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Laplace Transform-Chebyshev Collocation Method for Solving Third-Order Multi-Point Boundary Value Problems

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Abstract

In this paper, a new method for the approximate and exact solutions of third-order multi-point boundary value problems is presented. In this method, a new trial solution is constructed by the Laplace transform method which is later substituted into the original equation to obtain the residual equation. The new method is tested on linear and nonlinear multi-point problems and the computational results obtained are reported. The results obtained are also compared with solutions of other existing methods in the Mathematics literature to show the accuracy and computational efficiency of the method.

Keywords: Laplace transform method, collocation method, multi-point boundary value problems, linear and nonlinear problems

1.0 Introduction

In this study, we consider the following third-order multi-point boundary problem:

$$a_o(x)y'''(x)+a_1(x)y''(x)+a_2(x)y'(x)+a_3(x)y(x)=f(x) \tag{1.1}$$

subject to the boundary conditions:

$$y(\text{o})=\alpha_1 \tag{1.2}$$

$$y'(\text{o})=\alpha_2 \tag{1.3}$$

$$y'(1)=\alpha\, y'(\eta)+\lambda \tag{1.4}$$

where

$a_o(x),\ a_1(x),\ a_2(x),\ a_3(x)$ *and* $f(x)$ are continuous functions and $\alpha_1,\ \alpha_2,\ \alpha,\ \lambda$ are constants and $\eta\in(0,1)$ [1].

Multi-point boundary value problems play important roles in applied Mathematics, Physics, Engineering and other branches of Science. For instance, the modelling of physical problems involving vibrations occurring in a wire of uniform cross-section and composed of material with different densities, in the theory of elastic stability and fluid flow through porous media can be set up as multi-point boundary value problems.

Several authors have worked on different methods for the solution of multi-point boundary value problems. Therefore substantial amount of research work has been directed for the study of linear and nonlinear multi-point boundary value problems, such as the Chebyshev polynomial method [1], Reproducing kernel method [2-6], Adomian decomposition method [7], Shooting method [8,9], Pade approximations method [10], Homotopy perturbation method [11], Successive iteration method [12]. The authors [7] proposed a method for a class of second-order three-point boundary value problems, by converting the original problem into an equivalent integro-differential equation.

To the best of the authors' knowledge, the combination of Laplace transform and collocation methods has neither been implemented nor reported for either approximate or exact solution of boundary value problems.

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The rest of the paper is organized as follows: In the next section, we define Chebyshev polynomials and their properties. Section 3 presents the description of the Laplace transform- Chebyshev collocation method while section 4 is devoted to the implementation and applications of the proposed method. Finally, we give some concluding remarks in section 5.

2.0 Properties of Chebyshev Polynomial

Definition 2.1 [1, 13]: The Chebyshev polynomials (of the first kind) are defined by:

$$T_n(x) = \cos(\mathrm{n}\cos^{-1}(x)), \quad n = 0,1,2,...,N \tag{2.1}$$

The fundamental recurrence relation

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), \quad n = 2,3,... \tag{2.2}$$

together with the initial condition:

$$T_0(x) = 1, \quad T_1(x) = x \tag{2.3}$$

We may deduce from (2.2 and 2.3) that the first few Chebyshev polynomials are :

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_2(x) = 2x^2 - 1, \quad T_3(x) = 4x^3 - 3x \quad T_4(x) = 8x^4 - 8x^2 + 1$$

Definition 2.2 [1, 13]**:** The shifted Chebyshev polynomial $T_n^*(x)$ of the first kind on [0, 1] is a polynomial of degree **n** in x defined by

$$T_n^*(x) = T_n(2x-1) \tag{2.4}$$

Similarly, $T_n^*(x)$ satisfies the recurrence relation

$$T_n^*(x) = 2(2x-1)T^*_{n-1}(x) - T^*_{n-2}(x), \quad n \geq 2 \tag{2.5}$$

with the initial conditions

$$T_0^*(x) = 1, \quad T_1^*(x) = 2x - 1$$

Thus we have the following few polynomials:

$$T_0^*(x) = 1, \quad T_1^*(x) = 2x-1, \quad T_2^*(x) = 8x^2 - 8x + 1, \quad T_3^*(x) = 32x^3 - 48x^2 + 18x - 1, \quad \textit{see}\ [7]$$

