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Statistical Properties of The Exponentiated Skew-T Distribution

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Abstract

In this paper we provided another Exponentiated distribution of a one parameter or univariate skew-t distribution which will serves as a competitive model with two parameters. The statistical properties and some mathematical properties of the proposed Exponentiated skew-t distribution are provided and the method of estimating its parameters; by maximum likelihood estimator was also proposed.

Keywords: Skew-t, Exponentiated, Survival, Hazard and asymptotic behavior, Binomial expansion, Moments, Characteristic function

1. Introduction

Since the evolution of compound distributions where authors combined two similar distribution and make inferences from it; as give birth to the short form of combining distribution which some now called the "Distribution Generator".

In 1995, Gupta introduced the exponentiated distribution in modelling various kind of real life data, and one of them is the a two-parameter distribution, called the exponentiated Pareto distribution EP(θ, λ). Since then, many authors have developed various classes of the exponentiated distributions. The cumulative density function (CDF) of Exponentiated distribution was derived in [1] by raising the CDF of the arbitrary baseline distribution to an additional non-negative parameter

The exponentiated lomax distribution was proposed in [2] using different estimation methods, the exponentiated exponential distribution develop in, several exponentiated distributions such as the exponentiated Gamma, exponentiated Weibull, exponentiated Gumbel and exponentiated Frechet distribution was work in [4] but the exponentiated weibull was first introduced in [5] which the expansion in [5] was done in [4].

some exponentiated distribution was work on in [6] where they derived the moments, failure rate and the survival function of the distributions

In this paper we study the statistical properties of new Exponentiated distributions. The first to introduce the idea of the Exponentiated distribution (ED) was in [7] by discussing a new family of distribution termed as Exponentiated exponential distribution.

In this paper, we are going to propose a new model of the exponentiated skew-t distribution by computing its PDF, CDF and Quantile function with some statistical properties that is, improving the work in [7].

1.2. Skew-t distribution

The student-t distribution is a continuous statistical distribution used to sampled data, assumed to be symmetrical and normally distributed. The probability density function (PDF) is given as;

$$f(x)=\frac{\Gamma\left\{\frac{1}{2}(v+1)\right\}}{\sqrt{\pi v}\,\Gamma\left(\frac{1}{2}v\right)}\frac{1}{\left(1+\frac{x^2}{v}\right)^{\frac{v+1}{2}}}, \qquad x\in R \tag{1}$$

Introducing a scaling factor $\sqrt{(a+b)/2}$ on the 2 degrees of freedom on the distribution function was done in [8] and in [9] to the Cumulative distribution function (CDF). The new distribution function of the skew-t distribution is given as;

$$F(x)=\frac{1}{2}\left(1+\frac{x}{\sqrt{a+b+x^2}}\right), -\infty<x<\infty \tag{2}$$

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The PDF of the skew-t distribution was computed by taking the differential equation of the CDF in equation (2)

$$f(x)=\frac{\lambda}{2\left(\lambda+x^2\right)^{3/2}},\quad -\infty<x<\infty \tag{3}$$

where λ, $a+b$, a and b are the shape parameters and $x>0$, $\lambda>0$

1.3. Exponentiated distribution

The PDF and CDF of the Exponentiated G distribution was develop in [7] where they compute and introduced a class of Exponentiated distributions based on cumulative distribution function CDF for the exponential distribution. The CDF and PDF was specified in [7] as follow;

$$F(x)=G(x)^u;\ u>0 \tag{4}$$

where $G(x)$ is the CDF of any baseline distributions. We obtain the PDF of the above CDF by taking the derivative of the equation with respect to x to give;

$$f(x)=uG(x)^{u-1}g(x) \tag{5}$$

where $u>0$, the shape parameter and $g(x)=\frac{dG(x)}{dx}$

2.0 Exponentiated skew-t distribution (ESTD)

Let X be a random variable from an arbitrary baseline distribution. Hence, the proposed Exponentiated skew-t distribution (ESTD) is derived by substituting equation (2) and (3) into (5). Therefore, if a continuous non-negative random variable X is such that;

$X\sim ESTD(u,\lambda)$, its PDF is given by;

$$g(x)=u\left\{\frac{1}{2}\left(1+\frac{x}{\sqrt{\lambda+x^2}}\right)\right\}^{u-1}\frac{\lambda}{2\left(\lambda+x^2\right)^{\frac{3}{2}}} \tag{6}$$

