Transactions of the Nigerian Association of Mathematical Physics
***Volume 4,* (*July, 2017*), pp211 – 216**
© Trans. of NAMP

The Impact of Shocks Correlation On The Optimal Asset Allocation For An Investor With Ornsten-Uhlenbeck Stochastic Interest Rate Model

S. A. Ihediohia[1], B. I. Oruh[2] and B. O. OSU[3]

[1]Department of Mathematics, Plateau State University Bokkos, P.M.B 212 Jos, Plateau state, Nigeria.
[2,3]Department of Mathematics, Michael Okpara University of Agriculture Umadike Umuahia, Abia State, Nigeria.

Abstract

In this paper, we investigated and obtained a closed form solution to an investment and consumption decision problem with risk-free asset having a rate of return that is driven by the Ornstein- Uhlenbeck process. To easily handle the HJB equation derived (which was a nonlinear second other PDE), we transformed the PDE into an ODE by using elimination of dependency of variable as in literature. It was found that the optimal investment strategy on the risky asset becomes totally dependent on the relative risk aversion coefficient and the total amount available for investment (only) if there is no correlation. While it becomes totally dependent on the relative risk aversion coefficient , the total amount available for investment , the correlation coefficient of the Brownian motions, the constant volatility of the interest rate, the diffusion parameter of the risky asset and rate of return of the risk free asset if there is correlation.

Keywords: Impact, shocks correlation, asset allocation, maximum principle, Ornstein-Uhlenbeck, stochastic interest rate model

1.0 Introduction

The classical Merton's portfolio optimization problems shows that an investor dynamically allocates his wealth between one risk asset and one risk-free asset and chooses an optimal consumption rate to maximize total expected discounted utility of consumption [1, 2]. In this Merton's model, there are no, transaction costs, borrowing and shorting constraints. Hundreds of literally extensions and applications on investment and consumption problems have been inspired by this pioneer work of Merton. For example, the introduction of transaction costs into the investment and consumption problems, one can refer to [3-5]. In investigating the optimal consumption problem with borrowing constraints authors in [6-9] have made very useful contributions. However, the above mentioned models generally were studied under the assumption that risky asset price dynamics was driven by a geometric Brownian motion (GBM) and the risk-free asset with a rate of return that is assumed constant. Some authors have studied the problem under the extension of geometric Brownian motion (GBM) called the constant elasticity of variance (CEV) model which is a natural extension of the GBM. The constant elasticity of variance (CEV) model has an advantage that the volatility rate has correlation with the risky asset price. Cox and Ross [10] originally proposed the use of constant elasticity of variance (CEV) model as an alternative diffusion process for pricing European option. This has also been applied to analyze the option pricing formula [11-14]. Further applications of the constant elasticity of variance (CEV) model, in the recent years, has been in the areas of annuity contracts and the optimal investment strategies in the utility framework using dynamic programming principle. Detailed discussions can be found in [15-22].
This paper aims at investigating and giving a closed form solution to an investment and consumption decision problem where the risk-free asset has a rate of return that is driven by Ornstein-Uhlenbeck Stochastic interest rate of return model. Dynamic programming principle, specifically, the maximum principle is applied to obtain the HJB equation for the value function.
Owing to the introducing of consumption factor and the Ornstein-Uhlenbeck Stochastic interest rate of return, the HJB equation derived is much more difficult to deal with than the one obtained in [16].

Correspondence Author: B.O. Osu., Email: megaobrait@hotmail.com, Tel: +2348032628251

Inspired by the techniques in [16] and [23], we transform the nonlinear second-order partial differential equation into an ordinary differential equation using elimination of dependency on variables, which is easy to tackle.
The rest of this paper is organized as follows. In section 2 is the problem formulation of the financial market and the proposed optimization problem. In section 3, dynamic programming principle is applied to obtain the HJB equation and the optimal investment and consumption strategies in the power utility preference case investigated and the findings given. Section 4 concludes the paper.