3. Description of the Laplace Transform-Chebyshev Collocation Method

In this section, we will describe the numerical solution technique for the solution of third-order multi-point boundary value problem given in (1.1). Equation (1.1) can be rewritten as

$$y'''(x) + \beta_{1,0}(x)y''(x) + \beta_{2,0}(x)y'(x) + \beta_{3,0}(x)y(x) = \beta_{4,0}f(x) \tag{3.1}$$

where $\beta_{1,0}(x) = \dfrac{a_1(x)}{a_0(x)}, \quad \beta_{2,0}(x) = \dfrac{a_2(x)}{a_0(x)}, \quad \beta_{3,0}(x) = \dfrac{a_3(x)}{a_0(x)}, \quad \textit{and} \quad \beta_{4,0}(x) = \dfrac{1}{a_0(x)}$

Suppose x is the independent variable, applying Laplace transform to both sides of (3.1), we have

$$L[y'''(x)] + L[\beta_{1,0}(x)y''(x)] + L[\beta_{2,0}(x)y'(x)] + L[\beta_{3,0}(x)y(x)] = L[\beta_{4,0}(x)y(x)] \tag{3.2}$$

This implies

$$\begin{aligned} s^3\,\mathrm{Y}(s) - \mathrm{s}^2\mathrm{y}(0) - \mathrm{sy}'(0) - \mathrm{y}''(0) + \mathrm{L}[\beta_{1,0}(x)y''(x)] + \mathrm{L}[\beta_{2,0}(x)y'(x)] + \mathrm{L}[\beta_{3,0}(x)y(x)] \\ = \mathrm{L}[\beta_{4,0}(x)\,\mathrm{f}(x)] \end{aligned} \tag{3.3}$$

Replacing all the derivatives in (3.3) with new derivatives which are obtained through successive integration of third-order derivative to obtain other lower derivatives and the function y itself. Thus, the process of integration is as follows:

$$y'''(x) = \sum_{i=0}^{N} a_i T_i(x) \tag{3.4}$$

$$y''(x) = \sum_{i=0}^{N+1} \delta_{i,1}\phi_i^{[1]}(x) \tag{3.5}$$

$$y'(x) = \sum_{i=0}^{N+2} \delta_{i,2}\phi_i^{[2]}(x) \tag{3.6}$$

and

$$y(x) = \sum_{i=0}^{N+3} \delta_{i,3}\phi_i^{[3]}(x) \tag{3.7}$$

Substituting (3.4) - (3.7) into (3.3), we have

$$s^3\mathrm{Y}(s)-s^2(\sum_{i=0}^{N+3}\delta_{i,3}\phi_i^{[3]}(0))-s(\sum_{i=0}^{N+2}\delta_{i,2}\phi_i^{[2]}(0))-(\sum_{i=0}^{N+1}\delta_{i,1}\phi_i^{[1]}(0))+L(\beta_{1,0}(x)(\sum_{i=0}^{N+1}\delta_{i,1}\phi_i^{[1]}(x)))$$

$$+L(\beta_{2,0}(x)(\sum_{i=0}^{N+2}\delta_{i,2}\phi_i^{[2]}(x)))+L(\beta_{3,0}(x)(\sum_{i=0}^{N+3}\delta_{i,3}\phi_i^{[3]}(x)))=L(\beta_{4,0}(x)\,\mathrm{f}(x)) \quad (3.8)$$

Rearranging (3.8) and after simple algebraic simplification, we obtain

$$\mathrm{Y}(s)=\frac{1}{s^3}\left(\begin{array}{l}\sum_{i=0}^{N+1}\delta_{i,1}\phi_i^{[1]}(0)+s[\sum_{i=0}^{N+2}\delta_{i,2}\phi_i^{[2]}(0)]+s^2[\sum_{i=0}^{N+3}\delta_{i,3}\phi_i^{[3]}(0)]-L[\beta_{1,0}(x)(\sum_{i=0}^{N+1}\delta_{i,1}\phi_i^{[1]}(x))]\\ \qquad -L[\beta_{2,0}(x)(\sum_{i=0}^{N+2}\delta_{i,2}\phi_i^{[2]}(x))]-L[\beta_{3,0}(x)(\sum_{i=0}^{N+3}\delta_{i,3}\phi_i^{[3]}(x))]+L[\beta_{4,0}(x)(\mathrm{f}(x))]\end{array}\right) \quad (3.9)$$

