | | | |
| Sample mean $\bar{y}$ | 0 | 11.38 X10$^{-4}$ | 100 | 0 | 8.54 X10$^{-4}$ | 100 |
| Sukhatme [17] $t_1$ | 9.27X10$^{-4}$ | 27.33 X10$^{-4}$ | 41.656 | 6.05X10$^{-4}$ | 20.31X10$^{-4}$ | 42.04 |
| Srivenkataramana [10] | 5.39X10$^{-4}$ | 19.52 X10$^{-4}$ | 58.311 | 81.7X10$^{-4}$ | 9.77X10$^{-4}$ | 87.374 |
| Malik and Tailor [9] | 10.8X10$^{-4}$ | 29.75 X10$^{-4}$ | 38.260 | 8.78X10$^{-4}$ | 25.05 X10$^{-4}$ | 34.079 |
| Shukla [3] | 0.24 X10$^{-4}$ | 11.97X10$^{-4}$ | 95.134 | 29.1X10$^{-4}$ | 9.31 X10$^{-4}$ | 91.704 |
| $\bar{y}_{FTAA}^{(d)}$ | -15X10$^{-4}$ | 11.07 X10$^{-4}$ | 102.86 | -12X10$^{-4}$ | 7.47X10$^{-4}$ | 114.29 |
| **CASE II** | | | | | | |
| Sample mean $\bar{y}$ | 0 | 11.38 X10$^{-4}$ | 100 | 0 | 8.54X10$^{-4}$ | 100 |
| Sukhatme [17] $t_1$ | 16.9X10$^{-4}$ | 39.64 X10$^{-4}$ | 28.717 | 16.4X10$^{-4}$ | 38.58X10$^{-4}$ | 22.129 |
| Srivenkataramana [10] | 13.1X10$^{-4}$ | 28.40 X10$^{-4}$ | 40.081 | 9.43X10$^{-4}$ | 17.51X10$^{-4}$ | 48.766 |
| Malik and Tailor [9] | 19.6X10$^{-4}$ | 44.06 X10$^{-4}$ | 25.838 | 24.2X10$^{-4}$ | 51.55X10$^{-4}$ | 16.561 |
| Shukla [3] | 0.39 X10$^{-4}$ | 12.29X10$^{-4}$ | 92.644 | 64.9X10$^{-4}$ | 10.2X10$^{-4}$ | 83.811 |
| $\bar{y}_{FTAA}^{(d)}$ | -22X10$^{-4}$ | 11.04 X10$^{-4}$ | 103.16 | -20X10$^{-4}$ | 7.12X10$^{-4}$ | 120.03 |

Table 8: Bias, MSE and PRE of $\bar{y}_{FTAA}^{(d)}$ and some related existing two-phase factor-type product estimators when $\rho_{xy} = -0.8$ and $\rho_{xy} = -0.99$

| Estimators | $\rho_{xy} = -0.8$ | | | $\rho_{xy} = -0.99$ | | |
|---|---|---|---|---|---|---|
| | Bias | MSE | PRE | Bias | MSE | PRE |
| **CASE I** | | | | | | |
| Sample mean $\bar{y}$ | 0 | 9.81 X10$^{-4}$ | 100 | 0 | 3.84 X10$^{-4}$ | 100 |
| Sukhatme [17] $t_1$ | 7.51 X10$^{-4}$ | 25.61 X10$^{-4}$ | 38.29 | 7.84 X10$^{-4}$ | 3.01 X10$^{-4}$ | 127.96 |
| Srivenkataramana [10] | -1.35X10$^{-4}$ | 7.76 X10$^{-4}$ | 126.27 | -2.63 X10$^{-4}$ | 1.54 X10$^{-4}$ | 248.58 |
| Malik and Tailor [9] | 13.8X10$^{-4}$ | 37.12 X10$^{-4}$ | 26.411 | 17.01 X10$^{-4}$ | 2.37 X10$^{-4}$ | 162.33 |
| Shukla [3] | 38.5X10$^{-4}$ | 10.88X10$^{-4}$ | 90.108 | 0.44 X10$^{-4}$ | 7.89X10$^{-4}$ | 84.138 |
| $\bar{y}_{FTAA}^{(d)}$ | -23.9X10$^{-4}$ | 6.91 X10$^{-4}$ | 141.8 | -25.1X10$^{-4}$ | 1.39 X10$^{-4}$ | 275.05 |
| **CASE II** | | | | | | |
| Sample mean $\bar{y}$ | 0 | 9.81X10$^{-4}$ | 100 | 0 | 6.64 X10$^{-4}$ | 100 |
| Sukhatme [17] $t_1$ | 21.6X10$^{-4}$ | 52.14X10$^{-4}$ | 18.800 | 14.4 X10$^{-4}$ | 3.65 X10$^{-4}$ | 181.78 |
| Srivenkataramana [10] | 8.81X10$^{-4}$ | 13.43X10$^{-4}$ | 73.005 | 3.674227 | 4.07 X10$^{-4}$ | 162.96 |
| Malik and Tailor [9] | 41.1X10$^{-4}$ | 85.93X10$^{-4}$ | 11.408 | 32.4 X10$^{-4}$ | 6.87 X10$^{-4}$ | 96.65 |
| Shukla [3] | 90.1 X10$^{-4}$ | 12.24X10$^{-4}$ | 80.106 | 0.71 X10$^{-4}$ | 8.62 X10$^{-4}$ | 77.106 |
| $\bar{y}_{FTAA}^{(d)}$ | -17.0X10$^{-4}$ | 5.73 X10$^{-4}$ | 171.18 | -15.3X10$^{-4}$ | 1.82 X10$^{-4}$ | 364.50 |

