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On Some Properties of A Three Parameter Generalized Lindley Distribution

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Abstract

In this paper, we introduced a new three parameter generalized Lindley distribution which has the Lindley distribution, Power Lindley distribution and a Generalized Gamma distribution as sub-models. Some of the mathematical properties of the new distribution such as the density function, cumulative distribution function, survival function, hazard rate function, moments, measures of skewness and kurtosis, quantile function, Renyi entropy and moment generating function were extensively studied. The maximum likelihood estimation method was used in estimating the parameters of the new distribution. Finally, we applied the new distribution alongside with some well known lifetime distributions to a real lifetime data set to examine its flexibility.

Keywords: Lindley Distribution; Power Lindley Distribution; Hazard Rate; Moments

1.0 Introduction

The density function of the classical one parameter Lindley distribution proposed in [1] is given by

$$f(x,\lambda)=\frac{\lambda^2}{\lambda+1}(1+x)e^{-\lambda x} \quad ; \; x>0, \, , \lambda>0 \tag{1}$$

and the corresponding cumulative distribution function defined as:

$$F(x,\lambda)= \quad 1-\left(\frac{\lambda+1+\lambda x}{(\lambda+1)}\right)e^{-\lambda x} \tag{2}$$

Equation (1) which is a two-component mixture of Exponential (λ) and Gamma (2, λ) can be expressed as:

$$f(x,\lambda)=pf_1(x)+(1-p)f_2(x)$$

where $f_1(x)$ and $f_2(x)$ are the pdf of the Exponential (λ) and Gamma (2, λ) distribution respectively and $p=\frac{\lambda}{\lambda+1}$ is the mixing proportion.

The properties of the one parameter Lindley distribution was studied in [2] and applied to a waiting time data. Considering some comparison criteria, it was shown that the distribution is a better model than the exponential distribution in modeling lifetime data. But due to the failure rate property of the one parameter Lindley distribution, there are some situations where the distribution fails to provide a good fit in modeling real lifetime data. Consequently, it has been observed that the flexibility of a model can be increased by addition of extra parameter(s), thus many researchers have proposed generalized forms of the classical one parameter Lindley distribution. Some of these generalizations can be found in the literature [3-14]. In this paper, we introduced a new three parameter generalized Lindley distribution (3PGLD) which is also an alternative model in modeling real lifetime data sets.

The remaining sections of this paper are organized as follows: In Section 2, we introduce the density function and cumulative distribution function of the proposed distribution, in Section 3, we present the sub-models of the proposed distribution. Sections 4-8 cover the survival function and hazard rate function, the quantile function, moments and related measures, moment generating function and Renyi entropy of the proposed distribution. We estimated the parameters of the distribution using maximum likelihood method in Section 9.

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An application of the proposed distribution to a real lifetime data set along side with some well known related lifetime distributions is given in section 10 and finally in section 11, we give a concluding remark.

2.0 PDF and CDF of the Proposed Distribution (3PGLD)

Consider a family of distributions whose cumulative distribution function is given by

$$F(x) = \frac{\lambda^2}{\lambda+1}\int_0^{G(x)}(1+t)e^{-\lambda t}\,dt\,,$$

$$= 1 - \frac{[1+\theta(1+G(x))]e^{-\theta[G(x)]}}{(\theta+1)}, \qquad x>0, \theta>0 \qquad (3)$$

and the corresponding density function defined by

$$f(x) = \frac{\lambda^2}{\lambda+1}(1+G(x))e^{-\lambda[G(x)]}g(x)\,, \quad x>0, \lambda>0 \qquad (4)$$

where $G(x,\phi)\in[0,\infty]$ is a non-negative monotonically increasing function depending on the parameter vector ϕ and also differentiable. Then from equation (3) and (4), on letting $G(x,\phi)=\frac{x^{\alpha}}{\beta}$, the cumulative distribution function and the density function of the three parameter generalized Lindley distribution (3PGLD) is given by equations (5) and (6) respectively as

$$F(x) = 1-\frac{\left[1+\lambda\beta+\lambda x^{\alpha}\right]e^{-\lambda x^{\alpha}}}{1+\lambda\beta}, \quad x>0,\ \lambda,\alpha,\beta>0 \qquad (5)$$

and

$$f(x) = \frac{\alpha\lambda^2(\beta+x^{\alpha})x^{\alpha-1}e^{-\lambda x^{\alpha}}}{1+\lambda\beta}, \quad x>0,\ \lambda,\alpha,\beta>0 \qquad (6)$$

The density function in equation (6) which is a two-component mixture of Weibull distribution with shape parameter (α) and scale parameter (λ) and a generalized gamma distribution with shape parameters $(2,\alpha)$ and scale parameter (λ) can be expressed as

$$f(x,\lambda)=pf_1(x)+(1-p)f_2(x)$$

where $f_1(x)$ and $f_2(x)$ are pdf of Weibull distribution and Generalized Gamma distribution respectively and $p=\frac{\lambda\beta}{1+\lambda\beta}$ is the mixing proportion.

The graphical presentation of the density function of 3PGLD for some fixed values of the parameters is shown in Figure 1.

