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On the Construction of High Accuracy Symmetric Super-Implicit Hybrid Formulas with Phase-Lag Properties

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Abstract

Symmetric super-implicit (SSI) formulas for the numerical solution of the special second-order ordinary differential equations (ODEs) have been a focus of attention in the last decade. The solution of the class of ODEs exhibit oscillatory behaviour. P-stability is necessary for the numerical integration of such ODEs. In this paper, we construct families of hybrid SSI formulas employing future solution values while taking into consideration their phase-lag properties. The developed SSI hybrid formulas are thus P-stable with smaller phase-lag error constant than that of its local truncation error constant. The results from the numerical experiments shows that the new hybrid formulas are suitable for the integration of special second order ODEs.

KEYWORDS: Symmetric, Super-Implicit, Hybrid LMM, Phase-Lag, Periodicity, P-Stability
AMS Mathematics Subject Classification: 65L05, 65L06

1.0 Introduction

Ordinary differential equations (ODEs) arise in the modeling of some physical phenomena such as celestial mechanics, engineering, and lot more. Many of these models cannot be solved analytically. This is why accurate numerical solution is essential. The consideration is on the numerical integration of the initial value problem (IVP) of the special second order ODE of the form,

$$y''(x) = f\big(x, y(x)\big); y(x_0) = y_0, y'(x_0) = y_0, \tag{1}$$

where $y(x) \in \Re^t$, $f: \Re \times \Re^t \to \Re^t$, and the first derivative does not appear explicitly. The numerical methods for integrating (1) have been a focus of attention because such problems often arise in mechanics, theoretical physics, and chemistry amongst other areas of applications. The IVP (1) have also been considered in [1,2,3]. We define our linear multi-step method (LMM) for solving the second order IVP (1) as,

$$\sum_{j=0}^{k} \alpha_j\, y_{n+j} = h^2 \sum_{j=0}^{k} \beta_j f_{n+j},\ \ \beta_k \neq 0,\ \ \alpha_k = 1. \tag{2}$$

The first and second characteristic polynomials are,

$$\rho(z) = \sum_{j=0}^{k} \alpha_j\, z^j, \quad \sigma(z) = \sum_{j=0}^{k} \beta_j z^j. \tag{3}$$

The LMM (2) has an associated local truncation error (LTE) difference operator

$$L[y(x); h] = \sum_{j=0}^{k} \alpha_j y(x + jh) - h^2 \sum_{j=0}^{k} \beta_j y''^{(x+jh)} = C_{p+2} h^{p+2} y^{(p+2)}(x_n) + O\big(h^{(p+3)}\big). \tag{4}$$

The LTE is $C_{p+2} h^{p+2} y^{(p+2)}(x_n)$ at the point x_n, p is the order of the method, and C_{p+2} is the error constant given by,

$$C_q = \frac{1}{q!} \sum_{j=0}^{k} j^{q+2}\big(j^2\alpha_j - q(q-1)\beta_j\big) - \sum_{k+1=0}^{k} \frac{j^{q-2}}{(q-2)!}\beta_j, \qquad q > 2. \tag{5}$$

An important subclass of the LMM (2) is the Stömer-Cowell formula, which the general form is given by,

$$\rho(E) = E^{k-2}(E^2 - 2E + 1). \tag{6}$$

The Stömer-Cowell formula with step number greater than two are known to suffer orbital instability [4]. For a test problem that describes uniform motion in a circular orbit, the numerical solutions generated by such method spiral inwards for all values of the step length *h*. The modification of the Stömer-Cowell method for the integration of orbits was suggested in [5]. This approach was to remove truncation errors and thus avoided the instability of the Stömer-Cowell schemes. Various modifications of the Stömer-Cowell formulas have been proposed to bypass this deficiency see [6 – 10]. An alternative strategy to eliminate the orbital instability that is inherent in the Stömer-Cowell formulas was proposed in [4]. The adoption of symmetric linear multi-step methods was suggested with,

$$\rho(E) = (E-1)^2(E-a)(E-b), \quad |a|,\ |b| < 1. \tag{7}$$

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Adding the condition of symmetry, we find that $a = i$ and $b = -i$. This type of polynomial was also considered in [11] recently. The method (2) is assumed to satisfy the following conditions,

1. $\alpha_k = 1, |\alpha_0| + |\beta_0| \neq 0$ ($\alpha_j,\ \beta_j$ are real for $j = 0(1)k$),
2. $(\rho,\ \sigma) = 1$ (ρ and σ are relatively prime),
3. $\rho(1) = \rho'(1) = 0,\ \rho''(1) = 2\sigma(1)$ (consistency),
4. zero-stable.