2.0 The problem formulation:

We assume that an investor trades two assets in an economy continuously-c riskless asset (bond) and a risky asset (stock), Let the price of the riskless asset be denoted by $B(t)$ with a rate of return $r(t)$ which is stochastic and driven by the Orinstein-Uhlenbeck model.That is

$$dB(t) = r(t)B(t)dt \tag{1}$$

where

$$dr(t) = \alpha\big(\beta - r(t)\big)dt + \sigma dz_1(t) : r(0) = r_0 \tag{2a}$$

then

$$r(t) = (r_0 - \beta)e^{-\alpha t} + \sigma \int_{t_0}^{t} e^{-\alpha(t-\alpha)} dz_1(\alpha) \tag{2b}$$

Where α is the speed of mean reversion,β the mean level attracting the interest rate and σ the constant volatility of the interest rate.$z_1(t)$ is a standard Brownian motion. Also, let the price of the risky asset be denoted by $s(t)$ with the process

$$ds(t) = s(t)[\mu dt + \lambda dz_2(t)], \tag{3a}$$

then

$$s(t) = s(0)\exp\left[\lambda z_2(t) + \left(\mu - \frac{\lambda^2}{2}\right)t\right], \forall\, t \in (0, \infty), \tag{3b}$$

where μ and λ are constants and μ the drift parameter while λ is the diffusion parameter.$z_2(t)$ is another standard Brownian motion.
Through this work, we assume a probability space $(\Omega, \mathcal{F}, \rho)$ and a filtration $\{\mathcal{F}_t\}$. Uncertainty in the models are generated by the Brownian motions $z_1(t)$ and $z_2(t)$.
Let $\pi(t)$ to the amount of money the investor decides to put in the risky asset at time t,then the balance $[w(t) - \pi(t)]$ is the amount to be invested in the riskless assets, where $w(t)$ is the total amount of money available for investment.

Assumption:

We assume that transaction cost, tax and dividend are paid on the amount invested in the risky asset at constant rates, $\sigma, \theta\ and\ d$ respectively. Therefore for any policy π, the total wealth process of the investor follows the stochastic differential equation (SDE)

$$dw^{\pi}(t) = \pi(t)\frac{ds(t)}{s(t)} + [w(t) - \pi(t)]\frac{dB}{B(t)} - (\vartheta + \theta - d)\pi(t)dt. \tag{4}$$

Applying (1) and (3a) in (4) gives

$$dw^{\pi}(t) = \{[(\mu + d) - (r(t) + \vartheta + \theta)]\pi(t) + r(t)w(t)\}dt + \lambda\pi(t)dz_2(t). \tag{5}$$

Suppose the investor has a utility function $U(.)$ which is strictly concave and continuously differentiable on $(-\infty, +\infty)$ and wishes to maximize his expected utility of terminal wealth, then his problem can therefore be written as

$$\max_{\pi} \epsilon[U(T)] \tag{6}$$

subject to (5).
The optimization problem is under consideration is for the power utility function given as

$$U(w) = \frac{w^{1-\phi}}{1-\phi}, \phi \neq 1 \tag{7}$$

where ϕ is a constant.

3.0 The Optimal investment strategy for the power utility function

Here we obtain the explicit solutions for the optimization problem using stochastic control and the maximum principle.

3.1 The general framework

Define the value function as

$$G(t, r, s, w) = \max_{\pi}[\epsilon(U(w)] = 0; U(T, W) = U(w), 0 < t < T$$

$$r(t) = r, w(t) = w, s(t) = s \tag{8}$$

The corresponding Hamilton-Jacobi-Bellman (HJB) equation using the maximum principle is

$$G_t + \alpha(\beta - r)G_r + \mu s G_s + \{[(\mu + d) - (r + \vartheta + \theta)]\pi + rw\}G_w + \lambda^2 s\pi G_{sw} + \frac{1}{2}[\sigma^2 G_{rr} + \lambda^2 s^2 G_{ss} + \lambda^2\pi^2 G_{ww}] = 0 \tag{9}$$

where the Brownian motions do not correlate, and

$$G_t + \alpha(\beta - r)G_r + \mu s G_s + \{[(\mu + d) - (r + \vartheta + \theta)]\pi + rw\}G_w + \lambda^2 s\pi G_{sw} + \rho\sigma s G_{rs} + \rho\sigma\lambda\pi G_{rw} + \frac{1}{2}[\sigma^2 G_{rr} + \lambda^2 s^2 G_{ss} + \lambda^2\pi^2 G_{ww}] = 0 \tag{10}$$

where the Brownian motions correlate with correlation co-efficient ρ. $G_t, G_s, G_w\ and\ G_r$, are first partial derivatives with respect to $t, s, w\ and\ r$ respectively. Also $G_{rs}, G_{rw}, G_{sw}, G_{rr} G_{ss}\ and\ G_{ww}$ are second partial derivatives, with the boundary condition that at the terminal time T,