Table 9(a): Efficiency comparison of single and two-phase factor-type ratio estimators, $\bar{y}_{FTAA}$ and $\bar{y}_{FTAA}^{(d)}$

| $\rho_{xy}$ | $MSE(\bar{y}_{FTAA})$ | $MSE(\bar{y}(d)_{FTAA}I)$ | $MSE(\bar{y}(d)_{FTAA}I)$ |
|---|---|---|---|
| 0.2 | $10.32X10^{-4}$ | $11.04X10^{-4}$ | $11.02X10^{-4}$ |
| 0.5 | $5.08X10^{-4}$ | $6.88X10^{-4}$ | $6.79X10^{-4}$ |
| 0.8 | $1.72X10^{-4}$ | $4.21X10^{-4}$ | $3.67X10^{-4}$ |
| 0.99 | $1.32X10^{-4}$ | $1.99X10^{-4}$ | $1.58X10^{-4}$ |

Table 9(b): Efficiency comparison of single and two-phase factor-type product estimators, $\bar{y}_{FTAA}$ and $\bar{y}_{FTAA}^{(d)}$

| $\rho_{xy}$ | $MSE(\bar{y}_{FTAA})$ | $MSE(\bar{y}(d)_{FTAA}I)$ | $MSE(\bar{y}(d)_{FTAA}I)$ |
|---|---|---|---|
| -0.2 | $10.34X10^{-4}$ | $11.07X10^{-4}$ | $11.04X10^{-4}$ |
| -0.5 | $5.28X10^{-4}$ | $7.472X10^{-4}$ | $7.12X10^{-4}$ |
| -0.8 | $3.529X10^{-4}$ | $6.91X10^{-4}$ | $5.73X10^{-4}$ |
| -0.99 | $1.32X10^{-4}$ | $1.39X10^{-4}$ | $1.82X10^{-4}$ |

## 5.0 Results and Discussion

Tables 1-4 show the biases, MSEs and PRE of $\bar{y}_{FTAA}$ and some related existing factor-type ratio and product estimators under single-phase simple random sample scheme when the study and auxiliary variables are positively and negatively correlated with $\rho_{xy} = \pm 0.2,\ \rho_{xy} = \pm 0.5,\ \rho_{xy} = \pm 0.8$ and $\rho_{xy} = \pm 0.99$ coefficients respectively. The results of the analysis revealed that the suggested single phase factor-type estimator $\bar{y}_{FTAA}$ has minimum MSEs and high PRE among all the related existing ratio and product estimators considered.

Tables 5-8 show the biases, MSEs and PRE of $\bar{y}_{FTAA}^{(d)}$ and some related existing ratio and product factor-type estimators under two-phase simple random sample scheme when the study and auxiliary variables are positively and negatively correlated with $\rho_{xy} = \pm 0.2,\ \rho_{xy} = \pm 0.5,\ \rho_{xy} = \pm 0.8$ and $\rho_{xy} = \pm 0.99$ coefficients under cases I and II. The results of the analysis revealed that the suggested two-phase factor-type estimator $\bar{y}_{FTAA}^{(d)}$ has minimum MSEs and high PRE.

Tables 9(a & b) revealed that of the two alternative estimators, $\bar{y}_{FTAA}$ and $\bar{y}_{FTAA}^{(d)}$ considered in this study, the alternative single-phase ratio and product factor-type estimator, $\bar{y}_{FTAA}$, is the most efficient because it has the least mean square error across all the correlation coefficients investigated.

Conclusively, the suggested alternative single and two-phase factor-type estimators ( $\bar{y}_{FTAA}$ and $\bar{y}_{FTAA}^{(d)}$ ) demonstrate high level of efficiency over the related existing single and two-phase factor-type estimators considered in this study, hence, they are improved version and therefore recommended for use but the single-phase factor-type estimator, $\bar{y}_{FTAA}$ is most preferred.

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