The LMM (2) is symmetric if $\alpha_j = \alpha_{k-j}$ and $\beta_j = \beta_{k-j}$ for $j = 0(1)k$. The stability of the periodicity properties of the LMM (2) can be examined by the application to the scalar test problem,

$$y'' + \omega^2 y = 0, \omega, y \in \Re, \tag{8}$$

which results in the stability polynomial properties equation,

$$\Pi(r, H^2) = \rho(r) + H^2\sigma(r) = 0, \quad H = \omega h. \tag{9}$$

The stability polynomial (9) is expected to satisfy the hypotheses in [8].

Definition 1 *[4]: The Method (2) is said to have an interval of periodicity* $(0, H_0^2)$, *if for all* H^2 *in this interval, the roots* r_i *of the stability polynomial (9) satisfy* $r_1 = e^{i\theta(H)},\ r_2 = e^{-i\theta(H)},\ |r_i| \leq 1,\ \ i = 3(1)k$ *for real* $\theta(H)$.

Definition 2 *[4]: The LMM (2) is said to be P-stable, if its interval of periodicity is (0,∞).*

The barrier theorem as regards the attainable order of P-stable LMM (2) have been independently established in [12,4]. The order barrier is the equivalent of that in [13] for the first order ODEs. To be precise, we state the result for the second order ODEs as in [2].

Theorem 3 *[2]: The LMM (2) is P-stable/unconditionally stable if*

1. It is implicit, $\beta_k \neq 0$

2. It is, at best of order $p = 2$.

The conditions in theorem (3) have been proved to be sufficient in [14]. The most accurate P-stable LMM (2) is given by,

$$y_{n+2} - 2y_{n+1} + y_n = \tfrac{1}{4}h^2(f_{n+2} + 2f_{n+1} + f_n), \tag{10}$$

which the equivalent one-leg method was derived in [17],

$$y_{n+2} - 2y_{n+1} + y_n = h^2\left(x_{n+1}, \tfrac{1}{4}f_{n+2} + \tfrac{1}{4}f_{n+1} + \tfrac{1}{4}f_n\right). \tag{11}$$

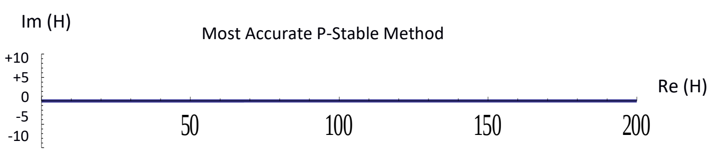

Figure 1: Stability plot of P-stable formula (10) and (11)

The method (10) and (11) have order $p = 2$, error constant $C_{p+2} = -\frac{1}{6}$, the interval of periodicity is (0, ∞), and the stability plot is given in Figure 1. The graph grows indefinitely in the positive direction on the real axis for an increasing H^2 which supports the P-stability definition 2. The introduction of hybrid two-step methods in [6] was to overcome the order barrier imposed on LMM (2). This method is thus P-stable as the stability plot is as in Figure 1.

$$y_{n+2} - 2y_{n+1} + y_n = h^2\left(\frac{-11}{360}(f_{n+2} + f_n) + \frac{3}{20}f_{n+1} + \frac{41}{90}\left(f_{n+\frac{3}{2}} + f_{n+\frac{1}{2}}\right)\right). \tag{12}$$

The order is $p = 4$, $C_{p+2} = \frac{17}{5760}$, with hybrid pair,

$$y_{n+\frac{3}{2}} = \frac{1}{4}y_{n+2} + y_{n+1} - \frac{1}{4}y_n + h^2\left(\frac{-3}{48}f_{n+2} + \frac{9}{48}f_{n+1}\right) \tag{13a}$$

and

$$y_{n+\frac{1}{2}} = \frac{1}{2}y_{n+1} + \frac{1}{2}y_n + h^2\left(\frac{9}{192}f_{n+2} - \frac{15}{96}f_{n+1} - \frac{1}{64}f_n\right). \tag{13b}$$

The use of Padé approximation to obtain P-stable one-leg formulas which is off great advantage in terms of function evaluation was suggested in [8,9,12]. Since then, there have been some interesting advances in the construction of new LMMs see for example the excellent work in [15,16,18]. P-stable super-implicit methods which are extension of the work in [19] was suggested in [20]. Examples of the methods derived in [20] are,

$$y_{n+2} - 2y_{n+1} + 2y_n - 2y_{n-1} + y_{n-2} = h^2\left(\frac{7411}{72576}f_n + \frac{362771}{453600}(f_{n+1} + f_{n-1}) + \frac{47057}{453600}(f_{n+2} + f_{n-2}) - \frac{2707}{453600}(f_{n+3} + f_{n-3}) + \frac{641}{1814400}(f_{n+4} + f_{n-4})\right),$$