$G(T,r,s,w)=u(w)$ (11)

The differentiation of (9) with respect to π gives

$[(\mu+d)-(r+\vartheta+\theta)]G_w+\lambda^2 sG_{sw}+\lambda^2\pi G_{ww}=0$ (12)

and the optimal strategy

$\pi^*_{d,\vartheta,\theta}=\frac{-[(\mu+d)-(r+\vartheta+\theta)]G_w}{\lambda^2 G_{ww}}-\frac{sG_{sw}}{G_{ww}}$, (13)

for the case where the Brownian motions do not correlate.

Differentiating (10) with respect to π gives

$[(\mu+d)-(r+\vartheta+\theta)]G_w+\lambda^2 G_{sw}+\rho\sigma\lambda G_{rw}+\lambda^2\pi G_{ww}=0$ (14)

and the optimal strategy

$\pi^*_{d,\vartheta,\theta}=\frac{-[(\mu+d)-(r+\vartheta+\theta)]G_w}{\lambda^2 G_{ww}}-\frac{sG_{sw}}{G_{ww}}-\frac{\rho\sigma\lambda G_{rw}}{\lambda^2 G_{ww}}$, (15)

for the case where the Brownian motions correlate.

3.2. The optimal strategy for the power utility

We consider two cases-when the Brownian motions do not correlate and when they correlate with correlation co-efficient ρ.

3.2.1. Case 1: When the Brownian motions do not correlate.

We consider power utility function described by (7).To eliminate the dependency on w, let the solution to the HJB equation (9) be

$G(t,r,s,w)=H(t,r,s)\frac{w^{1-\phi}}{1-\phi}$, (16a)

with boundary condition

$H(T,r,s)=1$, (16b)

then

$G_t=\frac{w^{1-\phi}}{1-\phi}H_t, G_r=\frac{w^{1-\phi}}{1-\phi}H_r, G_s=\frac{w^{1-\phi}}{1-\phi}H_s, G_{sw}=w^{-\phi}H_s, G_w=w^{-\phi}H_s.$ (16c)

Applying (17a) and (16c) to (13) gives

$\pi^*_{d,\vartheta,\theta}=\frac{[(\mu+d)-(r+\vartheta+\theta)]w}{\lambda^2}+\frac{s_w H_s}{G_{ww}}$. (17)

Applying (16a). (16c) and (17) to (9) and simplifying yields

$$H_t+\alpha(\beta-r)H_r+\mu sH_s+\left[(1-\phi)r+\frac{[(\mu+d)-(r+\sigma+\theta)]^2}{2\lambda^2}\right]H+\frac{\sigma^2}{2}H_{rr}+\frac{\lambda^2 s^2}{2}H_{ss}$$

$$-\frac{(1-\phi)\pi^2 s^2}{2\phi^2}\frac{{H_s}^2}{H}=0, \quad (18)$$

another second order partial differential equation.

To eliminate dependency on s, we further conjecture that

$H(t,r,s)=\frac{s^{1-\phi}}{1-\phi}I(t,r)$ (19a)

where

$I(T,r)=\frac{1-\phi}{s^{1-\phi}}$ (19b)

We obtain the following from (19a)

$H_t=\frac{s^{1-\phi}}{1-\phi}I_t, H_r=\frac{s^{1-\phi}}{1-\phi}I_r, H_s=s^{-\phi}I, H_{rr}=\frac{s^{1-\phi}}{1-\phi}I_{rr}$ and $H_{ss}=-\phi s^{-\phi-1}I$ (19c)

Applying (19a) and the equivalents of H_s from (19c) to (17) gives

$\pi^*_{d,\vartheta,\theta}=w\left[\frac{[(\mu+d)-(r+\vartheta+\theta)]}{\lambda^2}+\frac{1-\phi}{\phi}\right]$ (20)

Also the application of (19a) and (19c) to (18) yields, on simplification,

$I_t+\alpha(\beta-r)I_r+[(1-\phi)(\mu-\lambda^2)]I+\frac{\sigma^2}{2}I_{rr}=0$ (21)

Equation (21) is also a second order partial differential equation, so we conjecture that