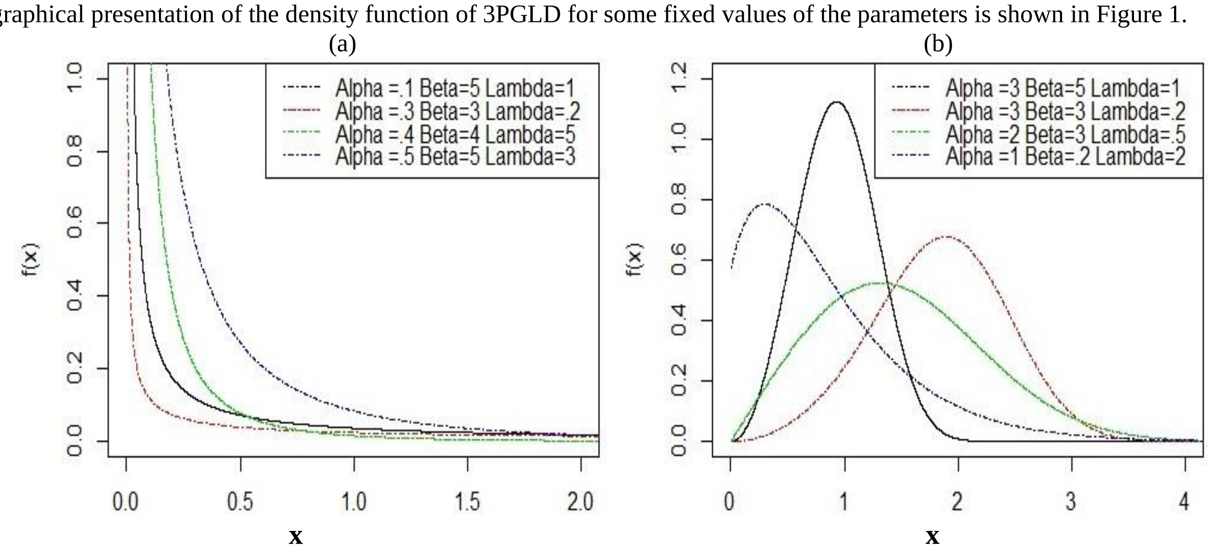

Figure 1: Density function of the 3PGLD for some fixed values of the parameters

From Figure 1(a), we observed that the density function of the 3PGLD at $\alpha < 1$ is a decreasing function and uni-modal for $\alpha \geq 1$ as displayed in Figure 1(b).

3.3 Sub-models of 3PGLD

3.3.1 Lindley Distribution

For $\alpha = \beta = 1$, the 3PGLD reduces to the classical one parameter Lindley distribution with probability density function given by,

$$f(x) = \frac{\lambda^2}{\lambda+1}(1+x)e^{-\lambda x}, \quad x > 0, \lambda > 0 \tag{7}$$

and the corresponding cumulative distribution function defined by

$$F(x) = 1 - \frac{(1+\lambda+\lambda x)e^{-\lambda x}}{\lambda+1}, \quad x > 0, \lambda > 0 \tag{8}$$

3.3.2 Power Lindley Distribution

For $\beta = 1$, the 3PGLD reduces to the Power Lindley distribution with density function given by,

$$f(x) = \frac{\alpha\lambda^2}{\lambda+1}\left(1+x^{\alpha}\right)x^{\alpha-1}e^{-\lambda x^{\alpha}}, \quad x > 0, \ \alpha, \lambda > 0 \tag{9}$$

and cumulative distribution function defined by,

$$F(x) = 1 - \frac{(1+\lambda+\lambda x^{\alpha})e^{-\lambda x^{\alpha}}}{\lambda+1}, \quad x > 0, \ \alpha, \lambda > 0 \tag{10}$$

3.3.3 Generalized Gamma Distribution

For $\beta = 0$, the 3PGLD reduces to the Generalized Gamma distribution with density function defined by,

$$f(x) = \alpha\lambda^2 x^{2\alpha-1}e^{-\lambda x^{\alpha}}, \quad x > 0, \ \alpha, \lambda > 0 \tag{11}$$

and the corresponding cumulative distribution function given by,

$$F(x) = 1 - (1+\lambda x^{\alpha})e^{-\lambda x^{\alpha}}, \quad x > 0, \ \alpha, \lambda > 0 \tag{12}$$

where 2 and α are the shape parameters and λ is the scale parameter.

4. Survival and Hazard Rate Function of (3PGLD)

Let X be a continuous random variable with density function *f(x)* and cumulative distribution function *F(x)*. The survival (reliability) function and hazard rate (failure rate) function of the three parameter generalized Lindley distribution are defined by:

$$S(x) = 1 - F(x) = \frac{(1+\lambda\beta+\lambda x^{\alpha})e^{-\lambda x^{\alpha}}}{1+\lambda\beta}, \quad x > 0, \ \alpha, \lambda, \beta > 0 \tag{13}$$

and

$$h(x) = \frac{f(x)}{S(x)} = \frac{\alpha\lambda^2(\beta+x^{\alpha})x^{\alpha-1}}{(1+\lambda\beta+\lambda x^{\alpha})}, \quad x > 0, \ \alpha, \lambda, \beta > 0 \tag{14}$$

The graph of the hazard rate function of the 3PGLD for different values of the parameters is given in Figure 2.