The corresponding CDF is given by;

$$G(x)=\left\{\frac{1}{2}\left(1+\frac{x}{\sqrt{\lambda+x^2}}\right)\right\}^{u} \tag{7}$$

where $x>0, u>0, \lambda>0$ and

λ, u are a shape parameters

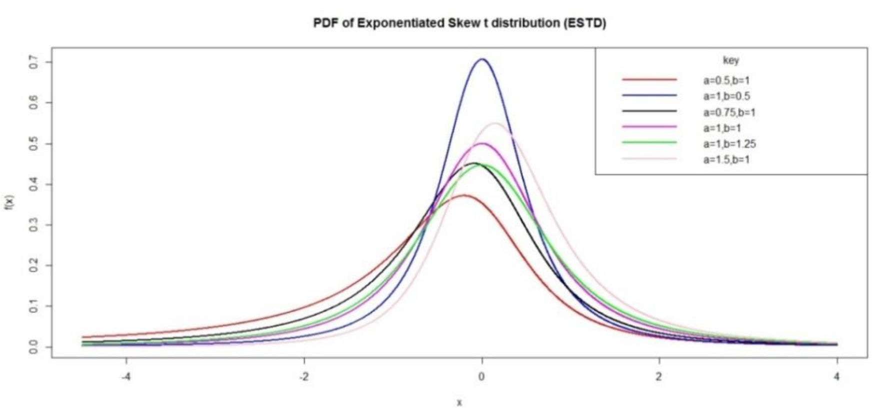

Figure 1: The graph of PDF of the Exponentiated skew-t distribution at different parameter value where $a=u, b=\lambda$

Figure 1 shows that an the parameter λ increased, the distribution tend to a normal distribution

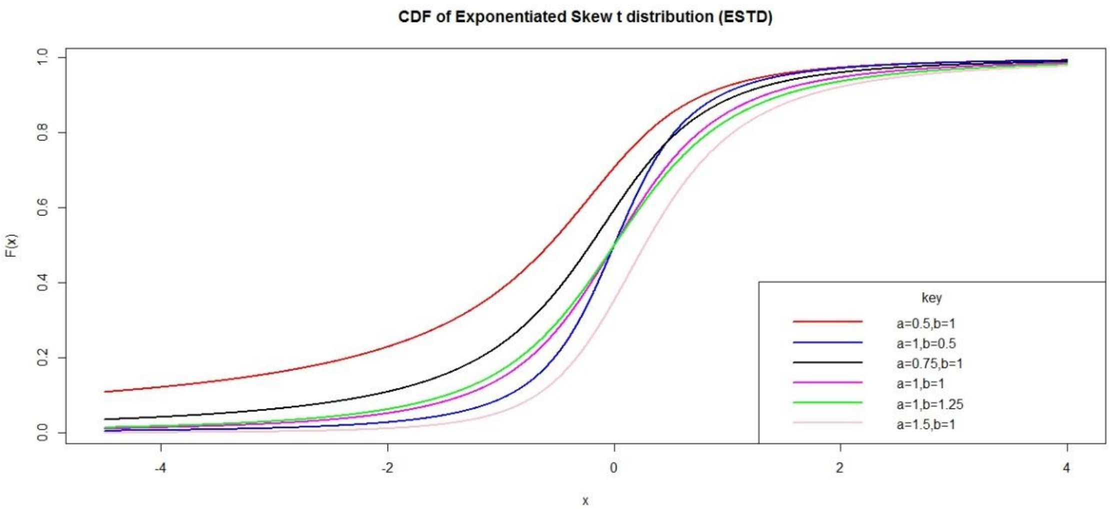

Figure 2: The graph of CDF of the Exponentiated skew-t distribution at different parameter value where $a=u, b=\lambda$

Figure 2 shows an ogive curve as the values started to increase, that is, a cumulative frequency computation output

1.4. Investigating the Validity of the probability density function of ESTD

Using appropriate transformation, we will verify that our proposed probability density function is a valid PDF. This will be achieved using integration and see whether the integral is equal to unity or otherwise. If it equal to unity (1), then density function is a proper probability density function otherwise it is not.

$$=\int_{-\infty}^{\infty} g(x)\,dx \tag{8}$$

$$=\int_{-\infty}^{\infty} u\left\{\frac{1}{2}\left(1+\frac{x}{\sqrt{\lambda+x^2}}\right)\right\}^{u-1}\frac{\lambda}{2\left(\lambda+x^2\right)^{\frac{3}{2}}}\,dx \tag{9}$$