$I(t,r)=\frac{r^{1-\phi}}{1-\phi}J(t)$ (22a)

to eliminate dependency on r such that at the terminal time T,

$I(T)=\frac{(1-\phi)^2}{(rs)^{1-\phi}}$ (22b)

From (22a) obtain,

$I_t=\frac{r^{1-\phi}}{1-\phi}\frac{dJ}{dt}, I_r=r^{-\phi}J$ and $I_{rr}=-\phi r^{-\phi-1}J.$ (22c)

The application of (22a) and (22c) on (21) gives

$\frac{r^{1-\phi}}{1-\phi}\frac{dJ}{dt}+\alpha(\beta-r)r^{-\phi}J+(1-\phi)(\mu-\lambda^2)\frac{r^{1-\phi}}{1-\phi}J+\frac{\sigma^2}{2}(-\phi)r^{-\phi-1}J=0,$ (23)

which simplifies to

$$\frac{dJ}{dt}+\left[\frac{(1-\phi)\propto(\beta-r)}{r}+(1-\phi)(\mu-\lambda^2)-\frac{\sigma^2}{2r^2}\phi(1-\phi)\right]J=0.$$

$$\frac{dJ}{dt}+(1-\phi)\left[\frac{2r\propto(\beta-r)+2r^2(\mu-\lambda^2-\sigma^2\phi}{2r^2}\right]J=0 \qquad (24)$$

Equation (24) becomes

$$\frac{dJ}{dt}+\zeta(t)J=0, \qquad (25a)$$

where

$$\zeta(t)=(1-\phi)\left[\frac{2r\propto(\beta-r)+2r^2(\mu-\lambda^2-\sigma^2\phi}{2r^2}\right]. \qquad (25b)$$

From equation (25a) we get

$$\frac{dJ}{J}=-\xi(t)dt, \qquad (26)$$

and on integration

$$J(t)=J(T)exp\left[\int_t^T\zeta(u)du\right]. \qquad (27)$$

Applying (22b) to (27) obtains

$$J(t)=\frac{(1-\phi)^2}{(rs)^{1-\phi}}exp\left[\int_t^T\zeta(u)du\right]. \qquad (28)$$

Therefore, the optimal value function for the investor`s problem is

$$G^*(t,r,s,w)=\frac{w^{1-\phi}}{1-\phi}exp\left[\int_t^T\zeta(u)du\right]. \qquad (29)$$

3.2.2. Case 2: When the Brownian motions correlate

Adopting (16a) and (16b), we obtain from (16a)

$$G_t=\frac{w^{1-\phi}}{1-\phi}H_t,\ G_w=w^{-\phi}H,\ G_{ww}=-\phi w^{-\phi-1}H,\ G_s=\frac{w^{1-\phi}}{1-\phi}H_s,\ G_{sw}=w^{-\phi}H_s, G_r=\frac{w^{1-\phi}}{1-\phi}H_r,\ G_{rr}=\frac{w^{1-\phi}}{1-\phi}H_{rr},\ G_{ss}=\frac{w^{1-\phi}}{1-\phi}H_{ss}\ and\ G_{rw}=w^{-\phi}H_r. \qquad (30)$$

Applying the equivalent of $G_w, G_{ww}, G_{sw}, and\ G_{rw}$ from equation (30) and (16a) to (15) gives

$$\pi^*_{d,\vartheta,\theta}=\frac{[(\mu+d)-(r+\vartheta+\theta)]w}{\lambda^2}+\frac{swH_s}{\phi H}+\frac{\rho\sigma wH_r}{\lambda\phi H}. \qquad (31)$$

