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On The Stability And Contraction Of Fixed Point Of The Solution Of Black-Scholes Equation In Hilbert Space

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Abstract

In this paper, we investigated the stability and contraction of fixed point of the solution of Black-Scholes equation in Hilbert space using the Lyapunov approach and method of integral equation. The Black-Scholes equation was reduced to Volterra integral equation of second kind and finally concluded that the solution is unique.

1.0 Introduction

Black-Scholes equation are frequently encountered in many fields of endeavor such as finance, financial engineering, option pricing theory, financial mathematics, economic, market analysis and stock exchange [1-5]. In general, the conceptual idea of the Black-Scholes equation of the form;

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 s^2\frac{\partial^2 V}{\partial s^2}= -rs\frac{\partial V}{\partial s}+rV, \quad (s,t)\in \mathbb{R}_+\times(0,T), \qquad (1.1)$$

(where σ^2 and r are constants), lies in the construction of a riskless portfolio taking positions in bonds (cash), option and the underlying stock [4] which may induce instability or poor performance of the price evolution of a European call. Therefore the stability problem for price option has attracted much attention during the past decades. In [6] Green function was used to study the Ulam-Hyers stability of Black-Scholes equation and concluded that the system is stable. See also [7-9].

Surveying in the literature, different methods aimed at solving Black-Scholes partial differential equation have appeared. The partial differential equations arising from the generalized option pricing model pose three challenges to the numerical approximation: the degeneracy of the equation, the coefficients being time and space-dependent and also unbounded in the space variables [10]. Also, many authors have worked on solution of Black-Scholes equation producing sound results [11-14], while little attention have been paid to contraction of fixed point of solution of Black-Scholes equation. Motivation by the above literature, the purpose of this paper is to investigate the stability and contraction of fixed point of the solution of homogenous Black-Scholes equation of the form

$$\frac{\partial V(s)}{\partial t}+\frac{1}{2}\sigma^2 s^2\frac{\partial^2 V(s)}{\partial s^2}+rs\frac{\partial V(s)}{\partial s}-rV(s)=0, \qquad (1.2)$$

where $V(s,t)$ is the price of an option, the independent variable s is the current option price of the stock, r is the annualized risk-free interest rate continuously compounded, t, the time in year generally use now $t=0$ at expiry $t=T$ and σ, volatility of an underlying asset. Equation (1.2) is of the parabolic form and can be considered as a diffusion equation. It provide quantitative information to continuously buy or sell assets to maintain a portfolio that grows at the riskless rate and thus provide insurance against downturns in the value of assets held long or protect against a rise in the value of assets held short. On the other hand, a quoted option price may be inconsistent with the value of the option as predicted by the Black-Scholes equation. In this case, it is possible to construct a portfolio which is guaranteed to outperform a riskless investment of the same magnitude. This possibility is called arbitrage.

In order to guarantee that the Black-Scholes equation has a unique solution one needs a boundary condition. With this condition imposed, the Black-Scholes equation is converted into an inhomogeneous equation of the form;

$$u(x)=f(x)+\lambda\int_a^x K(x,t)u(t)dt, \qquad (1.3)$$

where $u(x)$ is the unknown function, $K(x,t)$ is called the kernel or nucleus of the integral equation and λ is not an eigenvalue of the homogenous equation of the form;

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$$u(x) = \lambda \int_a^x K(x,t)u(t)dt, \quad (1.4)$$

which has at least one non-trivial solution corresponding to a particular value of λ. In this case, λ is an eigenvalue and the solution is an eigenfunction. If λ is an eigenvalue, the inhomogeneous equation has a solution if and only if

$$\int_a^x u(x)f(x)dx = 0 \quad (1.5)$$

for every function $f(x)$ [15].

2.0 Preliminaries

Definition 2.1: (Black-Scholes model) A mathematical formula design to price an option as a function of certain variables generally stock price, striking price, volatility, time to expiration dividends to be paid and the current risk-free interest rate for pricing European option on stocks [5].