$$p = 10,\ C_{p+2} = \frac{-4139}{79833600}, \tag{14}$$

$$y_{n+1} - 2y_n + y_{n-1} = h^2\left(\frac{57517}{72576}f_n + \frac{101741}{907200}(f_{n+1} + f_{n-1}) - \frac{8593}{907200}(f_{n+2} + f_{n-2}) + \frac{149}{129600}(f_{n+3} + f_{n-3}) - \frac{289}{3628800}(f_{n+4} + f_{n-4})\right),$$

$$p = 10,\ C_{p+2} = \frac{-317}{22809600}. \tag{15}$$

The methods (14,15) and like the hybrid methods to be proposed in this paper require additional formulas to predict the additional starting and future values.

2.0 P-Stable High Order Super-Implicit Hybrid LMM

In this section, we are particularly interested in the hybrid SSI formula of the form

$$\sum_{j=0}^{\frac{k}{2}} \alpha_j(y_{n+j}+y_{n-j}) = h^2 \sum_{j=0}^{\frac{k^*}{2}} \beta_j(f_{n+j}+f_{n-j}) + h^2\phi(f_{n+\lambda}+f_{n-\lambda}). \quad (16)$$

The method (16) is explicit for $k^* = k - 1$, implicit for $k^* = k$, and super-implicit for $k^* > k$ with $\lambda \in [0, 1]$ as in [6]. Here for $k = 4$, α_j are fixed, say $\alpha_2 = 1 = \alpha_0$, $\alpha_2 = -2$ to ensure symmetry and zero-stability conditions. The α_j are arbitrarily chosen. This was also considered in [7,11,20]. The constants $\beta_j = 0(1)\frac{k^*}{2}$ and ϕ are then determined. The formula (16) approximates the hybrid quantities $y_{n\pm\lambda}$ by an expression involving the quantities $\{(y_{n\pm j}; f_{n\pm j})\}$.

$$y_{n+\lambda} = \sum_{j=0}^{\frac{k}{2}} \psi_j(y_{n+j}+y_{n-j}) + h^2 \sum_{j=0}^{\frac{k^*}{2}} \gamma_j(f_{n+j}+f_{n-j}), \quad (17)$$

$$y_{n-\lambda} = \sum_{j=0}^{\frac{k}{2}} a_j(y_{n+j}+y_{n-j}) + h^2 \sum_{j=0}^{\frac{k^*}{2}} b_j(f_{n+j}+f_{n-j}). \quad (18)$$

Where, k^*, k are even and $\lambda \in [0, 1]$. Constants $\psi_j, \gamma_j, b_j, a_j$ are then determined. When $k = 4$ and $k^* = 6$, we have,

$$y_{n+2} - 2y_{n+1} + 2y_n - 2y_{n-1} + y_{n-2} = h^2(\beta_0 f_n + \beta_1(f_{n+1} + f_{n-1}) + \beta_2(f_{n+2} + f_{n-2}) + \beta_3(f_{n+3} + f_{n-3}) + \phi(f_{n+\lambda} + f_{n-\lambda})). \quad (19)$$

We chose our hybrid parameter $\lambda = \frac{1}{2}$ to ensure P-stability similar to the methods in [6]. Using MATHEMATICA v10 [21], we obtained the following values,

$$\emptyset = \frac{-82048}{70875}, \beta_0 = \frac{39967}{45360}, \beta_1 = \frac{22049}{18144}, \beta_2 = \frac{70529}{1134000}, \beta_3 = \frac{-1997}{2268000},$$

for method (19) with error constant $\frac{7967}{798336000}$ and order $p = 10$. Similarly, for the hybrid

$$y_{n+\lambda} = \psi_0 y_{n+2} + \psi_1 y_{n+1} + \psi_2 y_n + \psi_3 y_{n-1} + \psi_4 y_{n-2} + h^2(\gamma_0 f_n + \gamma_1 f_{n+1} + \gamma_2 f_{n+2} + \gamma_3 f_{n+3}), \quad (20)$$

we obtain the coefficients,

$\psi_0 = \frac{2583745}{3448832}, \psi_1 = \frac{-175635}{215552}, \psi_2 = \frac{1511055}{1724416}, \psi_3 = \frac{42085}{215552}, \psi_4 = \frac{-20223}{3448832}, \gamma_0 = \frac{-253989}{862208}, \gamma_1 = \frac{-299619}{431104}, \gamma_2 = \frac{-13833}{215552}, \gamma_3 = \frac{837}{431104}$ with LTE $= \frac{-132547y^{(9)}(x)h^9}{965672960}$ and for

