Transactions of the Nigerian Association of Mathematical Physics
Volume 4, (July, 2017), pp87 – 94
© Trans. of NAMP

2 and 4-Steps Block Hybrid Linear Multistep Methods for Solution of Special Second Order Ordinary Differential Equations.

S. D. Yakubu,[1*] Y. A. Yahaya[2] and K. O. Lawal[2]

[1*]Department of Mathematics and Computer Science, Ibrahim Badamasi Babangida University, Lapai, Nigeria
[2]Department of Mathematics, Federal University of Technology, Minna, Nigeria

Abstract

This paper examines a set of four (4) Implicit Hybrid Block Methods which are derived through multistep collocation method using power series as a basis function for generating solution of special second order initial value problem of ordinary differential equations. The derived continuous forms are evaluated at some grids and off-grid points of collocation and interpolation to form the block hybrid methods for step numbers k=2 and k= 4. The discrete schemes obtained posse uniformly high order and are found to be zero-stable, consistent and hence convergent. Some numerical examples are given to test the accuracy and efficiency advantages. The results of our evaluation show that our methods outperform reviewed work.

Keywords: Consistent, Order, Zero-stable, Convergence, Region of Absolute Stability, Stiff and Non-stiff Problems.

1.0 Introduction

Numerical methods are now prevalent and vital in applications of engineering and science due to the difficulties experienced in obtaining the analytical solution. Considering the special second order ordinary differential equation

$$y'' = f(x,y),\ y(a) = y_0,\ y'(a) = y'_0, \tag{1.1}$$

Hence, the necessity of numerical methods for such problem needs to be developed.

Many researchers that have worked extensively in this area [1,2]. The main aim of this research paper is to develop hybrid block methods of linear multistep method when k=2 and 4 with one off-grid point at both collocation and interpolation respectively that can be used to solve special second order ordinary differential equations.

Definition 1.1 Linear Multistep Method (LMM)

Linear multistep method is the computational method which is used to determine the sequence $[y_n]$ and it is a linear relationship between y_{n+j} and f_{n+j}, j = 0,1, 2,…k. [1]

A linear k-step method of order two is mathematically expressed as:

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^2 \sum_{j=0}^{k} \beta_j f_{n+j} \tag{1.2}$$

where $\alpha_k + \beta_k \neq 0$ Where $\beta_k = 0$ is an explicit scheme and $\beta_k \neq 0$ is an implicit scheme [1]

Definition 1.2 Hybrid Method

Hybrid method was as a result of the desire to increase the order without increasing the step number and without reducing the stability interval. Therefore, a k-step hybrid method is defined as:

$$\sum_{j=o}^{k} \alpha_j y_{n+j} = h^2 \left[\sum_{j=0}^{k} \beta_j f_{n+j} + \beta_v f_{n+v} \right] \tag{1.3}$$

Where α_k =1, just to remove arbitrariness, α_0 *and* β_0 are not both zero and $v \notin [0,1,2,...,k$. $f_{n+v} = f(x_{n+v}, y_{n+v})$ which is the off grid function of evaluation [1]

Definition 1.3 Order and Error Constant

The linear multistep method of type (1.2) is said to be of order p if $C_0 = C_1 = \dots C_{p+1} = 0$, but $C_{p+2} \neq 0$ and C_{p+2} is called the error constant, [1]

Definition 1.4 Convergences

The necessary and sufficient conditions for linear multistep method of type (1.2) is said to be convergent if and only if it is consistent and zero-stable.

Correspondence Author: Yakubu S.D., Email: saiduyakubu2014@gmail.com, Tel: Tel.:+2348068107232, +2348053438800

Definition 1.5 Stability Regions
The stability region of linear multistep method is part of the complex plane where the method when applied to the test equation $y'' = \lambda^2 y$ is absolute stable whose resultant finite difference equation has characteristics equation $\pi(z,r)=\rho(r)-z^2\sigma(r)$, $z=i\lambda h$, [3]

2.0 Derivation of the Methods

The methods are derived for the special second order ordinary differential equation base on the multistep collocation idea. The general form of a power series is given as follows

$$y(x)=\sum_{j=0}^{\infty}\alpha_j x^j$$

which is used as our basis function to produce an appropriate solution to (1.3) as follows

$$y(x)=\sum_{j=0}^{t+m-1}\alpha_j x^j \tag{2.1}$$

and

$$y''(x)=\sum_{j=2}^{t+m-1}j(j-1)\alpha_j x^{(j-2)} \tag{2.2}$$

where α_{js} are the parameters to be determined, t and m are the points of interpolation and collocation respectively. This process leads to (t+m-1) non-linear system of equations with (t+m-1) unknown coefficients, which are to be determined by the use of maple 13 mathematical software.