Using (16a) , (30) and (31) in (10) gives

$$\frac{w^{1-\phi}}{1-\phi}H_t+\frac{w^{1-\phi}}{1-\phi}\alpha(\beta-r)H_r+\left[(\mu+d)-(r+\vartheta+\theta)\left[\frac{[(\mu+d)-(r+\vartheta+\theta)]w}{\lambda^2}+\frac{swH_s}{\phi H}+\frac{\rho\sigma wH_r}{\lambda\phi H}\right]+rw\right]w^{-\phi}H+ \lambda^2 sw\left[\frac{[(\mu+d)-(r+\vartheta+\theta)]}{\lambda^2}+\frac{sH_s}{\phi H}+\frac{\rho\sigma H_r}{\lambda\phi H}\right]w^{-\phi}H_s+\rho\sigma s\frac{w^{1-\phi}}{1-\phi}H_{ss}+\rho\sigma\lambda w\left[\frac{[(\mu+d)-(r+\vartheta+\theta)]}{\lambda^2}+\frac{sH_s}{\phi H}+\frac{\rho\sigma H_r}{\lambda\phi H}\right]w^{-\phi}H_r+ \frac{1}{2}\sigma^2\frac{w^{1-\phi}}{1-\phi}H_{rr}+\lambda^2 sH_{ss}+\pi^2w^2\left[\frac{[(\mu+d)-(r+\vartheta+\theta)]}{\lambda^2}+\frac{sH_s}{\phi H}+\frac{\rho\sigma H_r}{\lambda\phi H}\right]^2\left(-\phi w^{-\phi-1}H\right)\Big]=0 \qquad (32)$$

That simplifies to

$$H_t+\left[\alpha(\beta-r)+\frac{(1-\phi^2)\rho\sigma[(\mu+d)-(r+\vartheta+\theta)}{\lambda\phi}\right]H_r+\left[(1-\phi)+\frac{[(\mu+d)-(r+\vartheta+\theta)]^2}{2\lambda^2}\right]H+\frac{(1-\phi)\lambda^2s^2}{2\phi}\frac{{H_s}^2}{H}+\frac{(1-\phi)\lambda s\rho\sigma}{\phi}\frac{H_sH_r}{H}+ \left[\frac{(1-\phi)(2\rho\sigma-1)\rho\sigma}{2\phi}\right]\frac{{H_r}^2}{H}+\frac{\sigma^2}{2}H_{rr}+\left[\frac{\lambda^2 s}{2}+\rho\sigma s\right]H_{ss}=0. \qquad (33)$$

Using (19a)-(19c and applying the equivalent of $H_s, H_r\ and\ H$ to (31) gives

$$\pi^*_{d,\vartheta,\theta}=\left[\frac{[(\mu+d)-(r+\vartheta+\theta)]}{\lambda^2}+\frac{1-\phi}{\phi}+\frac{\rho\sigma}{\lambda\phi}\frac{I_r}{I}\right]. \qquad (34)$$

In equation (33) obtains

$$\frac{s^{1-\phi}}{1-\phi}I_t+\left[\alpha(\beta-r)+\frac{(1-\phi^2)\rho\sigma[(\mu+d)-(r+\vartheta+\theta)}{\lambda\phi}\right]\frac{s^{1-\phi}}{1-\phi}I_r+\left[(1-\phi)r+\frac{[(1-\phi)(\mu+d)-(r+\vartheta+\theta)]^2}{2\lambda^2}\right]\frac{s^{1-\phi}}{1-\phi}I+\frac{(1-\phi)\lambda^2s^2}{2\phi}s^{-\phi-1}I+ \frac{(1-\phi)\lambda s\rho\sigma}{\phi}s^{-\phi}I_r+\left[\frac{(1-\phi)(2\rho\sigma-1)\rho\sigma}{2\phi}\right]\frac{s^{1-\phi}{I_r}^2}{I}+\frac{\sigma^2}{2}\frac{s^{1-\phi}}{1-\phi}I_{rr}+\frac{s(\lambda^2+2\rho\sigma}{2}\left(-\phi s^{-\phi-1}I\right)=0, \qquad (35)$$

which simplifies to

$$I_t+\left[\alpha(\beta-r)+\frac{(1-\phi^2)\rho\sigma[(\mu+d)-(r+\vartheta+\theta)]}{\lambda\phi}+\frac{(1-\phi)^2\lambda\rho\sigma}{\phi}\right]I_r+\left[(1-\phi)r+\frac{[(1-\phi)(\mu+d)-(r+\vartheta+\theta)]^2}{2\lambda^2}+\frac{(1-\phi)^3\lambda^2}{2\phi}- \frac{(1-\phi)\phi(\lambda^2+2\rho\sigma}{2s}\right]I+\frac{(1-\varphi)^2(2\rho\sigma-1)\rho\sigma}{2\phi}\frac{{I_r}^2}{I}+\frac{\sigma^2}{2}+I_{rr}=0. \qquad (36)$$

Equation (36) is yet a second order partial differential equation, so we conjecture as in (22a)-(22).