$$y_{n-\lambda} = a_0 y_{n+2} + a_1 y_{n+1} + a_2 y_n + a_3 y_{n-1} + a_4 y_{n-2} + h^2(b_0 f_n + b_1 f_{n-1} + b_2 f_{n-2} + b_3 f_{n-3}), \quad (21)$$

we obtain the following coefficients using MATHEMATICA v10 [21],

$a_0 = \frac{-20223}{3448832}, a_1 = \frac{42085}{215552}, a_2 = \frac{1511055}{1724416}, a_3 = \frac{-175635}{215552}, a_4 = \frac{2583745}{3448832}, b_0 = \frac{-253989}{862208}, b_1 = \frac{-299619}{431104}, b_2 = \frac{-13833}{215552}, b_3 = \frac{837}{431104}$ with LTE = $\frac{132547y^{(9)}(x)h^9}{965672960}$.

In a similar manner, for k = 4 and $k^* = 8$, we obtain the order p =12 method,

$$y_{n+2} - 2y_{n+1} + 2y_n - 2y_{n-1} + y_{n-2} = h^2(\beta_0 f_n + \beta_1(f_{n+1} + f_{n-1}) + \beta_2(f_{n+2} + f_{n-2}) + \beta_3(f_{n+3} + f_{n-3}) + \beta_4(f_{n+4} + f_{n-4}) + \phi(f_{n+\lambda} + f_{n-\lambda})), \quad (22)$$

with coefficients,

$\emptyset = \frac{-1059584}{1091475}, \beta_0 = \frac{603035}{798336}, \beta_1 = \frac{5728861}{4989600}, \beta_2 = \frac{343789}{4989600}, \beta_3 = \frac{-11887}{6985440}, \beta_4 = \frac{7967}{139708800}$ and error constant $\frac{-5367083}{5230697472000}$. Also, we obtain the coefficients of the hybrid

$$y_{n+\lambda} = \psi_0 y_{n+2} + \psi_1 y_{n+1} + \psi_2 y_n + \psi_3 y_{n-1} + \psi_4 y_{n-2} + h^2(\gamma_0 f_n + \gamma_1 f_{n+1} + \gamma_2 f_{n+2} + \gamma_3 f_{n+3} + \gamma_4 f_{n+4}), \quad (23)$$

as,

$\psi_0 = \frac{1641212183}{514490368}, \psi_1 = \frac{-87829569}{16077824}, \psi_2 = \frac{731654937}{257245184}, \psi_3 = \frac{7074877}{16077824}, \psi_4 = \frac{-5881545}{514490368}, \gamma_0 = \frac{-1924995069}{2572451840}, \gamma_1 = \frac{-955773}{358880}, \gamma_2 = \frac{-60322041}{183746560}, \gamma_3 = \frac{1646523}{80389120}, \gamma_4 = \frac{-3578769}{2572451840}$ with LTE $\frac{131972451y^{(9)}[x]h^9}{720286515200}$ and

$$y_{n-\lambda} = a_0 y_{n+2} + a_1 y_{n+1} + a_2 y_n + a_3 y_{n-1} + a_4 y_{n-2} + h^2(b_0 f_n + b_1 f_{n-1} + b_2 f_{n-2} + b_3 f_{n-3} + \gamma_4 f_{n-4}), \quad (24)$$

$a_0 = \frac{-5881545}{514490368}, a_1 = \frac{7074877}{16077824}, a_2 = \frac{731654937}{257245184}, a_3 = \frac{-87829569}{16077824}, a_4 = \frac{1641212183}{514490368}, b = \frac{-1924995069}{2572451840}, b_1 = \frac{-955773}{358880}, b_2 = \frac{-60322041}{183746560}, b_3 = \frac{1646523}{80389120}, b_4 = \frac{-3578769}{2572451840}$ with LTE $-\frac{131972451y^{(9)}[x]h^9}{720286515200}$.

3.0 P-Stable High Order Störmer-Cowell Type Super-Implicit Hybrid LMM

This section present a particular case of (2) called the Störmer-Cowell type LMM,

$$\sum_{j=0}^{\frac{k}{2}} \alpha_j(y_{n+j}+y_{n-j}) = h^2 \sum_{j=0}^{\frac{k^*}{2}} \beta_j(f_{n+j}+f_{n-j}) + h^2\phi(f_{n+\lambda}+f_{n-\lambda}). \quad (25)$$

However, the hybrids of interest are,

$y_{n+\lambda} = \sum_{j=0}^{k} \psi_j \, y_{n-1+j} + h^2 \sum_{j=0}^{k^*} \gamma_j f_{n+j},$ (26)

$y_{n+\lambda} = \sum_{j=0}^{k} a_j \, y_{n-1+j} + h^2 \sum_{j=0}^{k^*} b_j f_{n-j},$ (27)

where, k and the super-implicit parameter $k^*$are even, and λ ∊ [0, 1] as in [6]. When k = 4 and k^*= 6, we have,