Definition 2.2: (Stochastic differential equation) Let (Ω, F, P) be a probability space and let $X_t, t \in \mathbb{R}_+$ be stochastic process $X: \Omega \times \mathbb{R}_t \to \mathbb{R}$. Moreover, assume that $a(X_t, t): \Omega \times \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ and $b(X_t, t): \Omega \times \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ are stochastically integrable functions of $t \in \mathbb{R}$. Then the equation

$$\mathrm{d}X_t = a(X_t, t)dt + b(X_t, t)dW_t, \quad (2.1)$$

is called stochastic differential equation.

Note that equation (2.1) has to be understood as a symbolic notation of the stochastic integral equation

$$X_t = X_0 + \int_0^t a(X_s, s)ds + \int_0^t b(X_s, s)dW_s. \quad (2.2)$$

The function $a(X_t, t)$and $b(X_t, t)$ are referred to as the drift term and the diffusion term respectively.

Theorem 2.3 [15]: Let $f(x) \in L_2[0,1]$ and suppose that $K(x,y)$ is continuous for $x, y \in [0,1]$ and therefore uniformly bounded say $|K(x,y)| \leq M$. Then the equation

$$\emptyset(x) - \lambda \int_0^x K(x,y)\emptyset(y)\, dy = f(x). \quad (2.3)$$

has a unique solution $\phi(x)$ for all λ and$f(x)$ in $L_2[0,1]$.

Definition 2.4 [16]: Let (X, ρ) be a metric space and let f be a map. A point $x^* \in X$ is called a fixed point of f if $f(x^*) = x^*$. f is called a strict contraction if there exist a constant $k \in [0,1)$ such that $\rho(f(x), f(y)) \leq k\rho(x,y)$ for all $x, y \in X$.

Definition 2.5: A Hilbert space is a vector space H with an inner product $\langle f, g \rangle$ such that the norm defined by $|f| = \sqrt{\langle f, f \rangle}$ turns H into a complete metric space. If the metric defined by the norm is not complete then H is instead known as an inner product space.

Theorem 2.6 [17]: Let f and F be real-valued function defined on a closed interval $[a, b]$ such that F is continuous on all $[a, b]$ and derivative of F is f for almost all points in $[a, b]$. That is f and F are functions such that for all $x \in (a, b)$ except for perhaps a set of measure zero in the interval. If f is Riemann integrable on $[a, b]$ then

$$\int_a^b f(x)dx = F(b) - F(a).$$

Definition 2.7: (Stability) The equilibrium point $x = 0$ is stable if for each $\epsilon > 0$ there exist a $\delta > 0$ such that $\|x(0)\| < \delta$ implies that $\|x(t)\| < \epsilon$for $t \leq 0$.

Theorem 2.8: Let the origin $x = 0 \in D \subset \mathbb{R}^n$ be an equilibrium point for $\dot{x} = f(x)$. Let $V: D \to \mathbb{R}$ be a continuously differentiable function such that

(i) $V(0) = 0$

(ii) $V(x) > 0, \forall\, x \in D\backslash\{0\}$

(iii) $\dot{V}(x) \leq 0 \forall\, x \in D$

Then $x = 0$ is stable. Moreover if $\dot{V}(x) < 0, \forall\, x \in D\backslash\{0\}$ then $x = 0$ is asymptotically stable.

3.0 Main Result

We consider the homogenous Black-Scholes equation of the form in equation (1.2) with boundary conditions $V(0,t) = c_1$ and $V_s(z,t) = c_2$, which can be written as

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 V}{\partial s^2} + rs\frac{\partial V}{\partial s} = rV. \quad (3.1)$$

Converting equation (3.1) to integral equation by changing the dummy variable $s\ to\ y$ and integrating from $0\ to\ z$ gives

$$\int_0^z \left[\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 V}{\partial s^2} + rs\frac{\partial V}{\partial s}\right] dy = \int_0^z rV dy. \quad (3.2)$$

Integrating term by term gives

$\int_0^z V_t\,(y,t)dy = V(z,t) - c_1,$ (3.3)

$\frac{1}{2}\sigma^2 \int_0^z s^2 \frac{\partial^2 V}{\partial s^2} dy = \frac{\sigma^2 z^2 c_2}{2} - \sigma^2 z V(z,t) + \sigma^2 \int_0^z V(s,t)ds,$ (3.4)

and

$\int_0^z ryV_s(y,t)dy = rzV(z,t) + \int_0^z r\,V(s.t)ds.$ (3.5)