Taking the Laplace transform of (3.9), we have

$$y_{LT}(x)=L^{-1}\left\{\frac{1}{s^3}\left(\begin{array}{l}\sum_{i=0}^{N+1}\delta_{i,1}\phi_i^{[1]}(0)+s[\sum_{i=0}^{N+2}\delta_{i,2}\phi_i^{[2]}(0)]+s^2[\sum_{i=0}^{N+3}\delta_{i,3}\phi_i^{[3]}(0)]-L[\beta_{1,0}(x)(\sum_{i=0}^{N+1}\delta_{i,1}\phi_i^{[1]}(x))]\\ \qquad -L[\beta_{2,0}(x)(\sum_{i=0}^{N+2}\delta_{i,2}\phi_i^{[2]}(x))]-L[\beta_{3,0}(x)(\sum_{i=0}^{N+3}\delta_{i,3}\phi_i^{[3]}(x))]+L[\beta_{4,0}(x)(\mathrm{f}(x))]\end{array}\right)\right\} \quad (3.10)$$

Thus (3.10) is a new trial solution obtained from (1.1). Substituting $y_{LT}(x)$ into (1.1), we have

$$a_0(x)\frac{d^3}{dx^3}(y_{LT}(x))+a_1(x)\frac{d^2}{dx^2}(y_{LT}(x))+a_2(x)\frac{d}{dx}(y_{LT}(x))+a_3(x)(y_{LT}(x))=\mathrm{f}(x) \quad (3.11)$$

We now collocate (3.11) at point x_j, we have

$$a_0(x_j)\frac{d^3}{dx_j^{\,3}}(y_{LT}(x_j))+a_1(x_j)\frac{d^2}{dx_j^{\,2}}(y_{LT}(x_j))+a_2(x_j)\frac{d}{dx_j}(y_{LT}(x_j))+a_3(x_j)(y_{LT}(x_j))=\mathrm{f}(x_j) \quad (3.12)$$

where

$$x_j=a+\frac{(b-a)j}{N+2}, \quad j=1,2,3,...,N+1$$

Thus, (3.12) gives (N+1) linear or nonlinear algebraic system of equations which can be solved by Gaussian elimination method or Newton's method, respectively to obtain the unknown coefficients. These coefficients are then substituted into (3.10) to obtain the required solution.

4.0 Computational Examples

In this section, we demonstrate the applicability of the LT-CCM; we have applied it to three multi-point boundary value problems of linear and nonlinear nature and computed the results for different values of N. All computations are carried out with Maple 14.

Problem 4.1: Consider the following variable coefficient non-homogeneous linear third-order boundary value problem [1,14]

$$y'''(x)+xy''(x)=-6x^2+3x-6, \quad 0\le x\le 1 \quad (4.1)$$

subject to the boundary conditions

$$y(0)=0, \quad y'(0)=1, \quad y'(1)=y'(\tfrac{1}{2})-\frac{3}{4} \quad (4.2)$$

The exact solution is $y(x)=\frac{3}{2}x^2-x^3+x$

We apply the proposed method with N=3 and we obtain the following new trial solution:

$$y_{LT}(x)=c_3+c_2x+\frac{1}{2}c_1x^2-x^3+(\frac{1}{8}-\frac{1}{24}c_1)x^4+(\frac{1}{60}a_1-\frac{1}{60}a_2+\frac{1}{60}a_3-\frac{1}{60}a_0-\frac{1}{10})x^5+$$

$$(\frac{1}{30}a_2-\frac{3}{40}a_3+\frac{1}{120}a_1)x^6+(\frac{8}{105}-\frac{4}{315}a_2)x^7-\frac{1}{42}a_3x^8 \quad (4.3)$$