$y_{n+1} - 2y_n + y_{n-1} = h^2\big(\beta_0 f_n + \beta_1(f_{n+1} + f_{n-1}) + \beta_2(f_{n+2} + f_{n-2}) + \beta_3(f_{n+3} + f_{n-3}) + \phi(f_{n+\lambda} + f_{n-\lambda})\big).$ (28)

We choose our hybrid parameter as $\lambda = \frac{1}{2}$ to ensure P-stability. Using MATHEMATICA v10 [21], we obtained the following values for method (28),

$\emptyset = \frac{18496}{70875}, \beta_0 = \frac{20017}{90720}, \beta_1 = \frac{671}{36288}, \beta_2 = \frac{-241}{2268000}, \beta_3 = \frac{13}{4536000}$ with error constant $\frac{-1}{25344000}$ and order p =10. Also, for the hybrid formula (26,27), we have,

$y_{n+\lambda} = \psi_2 y_{n+1} + \psi_1 y_n + \psi_0 y_{n-1} + h^2(\gamma_0 f_n + \gamma_1 f_{n+1} + \gamma_2 f_{n+2} + \gamma_3 f_{n+3}),$ (29)

$\psi_0 = \frac{63}{1216}, \psi_1 = \frac{241}{608}, \psi_2 = \frac{671}{1216}, \gamma_0 = \frac{-503}{4864}, \gamma_1 = \frac{-631}{7296}, \gamma_2 = \frac{223}{14592}, \gamma_3 = \frac{-1}{456}$ with LTE = $\frac{577y^{(7)}[x]h^7}{583680}$ and

$y_{n-\lambda} = a_2 y_{n+1} + a_1 y_n + a_0 y_{n-1} + h^2(b_0 f_n + b_1 f_{n-1} + b_2 f_{n-2} + b_3 f_{n-3}),$ (30)

$\psi_0 = \frac{671}{1216}, \psi_1 = \frac{241}{608}, \psi_2 = \frac{63}{1216}, \gamma_0 = \frac{-503}{4864}, \gamma_1 = \frac{-631}{7296}, \gamma_2 = \frac{223}{14592}, \gamma_3 = \frac{-1}{456}$ with LTE = $\frac{-577y^{(7)}[x]h^7}{583680}$. For k = 2 and k^*= 8, we have,

$y_{n+1} - 2y_n + y_{n-1} = h^2\big(\beta_0 f_n + \beta_1(f_{n+1} + f_{n-1}) + \beta_2(f_{n+2} + f_{n-2}) + \beta_3(f_{n+3} + f_{n-3}) + \beta_4(f_{n+4} + f_{n-4}) + \phi(f_{n+\lambda} + f_{n-\lambda})\big).$ (31)

The coefficients of method (31) of order p = 12 are given by,

$\emptyset = \frac{40576}{155925}, \beta_0 = \frac{353093}{1596672}, \beta_1 = \frac{187171}{9979200}, \beta_2 = \frac{-53}{399168}, \beta_3 = \frac{61}{9979200}, \beta_4 = \frac{-1}{4435200}$ and error constant $\frac{46507}{10461394944000}$. The hybrids and their coefficients are respectively given by,

$y_{n+\lambda} = \psi_2 y_{n+1} + \psi_1 y_n + \psi_0 y_{n-1} + h^2(\gamma_0 f_n + \gamma_1 f_{n+1} + \gamma_2 f_{n+2} + \gamma_3 f_{n+3} + \gamma_4 f_{n+4})$ (32)

$\psi_0 = \frac{89}{2304}, \psi_1 = \frac{487}{1152}, \psi_2 = \frac{1241}{2304}, \gamma_0 = \frac{-48103}{552960}, \gamma_1 = \frac{-3323}{34560}, \gamma_2 = \frac{7171}{276480}, \gamma_3 = \frac{-43}{5760}, \gamma_4 = \frac{577}{552960}$ with LTE=$\frac{-157123y^{(8)}[x]h^8}{278691840}$ and

$y_{n+\lambda} = \psi_2 y_{n+1} + \psi_1 y_n + \psi_0 y_{n-1} + h^2(\gamma_0 f_n + \gamma_1 f_{n-1} + \gamma_2 f_{n-2} + \gamma_3 f_{n-3} + \gamma_4 f_{n-4}),$ (33)

$\psi_0 = \frac{1241}{2304}, \psi_1 = \frac{487}{1152}, \psi_2 = \frac{89}{2304}, \gamma_0 = \frac{-48103}{552960}, \gamma_1 = \frac{-3323}{34560}, \gamma_2 = \frac{7171}{276480}, \gamma_3 = \frac{-43}{5760}, \gamma_4 = \frac{577}{552960}$ with LTE = $\frac{157123y^{(8)}[x]h^8}{278691840}$.