Let $p=\frac{1}{2}\left(1+\frac{x}{\sqrt{\lambda+x^2}}\right)$ (10)

$$\frac{dp}{dx}=\frac{2\sqrt{\lambda+x^2}.1-x\frac{1}{2}2\left(\lambda+x^2\right)^{-1/2}.2x}{4\left(\lambda+x^2\right)} \tag{11}$$

$$=\frac{2\sqrt{\lambda+x^2}-2x^2\left(\lambda+x^2\right)^{-1/2}}{4\left(\lambda+x^2\right)}=\frac{2\left(\lambda+x^2\right)^{1/2}-2x^2\left(\lambda+x^2\right)^{-1/2}}{4\left(\lambda+x^2\right)} \tag{12}$$

$$=\frac{2\left(\lambda+x^2\right)^{1/2}-\frac{2x^2}{\left(\lambda+x^2\right)^{1/2}}}{4\left(\lambda+x^2\right)}=\frac{2\left(\lambda+x^2\right)-2x^2}{4\left(\lambda+x^2\right)^{3/2}} \tag{13}$$

$$=\frac{2\lambda+2x^2-2x^2}{4\left(\lambda+x^2\right)^{3/2}}=\frac{2\lambda}{4\left(\lambda+x^2\right)^{3/2}} \tag{14}$$

$$\frac{dp}{dx}=\frac{\lambda}{2\left(\lambda+x^2\right)^{3/2}}\rightarrow dp\left(2\left(\lambda+x^2\right)^{3/2}\right)=\lambda dx \tag{15}$$

$$dx=\frac{2\left(\lambda+x^2\right)^{3/2}}{\lambda}dp \tag{16}$$

As $x\rightarrow\infty, p=\infty$ and $x\rightarrow-\infty, p=-\infty$

Making substitution into equation (3.19), then

$$= \int_{-\infty}^{\infty} u\{p\}^{u-1} \frac{\lambda}{2\left(\lambda + x^2\right)^{\frac{3}{2}}} * \frac{2\left(\lambda + x^2\right)^{\to 3/2}}{\lambda} dp \quad (17)$$

$$= \int_{-\infty}^{\infty} u\{p\}^{u-1} dp = u \frac{p^{u-1+1}}{u-1+1}\Big|_{-\infty}^{\infty} = p^u\Big|_{-\infty}^{\infty} \quad (18)$$

$$= p^u\Big|_{-\infty}^{0} + p^u\Big|_{0}^{1} + p^u\Big|_{1}^{\infty}$$

$$= [0-\infty] + [1-0] + [\infty - 1]$$

$$= -\infty + 1 + \infty$$

$$= 1$$

Hence, $g(x)$ is indeed a valid probability density function of a continuous distribution.

2. Properties of the Proposed Model

3.1. Asymptotic Behavior of Exponentiated Skew-t distribution (ESTD)

In order to investigate the behavior of the proposed distribution, we proceed thus,

Theorem 1: The limit of Exponentiated skew-t density function as $x \to \infty$ is 0 and the limit as $x \to -\infty$ is 0.

Proof:

These can be shown by getting the limit of the Exponentiated skew-t density function

For $x \to \infty$:

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} \left\{ u \frac{\lambda}{2\left(\lambda + x^2\right)^{\frac{3}{2}}} \left\{ \frac{1}{2}\left(1 + \frac{x}{\sqrt{\lambda + x^2}}\right)\right\}^{u-1} \right\} \quad (19)$$

$$= \lim_{x\to\infty}(u) \times \lim_{x\to\infty} \frac{\lambda}{2\left(\lambda + x^2\right)^{\frac{3}{2}}} \times \lim_{x\to\infty} \left\{ \frac{1}{2}\left(1 + \frac{x}{\sqrt{\lambda + x^2}}\right)\right\}^{u-1} \quad (20)$$

$$= \lim_{x\to\infty}(u) \times 0 \times 1 \quad (21)$$

Since $\lim_{x\to\infty} \frac{\lambda}{2\left(\lambda + x^2\right)^{\frac{3}{2}}}$ =0

Then

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} \left\{ u \frac{\lambda}{2\left(\lambda + x^2\right)^{\frac{3}{2}}} \left\{ \frac{1}{2}\left(1 + \frac{x}{\sqrt{\lambda + x^2}}\right)\right\}^{u-1} \right\} = 0 \quad (22)$$

and let $x \to -\infty$

$$\lim_{x\to-\infty} g(x) = \lim_{x\to-\infty} \left\{ u \frac{\lambda}{2\left(\lambda + x^2\right)^{\frac{3}{2}}} \left\{ \frac{1}{2}\left(1 + \frac{x}{\sqrt{\lambda + x^2}}\right)\right\}^{u-1} \right\} \quad (23)$$

$$= \lim_{x\to-\infty}(u) \times \lim_{x\to-\infty} \frac{\lambda}{2\left(\lambda + x^2\right)^{\frac{3}{2}}} \times \lim_{x\to-\infty} \left\{ \frac{1}{2}\left(1 + \frac{x}{\sqrt{\lambda + x^2}}\right)\right\}^{u-1} \quad (24)$$