Substituting for equations (3.3), (3.4) and (3.5) in equation (3.2) yield

$V(z,t) - \sigma^2 zV(z,t) + rzV(z,t) - c_1 + \frac{\sigma^2 z^2 c_2}{2} + \sigma^2 \int_0^z V(s,t)ds + \int_0^z r\,V(s.t)ds = \int_0^z rVdy.$

Simplification of the above equation we have

$V(z,t)k_1 - c_1 + \frac{\sigma^2 z^2 c_2}{2} + (\sigma^2 + r)\int_0^z V(s,t)ds = \int_0^z rVdy,$ (3.6)

where $k_1 = 1 - \sigma^2 z + rz$.

Integrating equation (3.6) from 0 to $x$we have

$\int_0^x V(z,t)k_1 dt - \int_0^x c_1\,dt + \int_0^x \frac{\sigma^2 z^2 c_2}{2} dt + (\sigma^2 + r)\int_0^x \int_0^z V(s,t)dsdt = \int_0^x \int_0^z rVdsdt$

$\int_0^x V(z,t)dt = \frac{c_1 x}{k_1} - \frac{\sigma^2 z^2 c_2 x}{2k_1} + \frac{\sigma^2}{k_1}\int_0^x (x-z)V(s,t)ds$

$= f(z) + \alpha \int_0^x (x-z)V(s,t)ds,$ (3.7)

where $f(z) = \frac{c_1 x}{k_1} - \frac{\sigma^2 z^2 c_2 x}{2k_1}$ and $\alpha = \frac{\sigma^2}{k_1}$.

Applying theorem (2.6) to (3.7) we have

$V(z,t) - c_1 = f(z) + \alpha \int_0^x (x-z)V(s,t)ds,$

or

$V(z,t) = F(z) + \alpha \int_0^x K(x,z)V(s,t)ds,$

better still

$V(z,t) - \alpha \int_0^x K(x,z)V(s,t)ds = F(z)$ (3.8)

which is the Volterra equation of the first kind.

Theorem 4.1: Let $f(x) \in L_2[0,1]$ and suppose that $K(x,y)$ is such that

$\int_0^1 \int_0^1 |K(x,y)|^2 dxdy < \infty,$

then

$V(x) = f(x) + \lambda \int_0^x K(x,y)V(y)dy,$

has a unique solution for all $\lambda \in L_2[0,1]$.

Proof: Suppose that

$g_1^2(x) = \int_0^x |K(x,y)|^2 dyandg_2^2(y) = \int_y^1 |K(x,y)|^2 dx$

then

$g_1(x) = \left(\int_0^x |K(x,y)|^2 dy\right)^{\frac{1}{2}}$ and $g_2(y) = \left(\int_y^1 |K(x,y)|^2 dx\right)^{\frac{1}{2}}.$

We see that $g_1(x)$ and $g_2(x)$ are integrable. Let P be a number such that

$\int_0^1 g_1^2\,(x)dx \le P$ and $\int_0^1 g_2^2\,(y)dy \le P$.

Furthermore, we defined the function $r(x)$ by $r(x) = \int_0^x g_1^2\,(y)dy$. So that $r(1) \le P$.

Now, consider the series representation of integral equation

$V(x) = f(x) + \lambda Kf + \ldots + \lambda^{n-1}K^{n-1}f + \lambda^n K^n V$ (3.9)

where

$K^n V = \int_0^x K_n(x,y)V(y)dy.$

To estimate$\|K_n\|$, we have

$K_2(x,y) = \int_y^x K(x,z)K(z,y)dz.$ (3.10)

By Cauchy-Schwartz inequality

$|K_2(x,y)|^2 \le \int_y^x |K(x,z)|^2\,dz \int_y^x |K(z,y)|^2\,dz$

$\le g_1^2(x)g_2^2(y).$

Similarly,

$K_3(x,y) = \int_y^x K(x,z)K_2(z,y)dy.$

So that

$|K_3(x,y)|^2 \le \int_y^x |K(x,z)|^2\,dz \int_y^x |K(z,y)|^2\,dz$

$\le g_1^2(x) g_2^2(y) \int_y^x g_1^2(z)dz$

$= g_1^2(x) g_2^2(y)[\int_0^x g_1^2(z)dz - \int_0^y g_1^2(z)dz]$

$= g_1^2(x) g_2^2(y)[r(x) - r(y)].$ (3.11)