4.0 Phase-Lag Properties of the Super-Implicit Hybrid LMM

When deriving efficient numerical methods (2) for the solution of (1) it is useful to consider the phase-lag (PL) order as well as the algebraic order of the method. The concept of PL was first introduced in [22]. However, several methods with high phase-lag order have been proposed for the numerical integration of the IVP (1). A new approach to constructing methods for the numerical integration of (1) through a rational approximation to the cosine was given in [23]. Since then, [25 - 28] among others have all considered the phase-lag analysis of LMM (2). A method with maximum order of the phase-lag has minimal phase-lag error [10]. The phase-lag formula for the generalized symmetric LMM (2) have been considered in [27]. Following the idea in [27], the phase-lag for the SSI hybrid LMM is given by,

$$PL(H) = \frac{2A_{\frac{k^*}{2}}(H)cos(jH) + \cdots + 2A_j(H)cos(jH) + \cdots + 2A_0(H)}{2\left(\frac{k^*}{2}\right)^2 A_{\frac{k^*}{2}}(H) + \cdots + 2j^2 A_j(H) + \cdots + 2A_1(H)} = -c_{d+2}H^{d+2} + O(H^{d+4}). \quad (34)$$

Where $A_j(H) = \alpha_{\frac{k^*}{2}-j} + H^2\beta_{\frac{k^*}{2}-j}$, $1 \le j \le \frac{k^*}{2}$, $H = \omega h$, d is the order of the phase-lag and c is the phase-lag error constant. We use formula (34) in the sense of the work in [27] to obtain the phase-lag for methods (16,25). Upon applying (16) on the test problem (8), for k = 4 and k^* = 6, we have the phase-lag expression as,

$$PL(H) = \frac{T_1}{T_2} = -c_{d+2}H^{d+2} + O(H^{d+4}). \quad (35)$$

Here,

$$T_1 = 2(\alpha_0 + H^2\beta_0) + \big(2(\alpha_1 + H^2\beta_1)cos(H) + (\alpha_2 + H^2\beta_2)cos(2H) + (\alpha_3 + H^2\beta_3)cos(3H) + (\alpha_\lambda + H^2\beta_2)cos(\lambda H)\big) \quad (36)$$

$$T_2 = 2\big((\alpha_1 + H^2\beta_1) + 4(\alpha_2 + H^2\beta_2) + 9(\alpha_3 + H^2\beta_3)cos(3H) + (\alpha_\lambda + H^2\beta_2)cos(\lambda H)\big). \quad (37)$$

In particular, for the hybrid SSI method (19) with $= \frac{1}{2}$, we obtain the phase-lag order d = 10 with phase-lag error constant as in,

$$PL(H) = \frac{-7967}{1596672000} H^{12} + O(H^{14}) = -c_{d+2}H^{d+2} + O(H^{d+4}), \quad (38)$$

using MATHEMATICA v10 [21]. Similarly, for Störmer-Cowell type hybrid SSI method (28) with $\lambda = \frac{1}{2}$, we obtain the phase-lag order d = 10 with phase-lag error constant as in,

$$PL(H) = \frac{-1}{50688000} H^{12} + O(H^{14}) = -c_{d+2}H^{d+2} + O(H^{d+4}). \quad (39)$$

The summary of the algebraic and phase-lag properties of the proposed hybrid methods are presented in Tables 1 and 2.

Table 1: Summary of the algebraic and phase-lag properties of hybrid SILMM for $\lambda = \frac{1}{2}$

Methods	k	k^*	Order of PL d /Order of the Methods p, $d = p$	PL Error Constant c_{d+2}	LTE Constant C_{p+2}	Stability
(19)	2	2	6	$\frac{-7967}{1596672000}$	$\frac{7967}{798336000}$	P-stable
(22)	2	4	8	$\frac{-5367083}{10461394944000}$	$\frac{-5367083}{5230697472000}$	P-stable

Table 2: Summary of the algebraic and phase-lag properties of the Störmer-Cowell type hybrid SILMM for $\lambda = \frac{1}{2}$

Methods	k	k^*	Order of PL d /Order of the Methods p, $d = p$	PL Error Constant c_{d+2}	LTE Constant C_{p+2}	Stability
(28)	2	6	10	$\frac{-1}{50688000}$	$\frac{-1}{25344000}$	P-stable
(31)	2	8	12	$\frac{-46507}{20922789888000}$	$\frac{46507}{10461394944000}$	P-stable