Therefore equation (34) becomes

$$\pi^*_{d,\vartheta,\theta}=w\left[\frac{[(\mu+d)-(r+\vartheta+\theta)]}{\lambda^2}+\frac{1-\phi}{\phi}+\frac{(1-\phi)\rho\sigma}{\lambda\phi r}\right], \qquad (37)$$

the optimal investment in the risky asset.
Substituting for the values of $I_t, I_r, and\ I_{rr}$ in (36) using (22c) gives,

$$\frac{r^{1-\phi}}{1-\phi}\frac{dJ}{dt} + \left[\alpha(\beta - r) + \frac{(1-\phi^2)\rho\sigma[(\mu+d)-(r+\vartheta+\theta)]}{\lambda\phi} + \frac{(1-\phi)^2\lambda\rho\sigma}{\phi}\right] r^{-\phi}J + \left[(1-\phi)r + \frac{[(1-\phi)(\mu+d)-(r+\vartheta+\theta)]^2}{2\lambda^2} + \frac{(1-\phi)^3\lambda^2 s^2}{2\phi} - \frac{(1-\phi)\phi(\lambda^2+2\rho\sigma)}{2s}\right]\frac{r^{1-\phi}}{1-\phi}J + \frac{(1-\varphi)^2(2\rho\sigma-1)\rho\sigma}{2\phi}\frac{(1-\phi)}{r^{1-\phi}}J + \frac{\sigma^2}{2}(-\phi)r^{-\phi-1}J = 0. \quad (38)$$

Dividing equation (38) by $\frac{r^{1-\phi}}{1-\phi}$ yields

$$\frac{dJ}{dt} + \left[\frac{(1-\phi)}{r}\left[\alpha(\beta - r) + \frac{(1-\phi^2)\rho\sigma[(\mu+d)-(r+\vartheta+\theta)]}{\lambda\phi} + \frac{(1-\phi)^2\lambda\rho\sigma}{\phi} + (1-\phi)r + \frac{[(1-\phi)(\mu+d)-(r+\vartheta+\theta)]^2}{2\lambda^2} + \frac{(1-\phi)^3\lambda^2 s^2}{2\phi} - \frac{(1-\phi)\phi(\lambda^2+2\rho\sigma)\phi}{2s} + \frac{(1-\phi)^2}{r^{2(1-\phi)}}\frac{(1-\varphi)^2(2\rho\sigma-1)\rho\sigma}{2\phi} - \frac{\phi\sigma^2}{2}\frac{(1-\phi)}{r^2}\right]J = 0. \quad (39)$$

Equation (39) becomes

$$\frac{dJ}{dt} + k(t)J = 0, \quad (40a)$$

where

$$k(t) = \left[\frac{(1-\phi)}{r}\left[\alpha(\beta - r) + \frac{(1-\phi^2)\rho\sigma[(\mu+d)-(r+\vartheta+\theta)]}{\lambda\phi} + \frac{(1-\phi)^2\lambda\rho\sigma}{\phi} + (1-\phi)r + \frac{[(1-\phi)(\mu+d)-(r+\vartheta+\theta)]^2}{2\lambda^2} + \frac{(1-\phi)^3\lambda^2 s^2}{2\phi} - \frac{(1-\phi)\phi(\lambda^2+2\rho\sigma)\phi}{2s} + \frac{(1-\phi)^2}{r^{2(1-\phi)}}\frac{(1-\varphi)^2(2\rho\sigma-1)\rho\sigma}{2\phi} - \frac{\phi\sigma^2(1-\phi)}{r^2}\right]. \quad (40b)$$

Solving (40a) yields

$$J(t) = \frac{(1-\phi)^2}{(rs)^{1-\phi}}\ exp\left[\int_t^T k(u)du\right]. \quad (41)$$

Therefore the optimal value function to the investor`s problem when the Brownian motions correlate is given by

$$G^*(t, r, s, w) = \frac{w^{1-\phi}}{1-\phi} exp\left[\int_t^T k(u)du\right] \quad (42)$$