Since $\lim_{x\to\infty} \frac{\lambda}{2\left(\lambda + x^2\right)^{\frac{3}{2}}}$ =0

Then

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} \left\{ u \frac{\lambda}{2\left(\lambda + x^2\right)^{\frac{3}{2}}} \left\{ \frac{1}{2}\left(1 + \frac{x}{\sqrt{\lambda + x^2}}\right)\right\}^{u-1} \right\} = 0 \quad (25)$$

2.2. Hazard Rate Function of Exponentiated Skew-t distribution (ESTD)

The hazard rate function of a random variable X with the probability density function $g(x)$ and a cumulative distribution function $G(x)$ denoted by $h(x)$ is defined as the ratio of the density function $g(x)$ to its survival function $1-G(x)$. So, the hazard rate function $h(x)$ of Exponentiated skew-t distribution is given by:

$$h(x)=\frac{g(x)}{1-G(x)} \tag{26}$$

For the Exponentiated skew-t distribution, the $g(x)$ and $G(x)$ are given respectively by their equations. Using these expressions, the hazard rate function may be expressed as

$$h(x)=\frac{u\left\{\frac{1}{2}\left(1+\frac{x}{\sqrt{\lambda+x^2}}\right)\right\}^{u-1}\frac{\lambda}{2\left(\lambda+x^2\right)^{\frac{3}{2}}}}{1-\left\{\frac{1}{2}\left(1+\frac{x}{\sqrt{\lambda+x^2}}\right)\right\}^{u}} \tag{27}$$

$$=\frac{u\lambda\left\{\frac{1}{2}\left(1+\frac{x}{\sqrt{\lambda+x^2}}\right)\right\}^{u-1}\frac{1}{2\left(\lambda+x^2\right)^{\frac{3}{2}}}}{1-\left\{\frac{1}{2}\left(1+\frac{x}{\sqrt{\lambda+x^2}}\right)\right\}^{u}}$$

$$=\frac{\dfrac{u\lambda\left\{\frac{1}{2}\left(1+\frac{x}{\sqrt{\lambda+x^2}}\right)\right\}^{u-1}}{2\left(\lambda+x^2\right)^{\frac{3}{2}}}}{1-\left\{\frac{1}{2}\left(1+\frac{x}{\sqrt{\lambda+x^2}}\right)\right\}^{u}}$$

$$=\frac{\dfrac{u\lambda 2^{-u}\left(1+\frac{x}{\sqrt{\lambda+x^2}}\right)^{u-1}}{\left(\lambda+x^2\right)^{\frac{3}{2}}}}{1-2^{-u}\left(1+\frac{x}{\sqrt{\lambda+x^2}}\right)^{u}}$$

$$=\frac{u\lambda 2^{-u}\left(1+\frac{x}{\sqrt{\lambda+x^2}}\right)^{u-1}}{\left(\lambda+x^2\right)^{\frac{3}{2}}\left(1-2^{-u}\left(1+\frac{x}{\sqrt{\lambda+x^2}}\right)^{u}\right)}$$

$$=\frac{u\lambda 2^{-u}\left(1+\frac{x}{\sqrt{\lambda+x^2}}\right)^{u}\times\frac{1}{\left(1+\frac{x}{\sqrt{\lambda+x^2}}\right)}}{\left(\lambda+x^2\right)^{\frac{3}{2}}\left(1-2^{-u}\left(1+\frac{x}{\sqrt{\lambda+x^2}}\right)^{u}\right)}$$

$$=\frac{\dfrac{u\lambda 2^{-u}\left(1+\frac{x}{\sqrt{\lambda+x^2}}\right)^{u}}{\left(1+\frac{x}{\sqrt{\lambda+x^2}}\right)}}{\left(\lambda+x^2\right)^{\frac{3}{2}}\left(1-2^{-u}\left(1+\frac{x}{\sqrt{\lambda+x^2}}\right)^{u}\right)}$$

$$=\frac{\dfrac{u\lambda 2^{-u}\left(1+\dfrac{x}{\sqrt{\lambda+x^2}}\right)^u}{\dfrac{\sqrt{\lambda+x^2}+x}{\sqrt{\lambda+x^2}}}}{\left(\lambda+x^2\right)^{\frac{3}{2}}\left(1-2^{-u}\left(1+\dfrac{x}{\sqrt{\lambda+x^2}}\right)^u\right)}$$

