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Effect of energy dissipation on group velocity evolution for mechanics of sea waves.

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Abstract

This investigation concerns the effect of dispersion on the evolution of group velocity for the mechanics of water waves. The partial differential equation so derived in this consideration is identical to that governing the process of solitary waves. It, thus, consist of three distinct parts i.e. time evolution, nonlinearity and dispersion terms.
These equations were studied under various assumptions. In brief, it is established that in dispersive medium, the profile of group velocity is described by error function. Progressive wave solution is also applied and complete solution derived. In each case, this approach prevents the explosive nature usually associated with intersection of characteristics identified with the solution of quasi linear partial differential equation of kinematic type.

Keywords: Heat and mass transfer, mixed convection, perturbation method, convective boundaries, vertical channel.

1.0 Introduction

The concept of group velocity is very fundamental to the theory of wave groups [1].Its role in the energy focusing, involving the inter crossing of monochromatic wave groups is fundamental in the related analysis [2,3].
The concept of group velocity wasrelated to that of kinematic wave equation [4]. This follows the development that the characteristic wave frequency will go in different direction [5,6,7].
Equally important to mention is that group velocity concept models a number of important world problems. Some of the examples are traffic flow on highways[5,8], flood flow in a narrow river[9], waves in inclined channels and other similar singular phenomena.
Our approach in this study will be identical to a method adopted in studying solitary waves events.

Outline of solution

Consider a function q(x,t) related to $G_P(x,t)$ such that

$$\frac{\partial q}{\partial x} = \frac{\partial G_F}{\partial t} \tag{1}$$

$G_P(x,t) = V_g$ (group velocity)

Take a model for which q is proportional to $\frac{\partial G_F}{\partial x}$ in the form

$$q = Q(G_F) - \upsilon \frac{\partial . G_F}{\partial x} \tag{2}$$

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υ is invariable dispersion coefficient

$$\frac{\partial q}{\partial x} = Q'(G_P)\frac{\partial G_F}{\partial x} - \frac{\partial^2 G_F}{\partial x^2} \quad (3)$$

$$= G_F\frac{\partial G_F}{\partial x} - \upsilon\frac{\partial^2 G_F}{\partial x^2} \quad (4)$$

Where $G_F = \frac{\partial Q}{\partial G_F}$ and $Q = G^2{}_F/2 + R,\ R =$ constant

Using (1) and (4), we obtain

$$\frac{\partial G_F}{\partial t} + G_F\frac{\partial G_F}{\partial x} - v\frac{\partial^2 G_F}{\partial x^2} = 0 \quad (5)$$

(5) is an equation, which is essentially nonlinear. That G_F satisfies (5) is an unexpected result though interesting. The equation models a balance between three processes namely: time evolution locally, nonlinearity and dispersion. The equation is essentially hyperbolic (i.e. bounded eigen-function); but the linear form of it, is parabolic.ie $G_F\frac{\partial G_F}{\partial x}$ is ignored

1.1 The steady state solution

In this case, $\frac{\partial G_F}{\partial t} = 0$, thus

$$\frac{d}{dx}\left[\frac{G_F^2}{2} - \frac{v\partial G_F}{\partial x}\right] = 0 \quad (6)$$