3.3. Comparison:

The optimal investment in the risky asset when the Brownian motions do not correlate is given by (20) and when the Brownian motions correlate by (37), we have

$$\pi^{*c}_{d,\vartheta,\theta} = \left[\frac{[(\mu+d)-(r+\vartheta+\theta)]}{\lambda^2} + \frac{1-\phi}{\phi} + \frac{(1-\phi)\rho\sigma}{\lambda\phi r}\right]w$$

$$= \left[\frac{[(\mu+d)-(r+\vartheta+\theta)]}{\lambda^2} + \frac{1-\phi}{\phi}\right]w + \frac{1-\phi\rho\sigma}{\lambda\phi r}w$$

$$\pi^{*c}_{d,\vartheta,\theta} = \pi^{*wc}_{d,\vartheta,\theta} + \frac{(1-\phi)\rho\sigma}{\lambda\phi r}w. \quad (43)$$

Notice from equation (43) that the optimal investment on the risky asset when the Brownian motions correlate is greater than or less than optimal investment on the risky asset when the Brownian motions do not correlate.

If ϕ is greater than one the optimal investment is the risky asset when Brownian motions correlate is less than that when the Brownian motions do not correlate by a fraction of the ratio of total amount for investment and the rate of return of the riskless asset.

If ϕ is less than one the optimal investment on the risky asset when the Brownian motions correlate is more than the optimal investment on the risky asset when Brownian motions do not correlate.

If $\phi = 1$, though not allowed, the optimal investment under both conditions are equal.

3.4. Findings

Equation (20) shows that in the case where the Brownian motions do not correlate if the sum of the drift parameter and dividend rate equals the sum of the tax rate, transaction cost rate and the rate of the return of the risk-free asset, then, the optimal investment strategy on the risky asset becomes totally dependent on the relative risk aversion coefficient $'\phi'$ and the total amount available for investment. Also, the investment strategy is horizon dependent as w, and r are horizon dependent.

Equation (37)

$$\pi^*_{d,\vartheta,\theta} = w\left[\frac{[(\mu+d)-(r+\vartheta+\theta)]}{\lambda^2} + \frac{1-\phi}{\phi}(1 + \frac{\rho\sigma}{\lambda r})\right],$$

the optimal investment in the risky asset, in the case where the Brownie motions correlate if the sum of the drift parameter and dividend rate equals the sum of the tax rate, transaction cost rate and the rate of the return of the risk-free asset, then, the optimal investment strategy on the risky asset becomes totally dependent on the relative risk aversion coefficient $'\phi'$ the total amount available for investment $'w'$, the correlation coefficient of the Brownian motions, $'\sigma'$ the constant volatility of the interest rate, $'\lambda'$ the diffusion parameter of the risky asset and rate of return of the risk free asset. The investment strategy is horizon dependent as w, and r are horizon dependent also.

4. Conclusions

In this work we investigated an investor's investment problem. It is assumed that the rate of return of the risk free asset is driven by Ornstein-Uhlenbeck Stochastic interest rate of return model.

The application of dynamic programming principles and the conjectures on elimination of variables obtained close-form solutions to the optimal investment strategies for the two cases considered (where the investor has a power utility preference and taxes, transaction costs and dividend payments are involve).
It was found that in the case where the Brownian motions do not correlate; if the sum of the drift parameter and dividend rate equal the sum of the tax rate, transaction cost rate and the rate of the return of the risk-free asset, then, the optimal investment strategy on the risky asset becomes totally dependent on the relative risk aversion coefficient $'\phi'$ and the total amount available for investment. While in the case where the Brownian motions correlate ; if the sum of the drift parameter and dividend rate equal the sum of the tax rate, transaction cost rate and the rate of the return of the risk-free asset, then, the optimal investment strategy on the risky asset becomes totally dependent on the relative risk aversion coefficient $'\phi'$ the total amount available for investment $'w'$, the correlation coefficient of the Brownian motions, $'\sigma'$ the constant volatility of the interest rate, $'\lambda'$ the diffusion parameter of the risky asset and rate of return of the risk free asset. In both cases the investment strategy is horizon dependent as w, and r are horizon dependent.

References

[1] R. C. Merton, "Lifetime portfolio selection under uncertainty: the continuous-time case," The Review of Economics and Statistics, vol. 51, no. 3, pp. 247–257, 1969.

[2] R. C. Merton, "Optimum consumption and portfolio rules in a continuous-time model," Journal of Economic Theory, vol. 3, no. 4, pp. 373–413, 1971.