$$=\frac{u\lambda 2^{-u}\left(1+\dfrac{x}{\sqrt{\lambda+x^2}}\right)^u\sqrt{\lambda+x^2}}{\left(\lambda+x^2\right)^{\frac{3}{2}}\sqrt{\lambda+x^2}+x\left(1-2^{-u}\left(1+\dfrac{x}{\sqrt{\lambda+x^2}}\right)^u\right)}$$

$$=\frac{u\lambda 2^{-u}\left(1+\dfrac{x}{\sqrt{\lambda+x^2}}\right)^u}{\left(\lambda+x^2\right)\sqrt{\lambda+x^2}+x\left(1-2^{-u}\left(1+\dfrac{x}{\sqrt{\lambda+x^2}}\right)^u\right)}$$

$$=\frac{\dfrac{u\lambda\left(1+\dfrac{x}{\sqrt{\lambda+x^2}}\right)^u}{2^u}}{\left(\lambda+x^2\right)\sqrt{\lambda+x^2}+x\left(1-2^{-u}\left(1+\dfrac{x}{\sqrt{\lambda+x^2}}\right)^u\right)}$$

$$=\frac{u\lambda\left(1+\dfrac{x}{\sqrt{\lambda+x^2}}\right)^u}{2^u}\times\frac{1}{\left(\lambda+x^2\right)\sqrt{\lambda+x^2}+x\left(1-2^{-u}\left(1+\dfrac{x}{\sqrt{\lambda+x^2}}\right)^u\right)} \qquad (28)$$

Simplify and collect like term of the above expression

$$=\frac{u\lambda\left(1+\dfrac{x}{\sqrt{\lambda+x^2}}\right)^u}{\left(\lambda+x^2\right)\sqrt{\lambda+x^2}+x\times 2^u\left(1-2^{-u}\left(1+\dfrac{x}{\sqrt{\lambda+x^2}}\right)^u\right)}$$

$$=\frac{u\lambda\left(1+\dfrac{x}{\sqrt{\lambda+x^2}}\right)^u}{\sqrt{\lambda+x^2}+x\times\left(\lambda+x^2\right)\times\left(2^u-2^{u-u}\left(1+\dfrac{x}{\sqrt{\lambda+x^2}}\right)^u\right)}$$

$$h(x)=\frac{u\lambda\left(1+\dfrac{x}{\sqrt{\lambda+x^2}}\right)^u}{\left(\sqrt{\lambda+x^2}+x\right)\left(\lambda+x^2\right)\left(2^u-\left(1+\dfrac{x}{\sqrt{\lambda+x^2}}\right)^u\right)} \qquad (29)$$

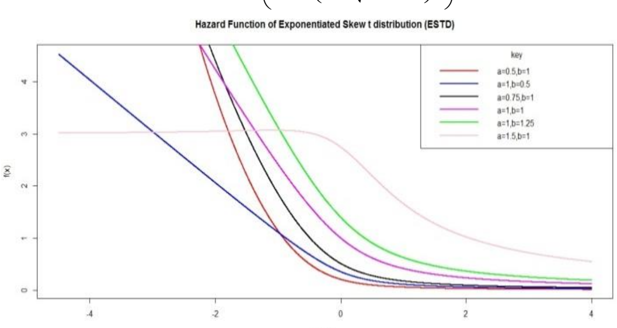

Figure 3: Hazard function of ESTD

Figure 3 shows the failure rate of the proposed distribution indicating decreases as the value increased

2.3. The Survival Function of Exponentiated Skew-T Distribution

The survival function is the probability that a stock price, device, or other object of interest will survive beyond a specified time. It is also known as the survivor function or reliability function.

The survival function is given as

$$s(x_t) = 1 - G(x_t) \tag{30}$$

$$= 1 - \left\{ \frac{1}{2}\left(1 + \frac{x_t}{\sqrt{\lambda + x_t^{\,2}}}\right)\right\}^u \tag{31}$$