Eqn (6) simplifies to

$$\frac{\partial G_F}{\partial x} = \frac{G_F^2}{2\upsilon} = A_0^2 \quad (7)$$

$$A_0 = U_1\ H(x) + U_2\ H(-x), \qquad u_1 > u_2 \quad (8)$$

H(x) = $\begin{matrix} 1, & x > 0 \\ 0, & x < 0 \end{matrix}$

H(-x) = $\begin{matrix} 1, & x < 0 \\ 0, & x > 0 \end{matrix}$

$H(x)$ is the Heaviside function

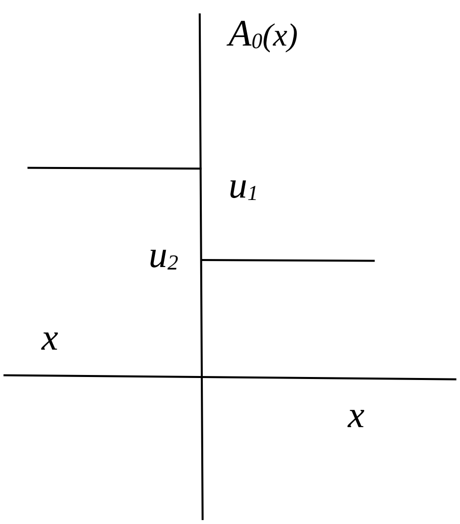

Fig 1. Profile of A_0 (x) in eqn (8)

(7) is nonlinear but has variable separable solution in the form

$$\ln\left[\frac{G_F - A_0}{G_F + A_0}\right] = \frac{R_o x}{\upsilon} \tag{9}$$

From (9), the solution applicable to the present analysis gives

$$G_F(x) = u_2 + (u_1 - u_2)\tanh\left(\frac{A_0 x}{\upsilon}\right) \tag{10}$$

$$= u_1\left[2 - \tanh\left(\frac{A_0 x}{\upsilon}\right)\right],\ u_2 = 2u_1 \tag{11}$$

Note that $u_1\, and\, u_2$ are given Cauchy initial datae, $G_F = u_2, 0 < x,$

$G(x,0) = u_1, x > 0$

That is, G(x,0)=u_1 H(x) + u_2 H(-x)

1.2 The linear cases

$$\frac{\partial G_F}{\partial t} = \upsilon\frac{\partial^2 G_F}{\partial x^2}, \tag{12}$$

$$G_F(x,0) = G(x) \tag{13}$$

(13) is also the initial specified data The Fourier transform to be employed is defined as

$$G(k,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G_F(x,t)\exp[-ikx]dx \tag{14}$$

With the inverse in the form

$$G_F(x,t) = \int_{-\infty}^{\infty} G(k,t)\exp[-ikx]dk \tag{15}$$

$$\overline{G}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G_F(x,0)\exp[-ikx]dx \tag{16}$$

$$-k^{2-}G(k,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[e^{+ikx}\frac{\partial G_F}{\partial x^2}(x,t)\right]dx \tag{17}$$

Applying (14) to (12) gives the ordinary differential equation of the form

$$\frac{d\overline{G}_F}{dt}(k,t) = -\nu k^2\,\overline{G}_F(k,t) \tag{18}$$

Using (18) and (Type equation here.16), we obtain the solution of (18) in the form

$$\overline{G}_F(k,t) = \overline{G}(k)\,\ell^{-\nu k^2 t} \tag{19}$$

From which (15) gives

$$G_F(x,t) = \int_{-\infty}^{\infty}\overline{G}(k)\,\ell_{dk}^{\left(ikx-\nu k^2 t\right)} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\xi)g(x-\xi)d\xi \tag{20}$$

By convulsion theorem for Fourier transform (20) is obtained.
We apply Fourier transform operator F defined by

$F\{g(x)\} = \int_{-\infty}^{\infty}\exp[\upsilon k^2 t]\, f(x)dx = \overline{G}(k)$, Where $f(x) = F^{-1}\{\overline{G}(k)\}$

Applying convolution theorem (eqn (20)) for Fourier transform, and noting the following,

$g(x) = F^{-1}\left\{\ell^{-\nu k^2 t}\right\} = \frac{1}{\sqrt{2}}\exp\left\{\frac{-x^2}{4\upsilon t}\right\}$, then

$$G_F(x,t) = \frac{1}{2\sqrt{\pi \upsilon t}} \int_{-\infty}^{\infty} \bar{G}(\xi) \exp\left[-\frac{(x-\xi)^2}{4\upsilon t}\right] d\xi$$

Take $r = \dfrac{\xi - x}{2\sqrt{\upsilon t}}$, $dr = \dfrac{d\xi}{2\sqrt{\upsilon t}}$, $\xi = x + 2r\sqrt{\upsilon t}$