5.0 Implementation of Hybrid SSILMM

We consider the implementation of our new hybrid methods derived to show the applicability of these methods in solving the almost periodic problem of (1) using MATLAB v7.5 [29]. We are faced with the problem of resolving the implicitness in the derived hybrid methods. However, we consider implementing methods (19,22,28,31) following the ideas in [6,9]. Assume that (1) is Lipschitz continuous with reference to $y(x)$ for all $x \in [a,b]$

$$\|f(x,y) - f(x,y^*)\| \leq L\|y - y^*\|, \tag{40}$$

where L is the Lipschitz constant. The approach of Newton-Raphson iterative method is used to resolve the implicitness in our methods. We use the predictor,

$$y_{n+2} - 2y_{n+1} + y_n = h^2 f_{n+1}, \tag{41}$$

of order $p = 2$, as the starter for the Newton-Raphson iteration with LTE = $\frac{1y^{(4)}[x]h^4}{12}$. The P-stable method,

$$y_{n+2} - 2y_{n+1} + y_n = \frac{h^2}{4}(f_{n+2} - 2f_{n+1} + f_n), \tag{42}$$

is employ to generate the future solution values $\{y_{n\pm j}\}_{j=2,3}$ in the case of (19,28) and $\{y_{n\pm j}\}_{J=2,3,4}$ in the case of (22,31) respectively. So that the Newton-Raphson iteration becomes,

$$y^{[t+1]}_{n+1} = y^{[t]}_{n+1}\left(J\left(y^{[t]}_{n+1}\right)\right)^{-1} F\left(y^{[t]}_{n+1}\right), t = 0,1,\dots,w, \tag{43}$$

where the Jacobian is given by,

$$J(y) = \frac{\partial F(y)}{\partial y}. \tag{44}$$

The numerical methods (19,22,28,31) are applied to solve almost periodic orbital problem as described in [5]. In the case of the P-stable method in (19),

$$F\left(y^{[t+1]}_{n+2}\right) = y^{[t]}_{n+2} - 2y_{n+1} + 2y_n - 2y_{n-1} + y_{n-2} = h^2\left(\frac{39967}{45360}f_n + \frac{22049}{18144}(f_{n+1} + f_{n-1}) + \frac{70529}{1134000}\left(f^{[t]}_{n+2} + f_{n-2}\right) - \frac{1997}{2268000}(f_{n+3} + f_{n-3}) - \frac{82048}{70875}\left(f^{[t]}_{n+\lambda} + f^{[t]}_{n-\lambda}\right)\right). \tag{45}$$

Also, in the case of the P-stable Störmer-Cowell type hybrid method in (22),

$$F\left(y^{[t+1]}_{n+1}\right) = y^{[t]}_{n+1} - 2y_n + y_{n-1} = h^2\left(\frac{20017}{90720}f_n + \frac{671}{36288}\left(f^{[t]}_{n+1} + f_{n-1}\right) - \frac{241}{2268000}(f_{n+2} + f_{n-2}) + \frac{13}{4536000}(f_{n+3} + f_{n-3}) + \frac{18496}{70875}\left(f^{[t]}_{n+\lambda} + f^{[t]}_{n-\lambda}\right)\right). \tag{46}$$

Example 1: Almost periodic orbital problem (Source: [4,6,8,9,11])

$$\boldsymbol{y'' + y = 0.001e^{ix}}, \tag{47}$$
$$\boldsymbol{y(0) = 1, y'(0) = 0.9995i^2 = -1},$$

which the theoretical solution is,

$$\boldsymbol{y(x) = u(x) + iv(x)} = (\boldsymbol{u(x), v(x)}), \tag{48}$$

where $\boldsymbol{u(x) = cosx + 0.0005xsinx}$ and $\boldsymbol{v(x) = i(sinx - 0.0005xcosx)}$. The IVP (47) represent motion on a perturbed circular orbit on the complex plane in which the path defined by the point $\boldsymbol{y}(x) = (\boldsymbol{u}(x), \boldsymbol{v}(x))$ spirals slowly outward such that its distance from the origin at any given time $\boldsymbol{x}$ is given by,

$$\Omega(x) = \sqrt{U(x)^2 + V(x)^2}\,, \tag{49}$$

The interval $\boldsymbol{0 < x \leq 40\pi}$ corresponds to 20 orbits of the point $\boldsymbol{y(x)}$,

$$\boldsymbol{\Omega(x_f) = |y(x_f)| = 1.00197197653449, x_f = 40\pi}. \tag{50}$$

The numerical result is generated using the step size $h = \frac{\pi}{2^q}$, $q = 3(1)13$, and can be seen in Table 3,4,5 and 6.