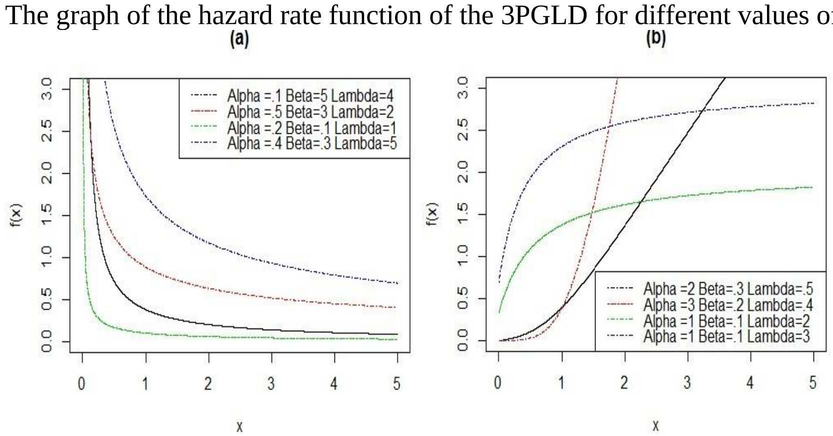

Figure 2: Hazard rate function of the 3PGLD for varying values of the parameters

Clearly from Figure 2(a), the 3PGLD exhibits monotone decreasing failure rate property and Figure 2(b) shows that the 3PGLD exhibits a monotone increasing failure rate property. It decreases monotonically when $\alpha < 1$ and increases monotonically when $\alpha \geq 1$.

5. Quantile Function of 3PGLD

Given the cumulative distribution function $F(x)$ defined by equation (5), the quantile function of the 3PGLD can be obtain as $Q_X(p) = F^{-1}(p)$. The quantile function of the Lindley family of distributions can be expressed in a closed form using the Lambert W function proposed in [15].

The p^{th} quantile function is obtained by solving $F(x) = p$ i.e.,

$$1 - \frac{\left[1+\lambda\beta+\lambda x^{\alpha}\right]e^{-\lambda x^{\alpha}}}{1+\lambda\beta} = p$$

$$\left[1+\lambda\beta+\lambda x^{\alpha}\right]e^{-\lambda x^{\alpha}} = (1+\lambda\beta)(1-p)$$

multiplying both sides by $e^{-(1+\lambda\beta)}$, we have

$$\left[-1-\lambda\beta-\lambda x^{\alpha}\right]e^{-(1+\lambda\beta+\lambda x^{\alpha})} = -(1+\lambda\beta)(1-p)e^{-(1+\lambda\beta)}$$

Clearly, we observe that $\left[-1-\lambda\beta-\lambda x^{\alpha}\right]$ is the Lambert W function of the real argument $-(1+\lambda\beta)(1-p)e^{-(1+\lambda\beta)}$. Thus, we have

$$W_{-1}\left[-(1+\lambda\beta)(1-p)e^{-(1+\lambda\beta)}\right] = \left[-1-\lambda\beta-\lambda x^{\alpha}\right]$$

$$\lambda x^{\alpha} = -1-\lambda\beta - W_{-1}\left[-(1+\lambda\beta)(1-p)e^{-(1+\lambda\beta)}\right]$$

$$x = \left[-\beta - \frac{1}{\lambda} - \frac{1}{\lambda}W_{-1}\left[-(1+\lambda\beta)(1-p)e^{-(1+\lambda\beta)}\right]\right]^{1/\alpha} \qquad (15)$$

where $p \in (0,1)$.

The median of the 3PGLD can be obtained by substituting $p = 1/2$ in equation (15) which yields,

$$Median = Q_2 = F^{-1}(1/2) = \left[-\beta - \frac{1}{\lambda} - \frac{1}{\lambda}W_{-1}\left[-1/2(1+\lambda\beta)\exp(-1-\lambda\beta)\right]\right]^{1/\alpha}$$

6. Moments and Some Related Measures of the 3PGLD

Let X be a continuous random variable with density function $f(x)$, then the r^{th} raw moment of X is defined by,

$$\mu_r' = E(X^r) = \int_0^{\infty} x^r f(x)dx \qquad (16)$$

Given the pdf in equation (6), the r^{th} raw moment of the 3PGLD is defined by,

$$\mu_r' = E(X^r) = \int_0^{\infty} x^r \frac{\alpha\lambda^2(\beta+x^{\alpha})x^{\alpha-1}e^{-\lambda x^{\alpha}}}{1+\lambda\beta}dx$$

$$= \frac{\alpha\lambda^2}{1+\lambda\beta}\left[\int_0^{\infty}\beta x^{\alpha+r-1}e^{-\lambda x^{\alpha}}dx + \int_0^{\infty}x^{2\alpha+r-1}e^{-\lambda x^{\alpha}}dx\right] \qquad (17)$$