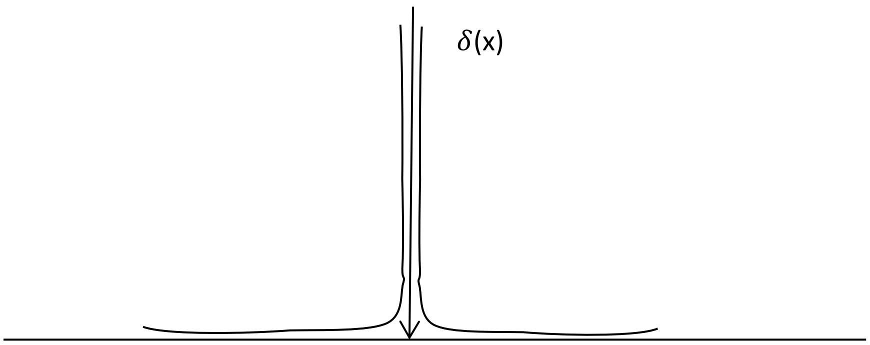

Fig 2. Delta function

Thus, $$G_F\,(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} G\left(x + 2r\sqrt{\upsilon t}\right) \ell^{-r^2} dr \qquad (21)$$

(a) $\bar{G}(x) = \delta(x)$; where $\delta(x)$ is delta function (fig.2)

$$\delta(x) = \begin{matrix} \infty, & x = 0 \\ 0, & x \neq 0 \end{matrix}$$

(b) $$G_F(x,t) = \frac{1}{2\sqrt{\pi \upsilon t}} \int_{-\infty}^{\infty} \delta(\xi) \exp\left[-\frac{(x-\xi)^2}{4\upsilon t}\right] d\xi$$

$$= \frac{1}{2\sqrt{\pi \upsilon t}} \exp\left[\frac{-x^2}{4\upsilon t}\right] \qquad (22)$$

$G(x) = T_0 H(x)$, $H(x)$ is Heaviside function, T_0 is a constant. (fig 1)

Let $r = \dfrac{\xi - x}{2\sqrt{vt}}$

$$G_F(x,t) = \frac{T_0}{\sqrt{\pi}} \int_{-\frac{x}{2\sqrt{\upsilon t}}}^{\infty} \ell^{-v^2}\, dr$$

$$= \frac{T_0}{2}\left[1 - erf\left(+\frac{x}{2\sqrt{\upsilon t}}\right)\right] = \frac{T_0}{2}\, erfc\left(\frac{x}{2\sqrt{\upsilon t}}\right) \qquad (23)$$

The solution coincides with initial data (b) (i.e t = 0)

Solution that corresponds to semi-infinite space

Instead of the case for which $-\infty < x < \infty,\ t > 0,$ what is considered here is that for which $0 < x < \infty,\ t > 0$

The initial condition gives $G_F\,(x,0) = H_0$

Instead as before, the half-space solution of eqn (12) is considered. Consequently the half range Fourier sine transform will applied. This is described by the relations: using sine-transform operator F_s, thus,

$$F_s\,\{f(x)\} = \int_0^{\infty} f(x) \sin kx\; dx = F(k) \qquad (24)$$

$$f(x) = \frac{2}{\pi} \int_0^{\infty} F(k) \sin kx\, dk \qquad (25)$$

Thus, if $F_s\ (f(x)) = f(k)$ (26)

and $F_s \left\{\frac{\partial^2 f}{\partial x^2}\right\} = \upsilon k\ \ f(0) - \upsilon k^2\ F(k)$ (27)

applying this to eqn (12) with $F_s\{G_F\ (x,t)\} = \overline{G}_F(k,t)$ and $G_F\ (x,0)=0,\ G_F\ (0,t)=u_0$