Table 3: Numerical Results of Method (19) at $x_f = 40\pi$

q	h	Method (19) (Ω)	Error $\lvert\Omega(x_f) - \Omega\rvert$
3	$\pi/2^3$	1.00321942175128	1.24744521678988e-003
4	$\pi/2^4$	1.00244005857149	4.68082037000883e-004
5	$\pi/2^5$	1.00209067667853	1.18700144044803e-004
6	$\pi/2^6$	1.00208276118813	1.10784653639229e-004
7	$\pi/2^7$	1.00199977717586	2.78006413707566e-005
8	$\pi/2^8$	1.00199784030696	2.58637724708244e-005
9	$\pi/2^9$	1.00197736701962	5.39048512671059e-006
10	$\pi/2^{10}$	1.00197688542805	4.90889355941881e-006
11	$\pi/2^{11}$	1.00197664465420	4.66811970589731e-006
12	$\pi/2^{12}$	1.00197409335020	2.1168157144924e-006
13	$\pi/2^{13}$	1.00197281831423	8.41779736138193e-007

Table 4: Numerical Results of Method (22) at $x_f = 40\pi$

q	h	Method (22) (Ω)	Error $\lvert\Omega(x_f) - \Omega\rvert$
3	$\pi/2^3$	1.00205910451501	8.71279805190195e-005
4	$\pi/2^4$	1.00201530580872	4.33292742272329e-005
5	$\pi/2^5$	1.00199358248877	2.16059542843539e-005
6	$\pi/2^6$	1.00198276484045	1.07883059601299e-005
7	$\pi/2^7$	1.00197736701962	5.39048512604445e-006
8	$\pi/2^8$	1.00197467086008	2.69432559174554e-006
9	$\pi/2^9$	1.00197332346804	1.34693355424709e-006
10	$\pi/2^{10}$	1.00197264994395	6.73409463525232e-007
11	$\pi/2^{11}$	1.00197231322489	3.36690401558926e-007
12	$\pi/2^{12}$	1.00197214487611	1.68341616424428e-007
13	$\pi/2^{13}$	1.00197206070440	8.41699103748539e-008

Table 5: Numerical Results of Method (28) at $x_f = 40\pi$

q	h	Method (28) (Ω)	Error $\lvert\Omega(x_f) - \Omega\rvert$
3	$\pi/2^3$	1.00203401920494	6.20426704474042e-005
4	$\pi/2^4$	1.00200287811217	3.09015776829291e-005
5	$\pi/2^5$	1.00198739738256	1.54208480669382e-005
6	$\pi/2^6$	1.00197967947316	7.70293866714233e-006
7	$\pi/2^7$	1.00197582613246	3.84959797039564e-006
8	$\pi/2^8$	1.00197390086563	1.92433114087898e-006
9	$\pi/2^9$	1.00197293858310	9.62048610331223e-007
10	$\pi/2^{10}$	1.00197245752955	4.80995061558076e-007
11	$\pi/2^{11}$	1.00197221702471	2.40490217517930e-007
12	$\pi/2^{12}$	1.00197209677777	1.20243278001198e-007
13	$\pi/2^{13}$	1.00197203665567	6.01211789241773e-008

Table 6: Numerical Results of Method (31) at $x_f = 40\pi$

q	h	Method (31) (Ω)	Error $\lvert\Omega(x_f) - \Omega\rvert$
3	$\pi/2^3$	1.00203401920494	6.20426704474042e-005
4	$\pi/2^4$	1.00200287811217	3.09015776829291e-005
5	$\pi/2^5$	1.00198739738256	1.54208480669382e-005
6	$\pi/2^6$	1.00197967947316	7.70293866714233e-006
7	$\pi/2^7$	1.00197582613246	3.84959797039564e-006
8	$\pi/2^8$	1.00197390086563	1.92433114087898e-006
9	$\pi/2^9$	1.00197293858310	9.62048610331223e-007
10	$\pi/2^{10}$	1.00197245752955	4.80995061558076e-007
11	$\pi/2^{11}$	1.00197221702471	2.40490217517930e-007
12	$\pi/2^{12}$	1.00197209677777	1.20243278001198e-007
13	$\pi/2^{13}$	1.00197203665567	6.01211789241773e-008

6.0 Conclusion

We have derived P-stable SSI hybrid methods based on (16, 25) with order as high as p = 10 and 12 in the case of (19,28) and (22,31) respectively which turns to be higher than that of the ones proposed in [20] for the same step number. In addition, we have also investigated the PL of the new SSI hybrid methods which interestingly have its order d coincides with the algebraic order p see tables 1 and 2. The order barrier in theorem 3 has been bypassed through the use of SSI hybrid methods. The efficiency and accuracy of the new hybrid methods have been tested on an almost periodic orbital problem as the results compares favourably with the theoretical solution see tables 3,4,5 and 6.

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