[3] S. E. Shreve and H. M. Soner, "Optimal investment and consumption with transaction costs," The Annals of Applied Probability, vol. 4, no. 3, pp. 609–692, 1994.

[4] M. Akian, J. L. Menaldi, and A. Sulem, "On an investmentconsumption model with transaction costs," SIAM Journal on Control and Optimization, vol. 34, no. 1, pp. 329–364, 1996.

[5] K. Janeˇcek and S. E. Shreve, "Asymptotic analysis for optimal investment and consumption with transaction costs," Finance and Stochastics, vol. 8, no. 2, pp. 181–206, 2004.

[6] W. H. Fleming and T. Zariphopoulou, "An optimal investment/ consumption model with borrowing," Mathematics of Operations Research, vol. 16, no. 4, pp. 802–822, 1991.

[7] J.L. Vila and T. Zariphopoulou, "Optimal consumption and portfolio choice with borrowing constraints," Journal of Economic Theory, vol. 77, no. 2, pp. 402–431, 1997.

[8] S. A. Ihedioha, "Investor's Power Utility Optimization with Consumption, Tax, Dividend and Transaction Cost under Constant Elasticity of Variance Model" Asian Research Journal of Mathematics, 4(2): 1-12, 2017.

[9] R. Yao and H. H. Zhang, "Optimal consumption and portfolio choices with risky housing and borrowing constraints," The Review of Financial Studies, vol. 18, no. 1, pp. 197–239, 2005.

[10] J. C. Cox andS. A. Ross, "The valuation of options for alternative stochastic processes," Journal of Financial Economics, vol. 3, no. 1-2, pp. 145–166, 1976.

[11] M. Schroder, "Computing the constant elasticity of variance option pricing formula," Journal of Finance, vol. 44, no. 1, pp. 211–219, 1989.

[12] C. F. Lo, P. H. Yuen, and C. H. Hui, "Constant elasticity of variance option pricing model with time-dependent parameters," International Journal of Theoretical and Applied Finance, vol. 3, no. 4, pp. 661–674, 2000.

[13] P. B. Phelim and S. T. Yisong, "Pricing lookback and barrier options under the CEV process," Journal of Financial and Quantitative Analysis, vol. 34, no. 2, pp. 241–264, 1999.

[14] D. Davydov and V. Linetsky, "Pricing and hedging path dependent options under the CEV process," Management Science, vol. 47, no. 7, pp. 949–965, 2001.

[15] J. Xiao, Z.Hong, and C.Qin, "The constant elasticity of variance (CEV) model and the Legendre transform-dual solution for annuity contracts," Insurance: Mathematics and Economics, vol. 40, no. 2, pp. 302–310, 2007.

[16] J. Gao, "Optimal investment strategy for annuity contracts under the constant elasticity of variance (CEV) model," Insurance: Mathematics and Economics, vol. 45, no. 1, pp. 9–18, 2009.

[17] J. Gao, "An extended CEV model and the Legendre transform dual- asymptotic solutions for annuity contracts," Insurance: Mathematics and Economics, vol. 46, no. 3, pp. 511–530, 2010.

[18] M. Gu, Y. Yang, S. Li, and J. Zhang, "Constant elasticity of variance model for proportional reinsurance and investment strategies," Insurance: Mathematics and Economics, vol. 46, no. 3, pp. 580–587, 2010.

[19] X. Lin and Y. Li, "Optimal reinsurance and investment for a jump diffusion risk process under the CEV model," North American Actuarial Journal, vol. 15, no. 3, pp. 417–431, 2011.

[20] A. Gu, X. Guo, Z. Li, and Y. Zeng, "Optimal control of excess of loss reinsurance and investment for insurers under a CEV model," Insurance: Mathematics and Economics, vol. 51, no. 3, pp. 674–684, 2012.

[21] E. J. Jung and J. H. Kim, "Optimal investment strategies for the HARA utility under the constant elasticity of variance model," Insurance: Mathematics and Economics, vol. 51, no. 1, pp. 667– 673, 2012.

[22] H. Zhao and X. Rong, "Portfolio selection problem with multiple risky assets under the constant elasticity of variance model," Insurance: Mathematics and Economics, vol. 50, no. 1, pp. 179–190, 2012.

[23] J. Liu, "Portfolio selection in stochastic environments," The Review of Financial Studies, vol. 20, no. 1, pp. 1–39, 2007.