$$\frac{d}{dt} G_F\ (k,t) = \upsilon k u_0 - \upsilon k^2\ G_F\ (k,t)$$

i.e $\frac{d}{dt}\overline{G}_F\ (k,t) + \upsilon k^2\ \ \overline{G}_F\ \ (k,t) = \upsilon k\, u_0$ (28)

from (28), $\overline{G}_F\ \ (k,t) = \frac{u_0}{k}\ \left[1 - e^{-\nu k^2 t}\right]$ (29)

$$G_F\ (x,t) = F_s^{-1}\ \{\overline{G}_F\ (k,t)\} = \frac{2u_0}{\pi}\int_0^\infty \left[1 - e^{-\nu k^2 t}\right]\frac{Sin\, kx}{k}\, dk \quad (30)$$

But $\int_0^\infty \frac{Sinkx}{k}\, dk\ = \pi/2$

Thus, $G_F\ (x,t) = u_0\left[1 - 2/\pi\ \int_0^\infty \ell^{-\nu k^2 t}\ \ \frac{\sin kx}{k}\, dx\right]$

$$= u_0\ \ erfc\ \left[\frac{-x}{2\sqrt{\nu t}}\right] \quad (31)$$

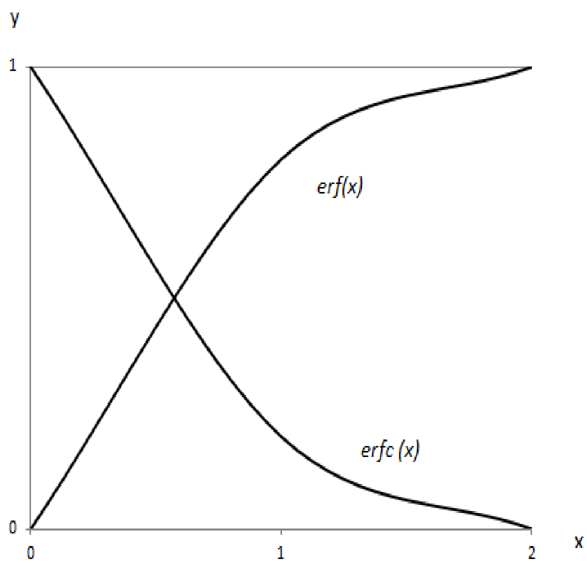

Fig 3. Error function

Equation (23) and (31) are identical. This is as one would expect.

The error function and its compliment are well tabulated in mathematical tables or can be deduced from table of normal probability distribution. The data required to provide numerical value of $G_F(x,t)$ is the values of dispersion coefficient υ. Note that $G_F(x,t)$ decreases with increasing υ. This result appears interesting in application

1.3 The case of finite space

$$\frac{\partial G_F}{\partial t} = \upsilon \frac{\partial^2 G}{\partial x^2}, \qquad 0 < x < L \qquad (32)$$

With $G_F(0,t) = h_1, G_F(L,t) = h_2,$ (32a)

$G_F(0,x) = h_3,$ (initial condition)

Let $G_F(x,t) = \phi(x,t) + H(x)$

Substitute in (32)

$$\upsilon\left(\frac{\partial^2 \phi(x,t)}{\partial x^2} + H(x)\right) = \frac{\partial \phi(x,t)}{\partial t}$$

For consistency, $H^{11}(x) = 0$

$H(x,0) = Ax + B$

$H(0) = h_1,\ H(L) = AL + B = AL + h_1 = h_2$

$A = \frac{h_2 - h_1}{L}$

$B = h_1$

$$H(x) = \frac{h_2 - h_1}{L}x + h_1 \qquad (32b)$$

$$\upsilon^2\left(\frac{\partial^2 \phi}{\partial x^2}\right) = \frac{\partial \phi}{\partial t} \qquad (33)$$

$\phi = \sum_n \phi_n = \sum_n X_n T_n \quad , X_n = X_n(x) \quad , T_n = T_n(t)$