Using the transformation $y = \lambda t^{\alpha}$, $t = \left(\frac{y}{\lambda}\right)^{1/\alpha}$, $dt = \frac{1}{\alpha\lambda}\left(\frac{y}{\lambda}\right)^{1/\alpha - 1}dy$

$$\beta\int_0^{\infty} x^{\alpha+r-1}e^{-\lambda x^{\alpha}}dx = \frac{\beta}{\alpha\lambda}\left[\int_0^{\infty}\left[\left(\frac{y}{\lambda}\right)^{1/\alpha}\right]^{\alpha+r-1}e^{-y}\left(\frac{y}{\lambda}\right)^{1/\alpha - 1}dy\right]$$

$$= \frac{\beta}{\alpha\lambda}\left[\int_0^{\infty}\frac{y^{r/\alpha}e^{-y}}{\lambda^{r/\alpha}}dy\right]$$

$$= \frac{\beta\,\Gamma(r/\alpha+1)}{\alpha\lambda^{r/\alpha+1}}$$

Similarly,

$$\int_0^\infty x^{2\alpha+r-1}e^{-\lambda x^\alpha}\,dx = \frac{\Gamma(r/\alpha+2)}{\alpha\lambda^{r/\alpha+2}}$$

upon substituting into equation (17), we have

$$\mu_r' = \frac{\alpha\lambda^2}{1+\lambda\beta}\left[\frac{\beta\,\Gamma(r/\alpha+1)}{\alpha\lambda^{r/\alpha+1}} + \frac{\Gamma(r/\alpha+2)}{\alpha\lambda^{r/\alpha+2}}\right]$$

$$= \frac{1}{1+\lambda\beta}\left[\frac{\lambda\beta\,\Gamma(r/\alpha+1)+\Gamma(r/\alpha+2)}{\lambda^{r/\alpha}}\right]$$

$$= \frac{r\,[\alpha(\lambda\beta+1)+r]\Gamma(r/\alpha)}{\alpha^2\lambda^{r/\alpha}(1+\lambda\beta)}, \qquad r = 1,\ 2,\ 3, 4,\ldots \tag{18}$$

From equation (18), the first four raw moments of the 3PGLD can be obtained as follows;

$$\mu_1' = \mu = \frac{[\alpha(\lambda\beta+1)+1]\Gamma(1/\alpha)}{\alpha^2\lambda^{1/\alpha}(1+\lambda\beta)}, \qquad \mu_2' = \frac{2[\alpha(\lambda\beta+1)+2]\Gamma(2/\alpha)}{\alpha^2\lambda^{2/\alpha}(1+\lambda\beta)}$$

$$\mu_3' = \frac{3[\alpha(\lambda\beta+1)+3]\Gamma(3/\alpha)}{\alpha^2\lambda^{3/\alpha}(1+\lambda\beta)}, \qquad \mu_4' = \frac{4[\alpha(\lambda\beta+1)+4]\Gamma(4/\alpha)}{\alpha^2\lambda^{4/\alpha}(1+\lambda\beta)}$$

Similarly, the k^{th} central moments of a random variable X is defined by,

$$\mu_k = E\{(X-\mu)^k\} = E\left\{\sum_{r=0}^{k}(-1)^r\binom{k}{r}X^{k-r}\mu^r\right\}$$

$$= \left\{\sum_{r=0}^{k}(-1)^r\binom{k}{r}E(X^{k-r})\mu^r\right\}$$

$$= \sum_{r=0}^{k}(-1)^r\binom{k}{r}\mu_{k-r}'\mu^r \tag{19}$$

where $\mu_1' = \mu$ and $\mu_0 = 1$.

Using equation (19), the 2nd, 3rd and 4th central moments can be obtained as

$$\mu_2 = \mu_2' - \mu^2; \qquad \mu_3 = \mu_3' - 3\mu_2'\mu + 2\mu^3; \qquad \mu_4 = \mu_4' - 4\mu_3'\mu + 6\mu_2'\mu^2 - 3\mu^4.$$

The variance (σ^2) and coefficient of variation (γ) of the 3PGLD are given by,

$$\sigma^2 = \mu_2' - \mu^2 = \frac{2[\alpha(\lambda\beta+1)+2]\Gamma(2/\alpha)}{\alpha^2\lambda^{2/\alpha}(1+\lambda\beta)} - \left[\frac{[\alpha(\lambda\beta+1)+1]\Gamma(1/\alpha)}{\alpha^2\lambda^{1/\alpha}(1+\lambda\beta)}\right]^2$$

$$= \frac{2\alpha^2(\lambda\beta+1)[\alpha(\lambda\beta+1)+2]\Gamma(2/\alpha) - [\alpha(\lambda\beta+1)+1]^2\Gamma^2(1/\alpha)}{\alpha^4\lambda^{2/\alpha}(1+\lambda\beta)^2} \tag{20}$$

$$\gamma = \frac{\sigma}{\mu} = \frac{\sqrt{2\alpha^2(\lambda\beta+1)[\alpha(\lambda\beta+1)+2]\Gamma(2/\alpha) - [\alpha(\lambda\beta+1)+1]^2\Gamma^2(1/\alpha)}}{[\alpha(\lambda\beta+1)+1]\Gamma(1/\alpha)} \tag{21}$$

Further substitution of the raw moments yields the measures of skewness and kurtosis as follows;