By substituting (4.3) into (4.2) ,the resulting residual is given by

$$R(x,a_i,c_i)=\frac{d^3}{dx^3}(y_{LT}(x))+x\frac{d^2}{dx^2}(y_{LT}(x))+6x-3x+6 \quad (4.4)$$

Collocating (4.4) at the point $x_j = \frac{j}{5}, \quad j = 1,2,3,4$ and using the boundary conditions, we obtain six linear algebraic equations which are solved by Gaussian elimination method. Thus we obtain the following values:

$$a_0 = 12, \quad a_1 = 24, \quad a_2 = 6, \quad a_3 = 0, \quad c_1 = 3, \quad c_2 = 1, \quad c_3 = 0$$

Substituting these values into (4.3), we get $y(x) = x + \frac{3}{2}x^2 - x^3$

which is the exact solution.

Problem 4. 2: Consider the following linear third-order boundary value problem [1,4, 14,15,16]

$$y'''(x) - xy''(x) = (x^3 - 2x^2 - 5x - 3)e^x \tag{4.5}$$

Subject to the boundary conditions,

$$y(0) = 0, \quad y'(0) = 1, \quad y'(1) = -e \tag{4.6}$$

Proceeding as discussed in Problem 4.1 for case N=5, the following values are obtained:

$$a_0 = -11.69194521, \quad a_1 = -10.63991222, \quad a_2 = -2.209560048, \quad a_3 = -0.285447894,$$

$$a_4 = -0.02507610879, \quad a_5 = -0.001223355897, \quad c_1 = 4.454759897 \times 10^{-10}, \quad c_2 = 1, \quad c_3 = 0$$

Substituting these values into the trial solution, we obtain the following approximate solution:

$$y(x) = x + 2.227379948 \times 10^{-10} x^2 - \frac{1}{2}x^3 - \frac{1}{3}x^4 - \frac{1}{8}x^5 - 0.03333333333x^6 - 0.006944442859x^7$$
$$-0.001190493127x^8 - 0.0001735390721x^9 - 0.0002220297859x^{10} - 0.000002295236702x^{11}$$
$$-3.600549075 \times 10^{-7} x^{12}$$

Comparison of the approximate solution of Problem 4.2 with the results of methods in [1, 4] is given in Table 2, which shows that the method is quiet efficient and accurate.

Problem 4.3: Consider the following non-homogeneous nonlinear third-order boundary value problem

$$y'''(x) + (y''(x))^2 = \sin^2 x - \cos x, \quad 0 \le x \le 1 \tag{4.7}$$

Subject to the boundary conditions

$$y(0) = 0, \quad y'(0) = 1, \quad y'(1) = y'(\tfrac{1}{2}) + \cos(1) - \cos(\tfrac{1}{2}) \tag{4.8}$$

The proposed method described in section 3 is also tested on the problem. We employed two approaches to solve this problem, namely; linearization approach and non linearization approach. In the case of linearization approach, the nonlinear boundary value problem (4.7) is converted into a sequence of linear boundary value problems generated by quasilinearisation technique [17].

$$y'''_{k+1} + 2y_k'' y''_{k+1} = (y_k'')^2 + \sin^2 x - \cos x, \quad k = 0 \tag{4.9}$$

subject to

$$y_{k+1}(0) = 0, \quad y'_{k+1}(0) = 1, \quad y'_{k+1} = y'_{k+1}(\tfrac{1}{2}) + \cos(1) - \cos(\tfrac{1}{2}) \tag{4.10}$$

With the initial approximation $y_0(x) = x + (\cos(1) - \cos(\frac{1}{2}))x^2$, the computational results for this problem at third iteration (i.e. k=2) at some selected points are presented in Table 3 for case N=6. Also, the results obtained are compared with those obtained by using non linearization approach. In nonlinearization approach, we obtained nonlinear system of algebraic equations which are solved by Newton's method. It is observed from the table that linearization approach gives better results than non-linearization approach.