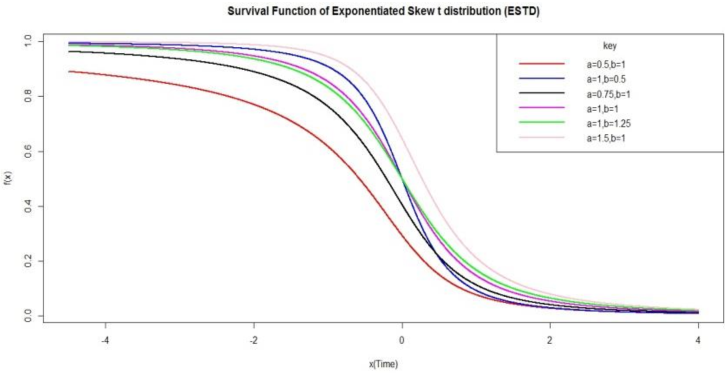

Figure 4: Survival function of ESTD

Figure 4 shows the survival function of the reliability plot of the proposed distribution

2.4. Quantile function and median of Exponentiated skew-t distribution

The Quantile function is given by;

$$Q(u) = G^{-1}(u) \tag{32}$$

Therefore, the corresponding quantile function for the Exponentiated skew-t model is given by;

$$G(x) = \left\{ \frac{1}{2}\left(1 + \frac{x}{\sqrt{\lambda + x^2}}\right)\right\}^u \tag{33}$$

$$\Rightarrow \left\{ \frac{1}{2}\left(1 + \frac{x}{\sqrt{\lambda + x^2}}\right)\right\}^u = q \tag{34}$$

Find the uth root of both side

$$\Rightarrow \frac{1}{2}\left(1 + \frac{x}{\sqrt{\lambda + x^2}}\right) = q^{1/u}$$

Multiply through by 2

$$\Rightarrow 1 + \frac{x}{\sqrt{\lambda + x^2}} = 2q^{1/u} \Rightarrow \frac{x}{\sqrt{\lambda + x^2}} = 2q^{1/u} - 1$$

$$\Rightarrow \frac{x}{\left(\lambda + x^2\right)^{1/2}} = 2q^{1/u} - 1$$

Find the square of both sides

$$\left(\frac{x}{\left(\lambda + x^2\right)^{1/2}}\right)^2 = \left(2q^{1/u} - 1\right)^2 \Rightarrow x^2 = \left(2q^{1/u} - 1\right)^2 \left(\lambda + x^2\right)$$

$$\Rightarrow x^2 = \lambda\left(2q^{1/u} - 1\right)^2 + x^2\left(2q^{1/u} - 1\right)^2 \Rightarrow x^2 - x^2\left(2q^{1/u} - 1\right)^2 = \lambda\left(2q^{1/u} - 1\right)^2$$

$$\Rightarrow x^2\left[1 - \left(2q^{1/u} - 1\right)^2\right] = \lambda\left(2q^{1/u} - 1\right)^2$$

$$x^2 = \frac{\lambda\left(2q^{1/u} - 1\right)^2}{\left[1 - \left(2q^{1/u} - 1\right)^2\right]} \Rightarrow x \pm \sqrt{\frac{\lambda\left(2q^{1/u} - 1\right)^2}{\left[1 - \left(2q^{1/u} - 1\right)^2\right]}} \tag{35}$$

First quantile is when $q=\frac{1}{4}$, Median quantile is when $q=\frac{2}{4}=\frac{1}{2}$ and Third quantile is when $q=\frac{3}{4}$

For example: Median quantile

$$x\pm\sqrt{\frac{\lambda\left(2q^{1/u}-1\right)^2}{\left[1-\left(2q^{1/u}-1\right)^2\right]}} \text{ Substitute q=0.5} \tag{36}$$

$$x\pm\sqrt{\frac{\lambda\left(2\times 0.5^{1/u}-1\right)^2}{\left[1-\left(2\times 0.5^{1/u}-1\right)^2\right]}}=x\pm\sqrt{\frac{\lambda\left(\frac{1}{u}-1\right)^2}{\left[1-\left(\frac{1}{u}-1\right)^2\right]}} \tag{37}$$

2.5. Binomial Expansion of the Proposed Probability Density Distributions

In this subsection, we will expand the PDF of the proposed distributions. Using the binomial series expansion. If "u" is a positive real non-integer and $|x| \leq 1$, we can considered the power series expansion as;

$$(1-x)^u=\sum_{i=0}^{\infty} b_i x^i \tag{38}$$

where $b_i=(-1)^i\binom{w-1}{i}$

Hence, using this above expression, we are going to obtain the probability density distribution of the proposed distribution.