Measure of Skewness (S_k) = $\dfrac{\mu_3}{\mu_2^{3/2}} = \dfrac{\mu_3^{'} - 3\mu_2^{'}\mu + 2\mu^3}{(\mu_2^{'} - \mu^2)^{3/2}}$ (22)

and

Measure of Kurtosis (K_s) = $\dfrac{\mu_4}{\mu_2^2} = \dfrac{\mu_4^{'} - 4\mu_3^{'}\mu + 6\mu_2^{'}\mu^2 - 3\mu^4}{(\mu_2^{'} - \mu^2)^2}$ (23)

Tables 1 and 2 show the theoretical moments of the 3PGLD for different values of the parameters. $\alpha = 2$

Table 1: Theoretical moments of 3PGLD for fixed value of the parameter ($\alpha = 2$)

$\mu_r^{'}$	$(\beta = 2,\ \lambda = 3)$	$(\beta = 2,\ \lambda = 4)$	$(\beta = 3,\ \lambda = 3)$	$(\beta = 3,\ \lambda = 4)$
$\mu_1^{'}$	0.54821	0.46773	0.53725	0.46015
$\mu_2^{'}$	0.38095	0.27778	0.36667	0.26923
$\mu_3^{'}$	0.31065	0.19386	0.29421	0.18534
$\mu_4^{'}$	0.28571	0.15278	0.26667	0.14423
σ^2	0.08042	0.05901	0.07803	0.05749
γ	0.69920	0.81951	0.71347	0.83210
S_k	0.59809	0.60974	0.61337	0.62006
K_s	3.16608	3.19188	3.20031	3.21638

Table 2: Theoretical moments of 3PGLD for fixed value of the parameter ($\alpha = 4$)

$\mu_r^{'}$	$(\beta = 2,\ \lambda = 3)$	$(\beta = 2,\ \lambda = 4)$	$(\beta = 3,\ \lambda = 3)$	$(\beta = 3,\ \lambda = 4)$
$\mu_1^{'}$	0.71331	0.65872	0.70593	0.65325
$\mu_2^{'}$	0.54821	0.46773	0.53725	0.46016
$\mu_3^{'}$	0.44638	0.35202	0.43342	0.34368
$\mu_4^{'}$	0.38095	0.27778	0.36667	0.26923
σ^2	0.03939	0.03381	0.03890	0.03342
γ	0.53737	0.58190	0.54298	0.58677
S_k	-0.11046	-0.10199	-0.09940	-0.09470
K_s	2.74277	2.74341	2.74383	2.74493

From Tables 1 and 2, we observed that the 3PGLD can be positively skewed and negatively skewed. Also, at some fixed values of the parameters, the distribution can be leptokurtic as well as platykurkic.

7. Moment Generating Function

Let X be a continuous random variable with density function $f(x)$, then the moment generating function of X is defined by,

$$M_X(t) = E[e^{tx}] = \int_0^\infty e^{tx} f(x)dx \quad (24)$$

Thus, we define the moment generating function of 3PGLD by,

$$M_X(t) = \int_0^\infty e^{tx} \frac{\alpha\lambda^2(\beta + x^\alpha)x^{\alpha-1}e^{-\lambda x^\alpha}}{1+\lambda\beta}dx$$

Using the transformation $e^{tx} = \sum_{n=0}^{\infty} \frac{t^n x^n}{n!}$, we have

$$M_X(t) = \frac{\alpha\lambda^2}{1+\lambda\beta}\left[\int_0^{\infty}\sum_{n=0}^{\infty}\frac{t^n}{n!}(\beta+x^{\alpha})\,x^{n+\alpha-1}e^{-\lambda x^{\alpha}}\,dx\right]$$

$$= \frac{\alpha\lambda^2}{1+\lambda\beta}\sum_{n=0}^{\infty}\frac{t^n}{n!}\left[\beta\int_0^{\infty}x^{n+\alpha-1}e^{-\lambda x^{\alpha}}\,dx + \int_0^{\infty}x^{n+2\alpha-1}e^{-\lambda x^{\alpha}}\,dx\right]$$

$$= \frac{\alpha\lambda^2}{1+\lambda\beta}\sum_{n=0}^{\infty}\frac{t^n}{n!}\left[\frac{\beta\,\Gamma(n/\alpha+1)}{\alpha\lambda^{n/\alpha+1}} + \frac{\Gamma(n/\alpha+2)}{\alpha\lambda^{n/\alpha+2}}\right]$$

$$= \sum_{n=0}^{\infty}\frac{t^n}{n!}\left[\frac{\lambda\beta\left(\frac{n}{\alpha}\right)\Gamma(n/\alpha) + \left(\frac{n}{\alpha}\right)\left(\frac{n}{\alpha}+1\right)\Gamma(n/\alpha)}{\lambda^{n/\alpha}(1+\lambda\beta)}\right]$$

$$= \sum_{n=0}^{\infty}\frac{t^n}{n!}\left[\frac{\left(\alpha\lambda\beta n+n^2+n\alpha\right)\Gamma(n/\alpha)}{\alpha^2\lambda^{n/\alpha}(1+\lambda\beta)}\right]$$

$$= \sum_{n=1}^{\infty}\frac{t^n}{(n-1)!}\left[\frac{(\alpha\lambda\beta+n+\alpha)\Gamma(n/\alpha)}{\alpha^2\lambda^{n/\alpha}(1+\lambda\beta)}\right] \qquad (25)$$

The first and second derivatives of equation (25) give the first and second raw moments of 3PGLD.