Substitute in (33), which is linear,

$\upsilon^2 T_n\left(\frac{d^2 X_n}{d x^2}\right) = X_n \frac{dT_n}{dt}$ ie $\frac{1}{X_n}\frac{d^2 X_n}{d X^2} = \frac{1}{\alpha T_n}\frac{dT_n}{dt} = -k_n^2$

Thus,

$$\frac{d^2 X_n}{d x^2} + k_n^2 T_n = 0 \qquad (34)$$

$$\frac{dT_n}{dt} + \upsilon k_n^2 T_n = 0 \qquad (35)$$

(34) gives $X_n(n) = A_n \cos k_n x + B_n \sin k_n x$

(35) gives $T_n(t) = R_n \exp[-\alpha k_n^2 t]$ (35a, b)

From (32a), $X_n(0) = A_n = 0.$

$X_n(x) = B_n \sin k_n x \quad , \qquad X_n(L) = 0 = B_n \sin k_n L,$

$$B_n \neq 0,\ \sin k_n L = 0, k_n L = n\pi, n = 0,1,2,3, \qquad (36)$$

$$k_n = \frac{n\pi}{L} \qquad (37)$$

$$\phi_n(x,t) = \sum_1^{\infty} E_n \sin\left(\frac{n\pi x}{L}\right)\exp[-\alpha_n t \qquad (38)$$

Where $E_n = R_n B_n$

$$\alpha_n = \left(\frac{n\pi}{L}\right)^2, \quad \alpha_m = \upsilon\left(\frac{(2m+1)\pi}{L}\right), m = n = 0,1,2,... \tag{39}$$

$$\phi(x,t) = \sum_1^{\infty} E_n \sin\left(\frac{n\pi x}{L}\right)\exp[-\alpha_n t] \tag{40}$$

$t = 0$, thus

$$\sum_1^{\infty} E_n \sin\left(\frac{n\pi x}{L}\right) = (h_3 - h_2) - \left(\frac{h_2 - h_1}{L}\right)x$$

$$E_n = \frac{1}{L}\int_0^L \left[(h_3 = h_1) = \left(\frac{h_2 = h_1}{L}\right)x\right]\sin\frac{n\pi x}{L}dx \tag{41}$$

$$= \frac{1}{L}\left\{(h_3 = h_1)\int_0^L \sin\frac{n\pi x}{L}dx - \frac{h_2 = h_1}{L}\int_0^L x\sin\frac{n\pi x}{L}dx\right\}$$

$$= \left\{a\int_0^L \sin\frac{n\pi x}{L}dx - b\int_0^L x\sin\frac{n\pi x}{L}dx\right\}$$

$$= = \frac{aL}{n\pi}\left[\left(-1^n\right) - 1\right] + \frac{bL}{n\pi}\cos n\pi$$

$$= \frac{Lb\left[(-1)^{n+1}\right]}{n\pi} + \frac{2aL}{\pi(2m+1)}, \quad \text{n= 1,2, … ; L,m =0,1,2, …}$$

$$a = h_3 - h_2, \; b = \frac{h_2 - h_1}{L}$$

$$G_F(x,t) = H(x) + \phi(x,t)$$

$$= \frac{h_2 - h_1}{L}x + h_1 + \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{L}e^{-(\alpha_n t)} = h_3$$

h_1, h_2, h_3 , L are numerical assigned constants

$$G_F(x,t) = \frac{h_2 = h_1}{L}x + h_1 + \frac{b}{\pi}\sum_{n=1}^{\infty}((-1))^n \sin\frac{n\pi x}{L}D_n(t) + \frac{2a}{\pi}\sum_{m<0}^{\infty}\left(\sin\left[\left(\frac{2m+1}{L}\right)x\right]\right)D_m(t). \tag{42}$$

$$D_n(t) = \exp[-\alpha_n t], \; D_m(t) = \exp[-\alpha_m t]$$

1.4 The travelling wave solution

The method solves certain types of nonlinear differential equation governing wave processes (Okeke, 1997 ,2015) and (Okeke and Opeh (2016). Instead of introducing directly, the spatial and time variables, combination of the two are utilized. Thus, we define $\xi = x - ct$, $G_F(x,t) = G_F(\xi)$, where c is a typical phase speed. Thus,