2.5.1. Binomial expansions of Exponentiated skew-t distribution

Recall the density distribution of Exponentiated skew-t distribution given as;

$X \sim ESTD(u,\lambda)$

$$g(x)=u\left\{\frac{1}{2}\left(1+\frac{x}{\sqrt{\lambda+x^2}}\right)\right\}^{u-1}\frac{\lambda}{2\left(\lambda+x^2\right)^{\frac{3}{2}}}$$

$$=\frac{u\lambda}{2^u}\left(1+\frac{x}{\sqrt{\lambda+x^2}}\right)^{u-1}\frac{1}{\left(\lambda+x^2\right)^{3/2}} \tag{40}$$

$$=\frac{u\lambda}{2^u}\sum_{i=0}^{u-1}\binom{u-1}{i}\left(\frac{x}{\sqrt{\lambda+x^2}}\right)^i\frac{1}{\left(\lambda+x^2\right)^{3/2}} \tag{41}$$

$$=\frac{u\lambda}{2^u}\sum_{i=0}^{u-1}\binom{u-1}{i}x^i\left(\lambda+x^2\right)^{-\left(\frac{3+i}{2}\right)} \tag{42}$$

Also applying binomial expansion to the above equation (42)

$$=\frac{u\lambda}{2^u}\sum_{i=0}^{u-1}\binom{u-1}{i}x^i\sum_{j=0}^{\infty}\binom{\frac{3+i}{2}}{j}x^{2j}(\lambda)^{-\left(\frac{3+i}{2}\right)-j} \tag{43}$$

$$=\frac{u\lambda}{2^u}(\lambda)^{-\left(\frac{3+i}{2}+j\right)}\sum_{i=0}^{u-1}\sum_{j=0}^{\infty}\binom{u-1}{i}\binom{\frac{3+i}{2}}{j}x^{i+2j} \tag{44}$$

Hence, the binomial expansion of Exponentiated skew-t distribution is given as;

$$g(x)=h_{u,\lambda}\,x^{i+2j} \tag{45}$$

where $h_{u,\lambda}=\frac{u\lambda}{2^u}(\lambda)^{-\left(\frac{3+i}{2}+j\right)}\sum_{i=0}^{u-1}\sum_{j=0}^{\infty}\binom{u-1}{i}\binom{\frac{3+i}{2}}{j}$

3.5. rth-Moments of Exponentiated skew-t distribution

If a continuous random variable X is such that;

$X \sim ESTD(u,\lambda)$

$$\mu_r = \int_{-\infty}^{\infty} x^r g\left(x\right) dx \tag{46}$$

where $g\left(x\right) = h_{u,\lambda}\, x^{i+2j}$, the simplified Exponentiated skew-t distribution using binomial expansions and

$$h_{u,\lambda} = \frac{u\lambda}{2^u}(\lambda)^{-\left(\frac{3+i}{2}+j\right)} \sum_{i=0}^{u-1} \sum_{j=0}^{\infty} \binom{u-1}{i} \binom{\frac{3+i}{2}}{j}$$

$$= \int_{-\infty}^{\infty} x^r h_{u,\lambda} x^{i+2j} dx = h_{u,\lambda} \int_{-\infty}^{\infty} x^r x^{i+2j} dx \tag{47}$$

Hence, the moments of Exponentiated skew-t distribution is given as;

$$\mu_r = h_{u,\lambda} \int_{-\infty}^{\infty} x^{r+i+2j} dx \tag{48}$$

3.6. Characteristic Function (CF) of Exponentiated skew-t distribution

Using the above equation, the CF of the ESTD is obtained as;

If a continuous random variable X is such that;

$$X \sim ESTD\left(u, \lambda\right)$$

$$g\left(x\right) = h_{u,\lambda}\, x^{i+2j} \tag{49}$$

where $h_{u,\lambda} = \frac{u\lambda}{2^u}(\lambda)^{-\left(\frac{3+i}{2}+j\right)} \sum_{i=0}^{u-1} \sum_{j=0}^{\infty} \binom{u-1}{i} \binom{\frac{3+i}{2}}{j}$ is the constant term, so

$$\varphi_X\left(t\right) = \mathrm{E}\left[e^{itX}\right] = \int_{-\infty}^{\infty} e^{itx} g\left(x\right) dx \tag{50}$$

$$\varphi_X\left(t\right) = h_{u,\lambda} \int_{-\infty}^{\infty} e^{itx} x^{i+2j} dx \tag{51}$$

Let $y = itx \quad \Rightarrow x = \frac{y}{it} \quad \Rightarrow \frac{dx}{dy} = \frac{1}{it}$

Making substitution of the above expression into equation (51)

$$\varphi_X\left(t\right) = h_{u,\lambda} \int_{-\infty}^{\infty} \frac{y^{i+2j}}{\left(it\right)^{i+2j}} e^{y} dy \tag{52}$$

The above equation (52) is the CF of ESTD.