8. Renyi Entropy

An entropy of a random variable X is a measure of variation of uncertainty associated with the random variable X. The Renyi entropy of X with density function $f(x)$ suggested in [16], is defined by,

$$\tau_R(\gamma) = \frac{1}{1-\gamma}\log\left[\int f^{\gamma}(x)dx\right], \qquad \gamma>0, \qquad \gamma\neq 1 \qquad (26)$$

Using equation (26), the Renyi entropy of the 3PGLD is defined by,

$$\tau_R(\gamma) = \frac{1}{1-\gamma}\log\left[\int_0^{\infty}\frac{(\alpha\lambda^2)^{\gamma}(\beta+x^{\alpha})^{\gamma}x^{\gamma(\alpha-1)}e^{-\gamma\lambda x^{\alpha}}}{(1+\lambda\beta)^{\gamma}}\,dx\right]$$

$$= \frac{1}{1-\gamma}\log\left[\left(\frac{\alpha\lambda^2}{(1+\lambda\beta)}\right)^{\gamma}\int_0^{\infty}(\beta+x^{\alpha})^{\gamma}x^{\alpha\gamma-\gamma}e^{-\gamma\lambda x^{\alpha}}\,dx\right]$$

From the series expansion $(a+b)^n = \sum_{j=0}^{\infty}\binom{n}{j}a^{n-j}b^j$, we have

$$= \frac{1}{1-\gamma}\log\left[\left(\frac{\alpha\lambda^2}{(1+\lambda\beta)}\right)^{\gamma}\sum_{j=0}^{\infty}\binom{\gamma}{j}\beta^{\gamma-j}\int_0^{\infty}x^{\alpha j+\alpha\gamma-\gamma}e^{-\gamma\lambda x^{\alpha}}\,dx\right]$$

but $$\int_0^{\infty}x^{\alpha j+\alpha\gamma-\gamma}e^{-\gamma\lambda x^{\alpha}}\,dx = \frac{\Gamma(j+\gamma-\gamma/\alpha+1/\alpha)}{\alpha(\gamma\lambda)^{j+\gamma-\gamma/\alpha+1/\alpha}}$$

so that

$$\tau_R(\gamma) = \frac{1}{1-\gamma}\log\left[\left(\frac{\alpha\lambda^2}{(1+\lambda\beta)}\right)^{\gamma}\sum_{j=0}^{\infty}\binom{\gamma}{j}\beta^{\gamma-j}\frac{\Gamma(j+\gamma-\gamma/\alpha+1/\alpha)}{\alpha(\gamma\lambda)^{j+\gamma-\gamma/\alpha+1/\alpha}}\right]$$

$$= \frac{1}{1-\gamma}\log\left[\left(\frac{\alpha^{\gamma-1}\lambda^{\gamma(1+1/\alpha)-1/\alpha}}{(1+\lambda\beta)^{\gamma}\,\gamma^{\gamma(1-1/\alpha)+1/\alpha}}\right)\sum_{j=0}^{\infty}\binom{\gamma}{j}\frac{\beta^{\gamma-j}}{(\gamma\lambda)^{j}}\Gamma(j+\gamma-\gamma/\alpha+1/\alpha)\right] \quad (27)$$

9. Maximum Likelihood Estimation

Let $(x_1, x_2, ..., x_n)$ be random samples from the 3PGLD, then the likelihood function is defined as,

$$L(x,\phi) = \prod_{i=1}^{n}\left[\frac{\alpha\lambda^2(\beta+x^{\alpha})\,x^{\alpha-1}e^{-\lambda x^{\alpha}}}{1+\lambda\beta}\right], \qquad \phi=(\alpha,\beta,\lambda) \quad (28)$$

with the log-likelihood function given by,

$$\ell(x,\phi) = \sum_{i=1}^{n}\log\left[\frac{\alpha\lambda^2(\beta+x^{\alpha})\,x^{\alpha-1}e^{-\lambda x^{\alpha}}}{1+\lambda\beta}\right],$$

$$= n\log\alpha + 2n\log\lambda + \sum_{i=1}^{n}\log(\beta+x_i^{\alpha}) + (\alpha-1)\sum_{i=1}^{n}\log(x_i) - \lambda\sum_{i=1}^{n}x_i^{\alpha} - n\log(1+\lambda\beta) \quad (29)$$

On differentiating the log-likelihood function with respect to the parameters, we obtain the score function as,

$$\frac{\partial\ell}{\partial\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\frac{x_i^{\alpha}\log x_i}{(\beta+x_i^{\alpha})} + \sum_{i=1}^{n}\log x_i - \lambda\sum_{i=1}^{n}x_i^{\alpha}\log x_i$$

$$\frac{\partial\ell}{\partial\beta} = \sum_{i=1}^{n}\frac{1}{(\beta+x_i^{\alpha})} - \frac{n\lambda}{(1+\lambda\beta)}$$

$$\frac{\partial\ell}{\partial\lambda} = \frac{2n}{\lambda} - \sum_{i=1}^{n}x_i^{\alpha} - \frac{n\beta}{(1+\lambda\beta)}$$

The maximum likelihood estimator $\hat{\phi}$ of ϕ can be obtained by solving the system of non-linear equation $\frac{\partial\ell}{\partial\phi} = 0$.