$$d/dx = \frac{d}{d\xi}\frac{d\xi}{dx} = \frac{d}{d\xi}, \; \frac{d^2}{dx^2} = \frac{d^2}{d\xi^2}$$

$$d/dt = \frac{d}{d\xi}\frac{d\xi}{dt} = -c\frac{d}{d\xi}$$

Equation (3) stated as

$$\frac{\partial G_F}{\partial t} + G_F\frac{\partial G_F}{\partial x} - \upsilon\frac{\partial^2 G_F}{\partial x^2} = 0$$

Transform to

$$-C\frac{dG_F}{d\xi}+\frac{1}{2}\frac{d}{d\xi}G_F^2-\frac{d}{d\xi}\left(\upsilon\frac{dG_F}{d\xi}\right)$$

$$=\frac{d}{d\xi}\left[-CG_F+\frac{G_F^2}{2}+\upsilon\frac{dG_F}{d\xi}\right]=0$$

i.e $$\upsilon\frac{dG_F}{d\xi}+\frac{G_F^2}{2}-CG_F=A \qquad (43)$$

Assume that when $G_F(\infty)=u_1,\ \frac{dG_F}{d\xi}(\infty)=0$

$G_F(-\infty)=u_2,\ \frac{dG_F}{d\xi}(-\infty)=0$. Thus, u_1 and u_2 satisfy the quadratic equation

$$\frac{u_1^2}{2}-uu_1=A=0$$

$$\frac{u_2^2}{2}-uu_2-A=0$$

i.e $$u_1^2-2uu_1-2A=0 \qquad (44)$$

$$u_2^2-2uu_2-2A=0 \qquad (45)$$

From (44) and (45)

$$u=\frac{u_1+u_2}{2},\ A=\frac{u_1u_2}{2} \qquad (46)$$

From (43)

i.e $$2\upsilon\frac{dG_F}{d\xi}+\left[G_F^2-(u_1+u_2)G_F+u_1u_2\right]=0$$

$$v\frac{dG_F}{d\xi}+\frac{G_F^2}{2}-\frac{1}{2}(u_1+u_2)G_F+\frac{u_1u_2}{2}=0$$

and finally

$$2\upsilon\frac{dG_F}{d\xi}+\left(G_F-u_1\right)\left(G_F-u_2\right)=0 \qquad (47)$$

This is now variable separable equation

$$\frac{dG_F}{\left(G_F-u_1\right)\left(G_F-u_2\right)}=\frac{1}{u_1-u_2}\left[\frac{dG_F}{G_F-u_1}-\frac{dG_F}{G_F-u_2}\right]=\frac{-d\xi}{2\upsilon}$$

Integrating, $\frac{1}{u_1-u_2}\left[\ln\left(\frac{G_F-u_1}{G_F-u_2}\right)\right]=\frac{-\xi}{2\upsilon}$

$$\frac{G_F-u_1}{G_F-u_2}=\exp\left(\frac{-\varepsilon\left(u_1-u_2\right)}{2\upsilon}\right)=A \qquad (48)$$

Thus (48) provides that

$$G_F=\frac{u_1-Au_2}{1-A}=\frac{u_1-u_2\ \exp\left[-\xi/2\upsilon\ (u_1-u_2)\right]}{1-\exp\left[-\xi/2\upsilon\ (u_1-u_2)\right.}$$

And finally

$$G_F(\xi) = \frac{u_1 \exp\left[\frac{\xi}{4\upsilon}(u_1 - u_2)\right] - u_2 \exp\left[\frac{-\xi}{4\upsilon}(u_1 - u_2)\right]}{2\coth\left[\frac{\xi}{4\upsilon}(u_1 - u_2)\right]} \tag{49}$$