Table 1: Comparison of Computational results for Problem 4.1

X	Exact Solution	LT-CCM Solution	Absolute errors	Error of [4] n=100
0	0.0000	0.0000	0.0000	0.0000
0.1	1.4000 E-002	1.4000 E-002	0.0000	5.17 E-09
0.2	5.2000 E-002	5.2000 E-002	0.0000	4.13 E-08
0.3	1.0800 E-001	1.0800 E-001	0.0000	1.39 E-07
0.4	1.7600 E-001	1.7600 E-001	0.0000	3.32 E-07
0.5	2.5000 E-001	2.5000 E-001	0.0000	6.53 E-07
0.6	3.2400 E-001	3.2400 E-001	0.0000	1.13 E-06
0.7	3.9200 E-001	3.9200 E-001	0.0000	1.81 E-06
0.8	4.4800 E-001	4.4800 E-001	0.0000	2.73 E-06
0.9	4.8600 E-001	4.8600 E-001	0.0000	3.94 E-06
1.0	5.0000 E-001	5.0000 E-001	0.0000	5.48 E-06

Table 2: Comparison of Computational results for Problem 4.2

X	Exact Solution	LT-CCM Solution	Absolute errors n=5	Error of [1] n=8	Error of [4] n=20
0	0.000000000	0.000000000	0.0000	2.8834 E-018	0
0.1	0.09946538262	0.09946538264	2.00 E-011	9.9859 E-008	8.29 E-07
0.2	0.1954244413	0.1954244414	1.00 E-011	2.3056 E-007	-
0.3	0.2834703497	0.2834703497	0.0000	3.5655 E-007	1.63 E-07
0.4	0.3580379275	0.3580379275	0.0000	4.8630 E-007	4.88 E-07
0.5	0.4121803178	0.4121803178	0.0000	6.1654 E-007	4.62 E-07
0.6	0.4373085120	0.4373085121	1.00 E-010	7.4853 E-007	-
0.7	0.4228880685	0.4228880687	2.00 E-010	8.8360 E-007	8.12 E-07
0.8	0.3560865485	0.3560865488	3.00 E-010	1.0187 E-006	-
0.9	0.2213642800	0.2213642801	1.00 E-011	1.1639 E-006	6.60 E-07
1.0	0.000000000	4.1700×10^{-11}	4.17 E-010	1.2782 E-006	0

Table 3: Comparison of Computational results for Problem 4.3

X	Exact Solution	LT-CCM Solution		Errors	
		(Linearization approach)	(nonlinearization approach)	(linearization approach)	(non-linearization approach)
0	0.000000000	0.000000000	0.000000000	0.00	0.00
0.1	0.09983341665	0.0998334171	0.09983342334	4.60 E-010	6.690 E-009
0.2	0.1986693308	0.1986693329	0.1986693576	2.10 E-009	2.680 E-008
0.3	0.2955202067	0.2955202116	0.2955202675	4.90 E-009	6.080 E-008
0.4	0.3894183423	0.3894183512	0.3894184517	8.90 E-008	1.094 E-007
0.5	0.4794255386	0.4794255530	0.4794257121	1.44 E-008	1.735 E-007
0.6	0.5646424734	0.5646424947	0.56464249283	2.13 E-008	2.549 E-007
0.7	0.6442176872	0.6442177169	0.6442180414	2.97 E-008	3.542E-007
0.8	0.7173560909	0.7173561308	0.7173565627	3.99 E-008	4.718 E-007
0.9	0.7833269096	0.7833269620	0.7833275098	5.24 E-008	6.002 E-007
1.0	0.8414709848	0.8414710500	0.8414716921	6.52 E-008	7.070 E-007

5. Conclusion

In this paper, we have introduced and applied new method for solving third-order multi-point boundary value problems. The new method is based on the Laplace transform and Chebyshev collocation methods. It is clearly seen that LT-CCM is an efficient and powerful method for obtaining exact and approximate solutions of multi-point boundary value problems. The computational results show that the new method can solve the problems efficiently and the comparison also shows that the results obtained are in good agreement with exact solution and existing results in the Mathematics literature. The present method will be extended to the numerical solution of integro-differential equations in the future work.

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