Using induction approach we have

$|K_n(x,y)|^2 \le g_1^2(x) g_2^2(y) \frac{[r(x)-r(y)]^{n-2}}{(n-2)!}, n \ge 2.$

Equation (3.9) can be rewritten in the form

$V = T^n V,$

where

$TV = f + \lambda KV$

and we can show that for large n, T^n is a contraction operator. That is

$|T^n V_1 - T^n V_1|^2 = \left|\int_0^x K_n(x,y)[V_1(y) - V_2(y)]dy\right|^2$ (3.12)

$\le \int_0^x |K_n(x,y)[V_1(y) - V_2(y)]|^2\,dy$

$\le \int_0^x \frac{g_1^2(x) g_2^2(y)[r(x)-r(y)]^{n-2} dy}{(n-2)!} \cdot \int_o^x |V_1(y) - V_2(y)|dy$

$\le \frac{g_1^2(x)[r(x)]^{n-2}}{(n-2)!} \int_0^1 g_2^2(y)dy \|V_1 - V_2\|^2.$

Hence

$\|T^n V_1 - T^n V_2\|^2 \le \frac{[r(1)]^{n-1}P}{(n-1)!} \|V_1 - V_2\|^2 \le \frac{P^n}{(n-1)!} \|V_1 - V_2\|^2.$ (3.13)

Therefore,

$\|T^n V_1 - T^n V_2\| \le \sqrt{\frac{P^n}{(n-1)!}} \|V_1 - V_2\|,$

so that T is a contraction operator if $\frac{P^n}{(n-1)!} < 1$. For large n, the Volterra equation and equation (3.10) will have a unique solution in $L_2[0,1]$.

4.0 Stability Analysis of Black-Scholes Equation

The equivalent system of equation (1.2) is

$\dot{v}_1 = v_2,$

$\dot{v}_2 = \frac{2rv_1}{\sigma^2 s^2} - \frac{2v_2(\alpha s+1)}{\sigma^2 s^2}.$

The energy function for the above system is $H =$ kinetic energy + potential energy

$H = \frac{1}{2}\dot{v}_1{}^2 + \int f(v_1)dv_1$

where

$f(v_1) = \frac{2rv_1}{\sigma^2 s^2}$

$v(v_1 v_2) = \frac{1}{2} v_2{}^2 + \frac{v_1{}^2 r}{\sigma^2 s^2}.$ (4.1)

To verify for stability by Lyapunov approach we test

$v(v_1 v_2) > 0,\ v(v_1 v_2) = 0 \text{as} v_1 = v_2 = 0$ and $\dot{v}(v_1 v_2) < 0.$

But

$v(v_1 v_2) > 0 \Longrightarrow \frac{1}{2} v_2{}^2 + \frac{v_1{}^2 r}{\sigma^2 s^2} > 0.$

$v(0,0) = 0$

$\dot{v}(v_1 v_2) = \frac{\partial v}{\partial v_1} . \frac{dv_1}{dt} + \frac{\partial v}{\partial v_2} . \frac{dv_2}{dt}$

$= \frac{2v_1 r}{\sigma^2 s^2} . v_2 + v_2 . (\frac{2rv_1}{\sigma^2 s^2} - \frac{2v_2(\alpha s+1)}{\sigma^2 s^2})$

$= \frac{2v_1 v_2 r}{\sigma^2 s^2} + \frac{2v_1 v_2 r}{\sigma^2 s^2} - \frac{2x_2^2(\alpha s+1)}{\sigma^2 s^2}$

$$= \frac{-2x_2(-2rx_1 + x_2(\alpha s + 1))}{\sigma^2 s^2} < 0. \quad (4.2)$$

Therefore $\dot{v}(v_1 v_2) < 0$. Hence the equilibrium point is stable.

Conclusion

From our result, stability and contraction of a solution of equation (1.2) in Hilbert space have been shown. Further investigation shows that the equilibrium point is stable while the conversion of Black-Scholes equation into Volterra equation of second kind yields equation (1.2) to a unique solution.

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