This equation can be solved by an iterative scheme known as Newton Raphson iterative scheme given by,

$$\hat{\phi} = \phi_k - H^{-1}(\phi_k)U(\phi_k), \qquad \hat{\phi} = (\hat{\alpha},\hat{\beta},\hat{\lambda})^T \quad (30)$$

where $U(\phi_k)$ is the score function and $H(\phi_k)$ is the Hessian matrix, which is the second derivative of the likelihood function. The Hessian matrix is defined by,

$$H(\phi_k) = \begin{pmatrix} \frac{\partial^2\ell}{\partial\alpha^2} & \frac{\partial^2\ell}{\partial\alpha\partial\beta} & \frac{\partial^2\ell}{\partial\alpha\partial\lambda} \\ \frac{\partial^2\ell}{\partial\beta\partial\alpha} & \frac{\partial^2\ell}{\partial\beta^2} & \frac{\partial^2\ell}{\partial\beta\partial\lambda} \\ \frac{\partial^2\ell}{\partial\lambda\partial\alpha} & \frac{\partial^2\ell}{\partial\lambda\partial\beta} & \frac{\partial^2\ell}{\partial\lambda^2} \end{pmatrix}$$

where

$$\frac{\partial^2 \ell}{\partial \alpha^2} = \beta \sum_{i=1}^{n} \frac{x_i^{\alpha} [\log x_i]^2}{(\beta + x_i^{\alpha})^2} - \lambda \sum_{i=1}^{n} x_i^{\alpha} [\log x_i]^2 - \frac{n}{\alpha^2}$$

$$\frac{\partial^2 \ell}{\partial \alpha \partial \beta} = \frac{\partial^2 \ell}{\partial \beta \partial \alpha} = -\sum_{i=1}^{n} \frac{x_i^{\alpha} \log x_i}{(\beta + x_i^{\alpha})^2}$$

$$\frac{\partial^2 \ell}{\partial \alpha \, \partial \lambda} = \frac{\partial^2 \ell}{\partial \lambda \, \partial \alpha} = -\sum_{i=1}^{n} x_i^{\alpha} \log x_i$$

$$\frac{\partial^2 \ell}{\partial \beta^2} = \frac{n\lambda^2}{(1+\lambda\beta)^2} - \sum_{i=1}^{n} \frac{1}{(\beta + x_i^{\alpha})}$$

$$\frac{\partial^2 \ell}{\partial \beta \, \partial \lambda} = \frac{\partial^2 \ell}{\partial \lambda \partial \beta} = -\frac{n}{(1+\lambda\beta)^2}$$

$$\frac{\partial^2 \ell}{\partial \lambda^2} = \frac{n\beta^2}{(1+\lambda\beta)^2} - \frac{2n}{\lambda^2}$$

10. Application of the 3PGLD

In this paper, we fit the proposed distribution to two real data sets alongside with some well known lifetime distributions with density function given by;

(i) Exponentiated Power Lindley Distribution (EPLD) reported in

$$f(x) = \frac{\alpha\lambda^2 \beta (1+x^{\alpha}) x^{\alpha-1} e^{-\lambda x^{\alpha}}}{1+\lambda} \left[1 - \left(1 + \frac{\lambda x^{\alpha}}{1+\lambda} \right) e^{-\lambda x^{\alpha}} \right]^{\beta-1}, \qquad x>0,\ \alpha,\beta,\lambda>0$$

(ii) Power Lindley Distribution (PLD) reported in [4],

$$f(x) = \frac{\alpha\lambda^2 \beta (1+x^{\alpha}) x^{\alpha-1} e^{-\lambda x^{\alpha}}}{1+\lambda}, \qquad x>0,\ \alpha,\lambda>0$$

(iii) Lindley-Exponential Distribution (LED) reported in [12],

$$f(x) = \frac{\lambda^2 \alpha e^{-\alpha x} (1-e^{-\alpha x})^{\lambda-1} (1-\log(1-e^{-\alpha x}))}{1+\lambda}, \qquad x>0,\ \alpha,,\lambda>0$$

(iv) Weibull Distribution reported in [17],

$$f(x) = \alpha\lambda x^{\alpha-1} e^{-\lambda x^{\alpha}}, \qquad x>0,\ \alpha,\lambda>0$$

(v) Lindley Distribution reported in [18],

$$f(x) = \frac{\lambda^2 (1+x) e^{-\lambda x}}{1+\lambda}, \qquad x>0,\ \lambda>0$$

Data Set: Table 3 shows the data set consisting of the times to failure of 50 devices put on life test at time 0, reported in [18].

Table 3: Time to Failure of 50 devices

0.10	0.20	1	1	1	1	1	2	3	6	7	11	12
18	18	18	18	18	21	32	36	40	45	46	47	50
55	60	63	63	67	67	67	67	72	75	79	82	82
83	84	84	84	85	85	85	85	85	86	86		

The comparison criteria considered in this work includes, the estimates of the parameters of the distribution, $-2\log(L)$, Akaike Information Criterion $[AIC = 2k - 2\log(L)]$, Bayesian Information Criterion $[BIC = k\log(n) - 2\log(L)]$ and Kolmogorov-Smirnov Statistic $(K-S)$. Where n is the number of observations, k is the number of estimated parameters and L is the value of the likelihood function evaluated at the parameter estimates.