3.7. Maximum Likelihood estimation of Exponentiated Skew-t distribution

Exponentiated skew-t distribution is given as:

$$g\left(x\right) = u\left\{\frac{1}{2}\left(1 + \frac{x}{\sqrt{\lambda + x^2}}\right)\right\}^{u-1} \frac{\lambda}{2\left(\lambda + x^2\right)^{\frac{3}{2}}} \tag{53}$$

$$L\left(\theta\right) = \prod_{i=1}^{n} g\left(x;\theta\right) = L\left(x;u,\lambda\right) = \prod_{i=1}^{n} g\left(x;u,\lambda\right) \tag{54}$$

where $\theta = \left(u, \lambda\right)$ are the unknown

$$= \prod_{i=1}^{n}\left[u\left\{\frac{1}{2}\left(1 + \frac{x}{\sqrt{\lambda + x^2}}\right)\right\}^{u-1} \frac{\lambda}{2\left(\lambda + x^2\right)^{\frac{3}{2}}}\right] \tag{55}$$

$$= u^n \lambda^n \prod_{i=1}^{n}\left[\left\{\frac{1}{2}\left(1 + \frac{x}{\sqrt{\lambda + x^2}}\right)\right\}^{u-1} \frac{1}{2\left(\lambda + x^2\right)^{\frac{3}{2}}}\right] \tag{56}$$

Taking the log transformation of the above equations, we have the log-likelihood given as

$$InL(x;u,\lambda)=n\log u+n\log\lambda-nu\log 2+(u-1)\sum_{i=1}^{n}\log\left(1+\frac{x}{\sqrt{\lambda+x^2}}\right)-\frac{3}{2}\sum_{i=1}^{n}\log\left(\lambda+x^2\right) \quad (57)$$

Differentiating the above equation (57) partially with respect to u and λ equating to zero.

$$\frac{dl}{du}=\frac{n}{u}-n\log(2)+\sum_{i=1}^{n}\log\left(1+\frac{x}{\sqrt{\lambda+x^2}}\right)=0 \quad (58)$$

$$\frac{dl}{d\lambda}=\frac{n}{\lambda}-\frac{3}{2}\sum_{i=1}^{n}\left(\frac{1}{\lambda+x^2}\right)+(u-1)\frac{\sum_{i=1}^{n}\frac{d}{d\lambda}\left(1+\frac{x}{\sqrt{\lambda+x^2}}\right)}{1+\frac{x}{\sqrt{\lambda+x^2}}}=0 \quad (59a)$$

$$=\frac{n}{\lambda}-\frac{3}{2}\sum_{i=1}^{n}\left(\frac{1}{\lambda+x^2}\right)+(u-1)\frac{\sum_{i=1}^{n}\left(\sqrt{\lambda+x^2}\left(\frac{1}{2(\lambda+x^2)}-\frac{\sqrt{\lambda+x^2}+x}{2(\lambda+x^2)^{3/2}}\right)\right)}{\sqrt{\lambda+x^2}+x}=0 \quad (59b)$$

That is $\frac{dl}{d\lambda}\log\left(1+\frac{x}{\sqrt{\lambda+x^2}}\right)$

Let p $\log\left(1+\frac{x}{\sqrt{\lambda+x^2}}\right)$, so we have $\frac{dl}{d\lambda}\log(p)$

$=\log(p)=\frac{1}{p}$, and $\frac{dp}{d\lambda}=\frac{-x}{2(\lambda+x^2)^{3/2}}$

$$\frac{1}{p}.\frac{dp}{d\lambda}=\frac{1}{1+\frac{x}{\sqrt{\lambda+x^2}}}.\frac{-x}{2(\lambda+x^2)^{3/2}}=\frac{\sum_{i=1}^{n}\left(\sqrt{\lambda+x^2}\left(\frac{1}{2(\lambda+x^2)}-\frac{\sqrt{\lambda+x^2}+x}{2(\lambda+x^2)^{3/2}}\right)\right)}{\sqrt{\lambda+x^2}+x}$$

The maximum likelihood estimates (MLEs) obtained of the proposed model cannot be solved algebraically except using any of the Newton Raphson method because the equation obtained is a nonlinear system equation.

4. Conclusion

We define a two parameter Exponentiated skew-t distribution. We provide explicit expressions for the validity of the proposed distribution, quantile function, hazard function, survival function, rth-moments and Characteristic function. We propose the use of the new model in situations where both negative and positive values exist, increases with time and then decreases. Further research would involve comparing the performance of this model to any two parameter distributions with the aid of real data set and most especially with stock data which exhibit extreme data values.

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