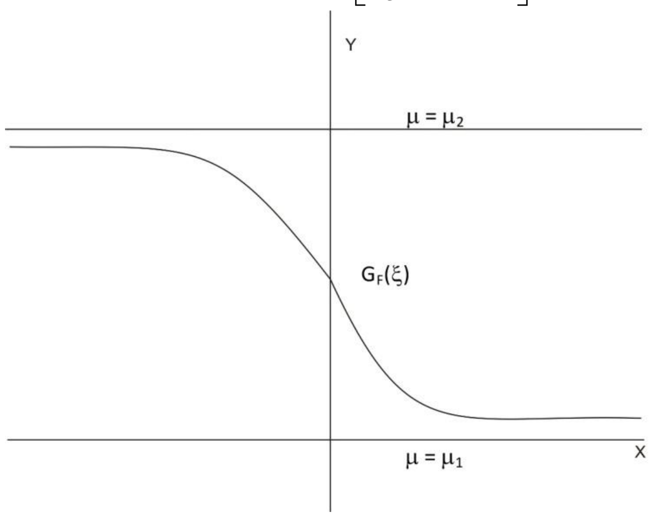

Fig 4: Travelling wave solution

Solution (49) is still valid even if $u_1 = u_2$. The constraint is that $u_1 > u_2$. The presence of dispersion coefficient υ is to prevent the solution from disintegrating, leading to breaking or other cartatrophic behaviour.

Conclusion

In view of the explosive behavior associated with quasi- linear differential equations earlier stated, this study established an identical equation but now with dispersive term. The resulting differential equation provides the balance between convention and diffusion forces leading to solution that are bounded in space and time.This equation is studied under two assumptions. Namely linear form and steady state form. In the linear form, the solution is provided for a range of values of x i.e (i) $-\infty < x < \infty$, (ii) $0 < x < \infty$, (iii) $0 < x < L$ (L=constant). The solutions appear to suggest that (i) and (ii) are completely the same, even though the mathematical details leading to the solutions are different. Each of course models the process described by a special function i.e error function for completeness, erf $(\theta) = \frac{1}{\sqrt{\pi}} \int_0^{\theta} \exp[-\eta^2] d\eta$) and its complement which is stated as erfc (θ)=1-erf (θ).

The function erf (θ) is tabulated in a number of mathematical table and has wide range of applications in marine physics e.g. solitary wave propagation. Error function models other process, which following this study now includes group velocity in a dispersive medium. This represents an important result from physical oceanographic point of view and is a credit to this investigation.

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References

[1] Lighthill, M. G (1965); Group velocity. *Inst. Math Appl.* **1:** 1 – 28.

[2] Estrada, R. (1994); Asymptotic analysis. Birkhause, Boston, U.S.A

[3] Matveen, V.B. (2002); Position slowly decreasing and analogue of solution. *Theoretical and mathematical Physics* **131(1):** 483-497.

[4] Kharif, C. (2001); Focusing of nonlinear wave group JETP Letters. **73(4):** 170 – 175.

[5] Whitham, G.B. (1974); Linear and non-linear waves, Wiley and sons, New York, London, Sydney, Toronto.

[6] Whitham, G.B. (2011): Linear and non-linear waves,onlinelibrarywiley.com.

[7] Johnson, R. S, (1997); A modern introduction to mathematical theory of Water waves, Cambridge University press (Cambridge, England).

[8] Greenberg, H. (1959); An Analysis of Traffic Flow. *Journal Operations Research Archieve.* **7(1):** 79-85.

[9] Rahman M. (2002); Water Waves. Clarendon Press. Oxford

[10] Okeke, E. O. (1997): The Effect of Higher Order Nonlinear term on the Shallow Water Finite Amplitude Wave, Bull. Di Geophy. Theory andApplication. 18, 247-254.

[11] Okeke, E. O. (2015); Further on non-linear Stochastic waves. Trans NAMP **1**: 151 -156.

[12] Okeke, E. O; and Opeh R (2016); On the non-linear Stochastic Family that Model water waves. *J. Nig. Assoc. Math. Physics***38:**31 -32.

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