Table 4: Summary Statistics for the Data Set

Models	*Estimates*	*-2logL*	*AIC*	*BIC*	*K-S*
3PGLD	$\alpha = 0.8817$ $\beta = 17.0543$ $\lambda = 0.0542$	479.5172	485.5172	491.2532	0.1850
EPLD	$\alpha = 0.4199$ $\beta = 0.5748$ $\lambda = 2.3968$	492.2642	498.2641	504.0002	0.2118
PLD	$\alpha = 0.1612$ $\lambda = 0.6640$	484.1746	488.1747	491.9987	0.1961
LED	$\alpha = 1.0392$ $\lambda = 0.1501$	484.0982	488.0982	491.9222	0.2107
WEIBULL	$\alpha = 0.9486$ $\lambda = 44.8940$	482.0036	486.0037	489.8277	0.1929
LINDLEY	$\lambda = 0.0429$	502.8606	504.8606	506.7726	0.1990

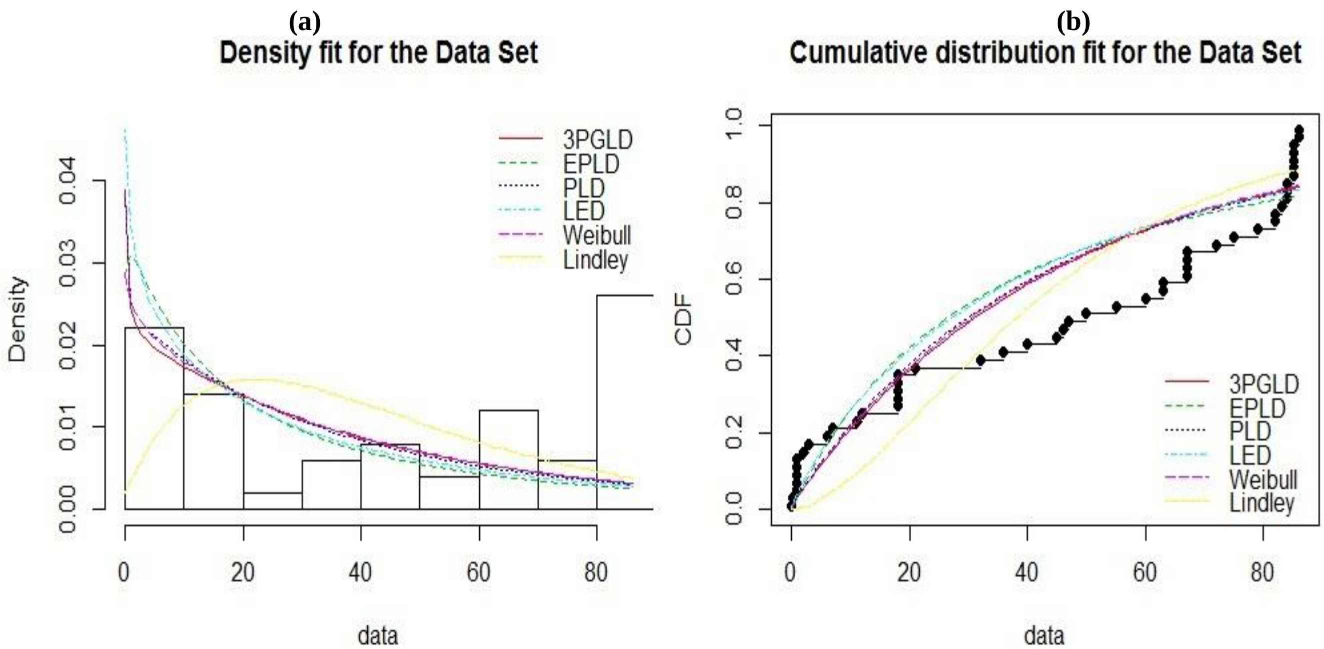

Figure 3: Density and Cumulative Distribution fit for the Data Set

The fit of the density and cumulative distribution fit of each distribution for the data set are given in the Figure 3(a) and Figure 3(b) respectively.

Table 4 shows that the three-parameter generalized Lindley distribution (3PGLD) has the least value of *-2logL, AIC, BIC and K-S Statistic*, which indicates that the 3PGLD demonstrates superiority over the Exponentiated Power Lindley distribution, Power Lindley distribution, Lindley Exponential distribution, Weibull distribution and the classical one parameter Lindley distribution in modeling the lifetime data sets under study. This claim was further supported by inspecting the density and cumulative distribution fit of the distributions for the real lifetime data set.

11. Conclusion

In this paper, a three parameter generalized Lindley distribution is introduced and the some of the mathematical properties were extensively studied. The maximum likelihood estimation method for estimating its parameters were also achieved. An application of the 3PGLD to a real lifetime data set reveals its superiority over the Exponentiated

Power Lindley distribution, Power Lindley distribution, Lindley Exponential distribution, Weibull distribution and the classical one parameter Lindley distribution in modeling the lifetime data sets under study.

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