2.1. Derivation of the hybrid block method when k=2 with one off-grid point at collocation.

Using equations (2.1) and (2.2) with t=2, m=4. The degree of our polynomial is (t+m-1). Equations (2.1) was interpolated at x= $x_{n+j,j=0,1}$ and (2.2) collocated at x= $x_{n+j,j=0,\frac{3}{5},1,2}$ which gives the following non-linear system of equations of the form as follows

$$\sum_{j=0}^{t+m-1}\alpha_j x_{n+i}^j = y_{n+i}, i=0,1 \tag{2.3}$$

$$\sum_{j=2}^{t+m-1}j(j-1)\alpha_j x_{n+i}^{(j-2)} = f_{n+i}, i=0,\frac{3}{5},1,2 \tag{2.4}$$

With maple 13 software, we obtain the continuous formulation of equations (2.3) and (2.4) and the continuous formulation is evaluated at x= x_{n+j} where j= $\frac{3}{5},2$ and its first derivative also evaluated at x= x_n gives the following set of discrete schemes that form the first hybrid block method when k=2 with one off-grid point at collocation.

$$\left.\begin{aligned}
&y_{n+2}-2y_{n+1}+y_n=\frac{h^2}{12}[f_n+10f_{n+1}+f_{n+2}]\\
&y_{n+\frac{3}{5}}-\frac{3}{5}y_{n+1}-\frac{2}{5}y_n=\frac{h^2}{125}[-\frac{247}{150}f_n-\frac{163}{12}f_{n+\frac{3}{5}}+\frac{31}{100}f_{n+1}-\frac{2}{25}f_{n+2}]\\
&hz_0-y_{n+1}+y_n=\frac{h^2}{3}[-\frac{13}{24}f_n-\frac{25}{21}f_{n+\frac{3}{5}}+\frac{1}{4}f_{n+1}-\frac{1}{56}f_{n+2}]
\end{aligned}\right\} \tag{2.5}$$

Equation (2.5) having uniform order four (4) with error constant as follows

$$\left(\frac{1}{240},\frac{177}{1250000},\frac{1}{800}\right)^T$$

2.2 Derivation of the second block method when k=2 with one off-grid point at interpolation

Equation (2.1) was interpolated at x= $x_{n+j}, j=0,\frac{3}{5},1$ and equation (2.2) collocated at $x=x_{n+j}, j=0,1,2$ which gives the system of non-linear equations of the form as follows:

$$\sum_{j=0}^{t+m-1}\alpha_j x_{n+u}^j = y_{n+u}, u=0,\frac{3}{5},1 \tag{2.6}$$

$$\sum_{j=2}^{t+m-1}j(j-1)\alpha_j x_{n+v}^{(j-2)} = f_{n+v}, v=0,1,2 \tag{2.7}$$

Adopting the previous procedure in the first block to generate the continuous formula and this continuous formula is evaluated at $x=x_{n+2}$. Its second derivative is evaluated at $x=x_{n+\frac{3}{5}}$ and first derivative is evaluated at $x=x_n$ gives the second hybrid block method when k=2 with one off-grid point at interpolation as follows:

$$\left.\begin{aligned}
&y_{n+2}-2y_{n+1}+y_n=\frac{h^2}{12}[f_n+10f_{n+1}+f_{n+2}]\\
&\frac{1500}{163}y_{n+\frac{3}{5}}-\frac{900}{163}y_{n+1}-\frac{600}{163}y_n=\frac{h^2}{4075}[-494f_n-4075f_{n+\frac{3}{5}}+93f_{n+1}-24f_{n+2}]\\
&hz_0-\frac{12500}{3423}y_{n+\frac{3}{5}}+\frac{1359}{1141}y_{n+1}+\frac{8423}{3423}y_n=\frac{h^2}{28}[-\frac{1209}{326}f_n+\frac{339}{163}f_{n+1}-\frac{33}{326}f_{n+2}]
\end{aligned}\right\} \qquad (2.8)$$

Equation (2.8) having uniform order four (4), with error constant as follows

$$\left(\frac{1}{240},\frac{531}{407500},\frac{669}{912800}\right)^T$$

2.3 Derivation of the third block hybrid method when k=4 with one off-grid point at collocation

Equation (2.1) was interpolated at x= $x_{n+j},\ j=0,1$ and equation (2.2) collocated at x= $x_{n+j},\ j=0,1\frac{6}{5},2,3,4$ which gives the system of non-linear equations of the form

$$\sum_{j=0}^{t+m-1}\alpha_j x_{n+r}^j=y_{n+r},\ r=0,1 \qquad (2.9)$$

$$\sum_{j=2}^{t+m-1}j(j-1)\alpha_j x_{n+q}^{(j-2)}=f_{n+q},\ q=0,1,\frac{6}{5},2,3,4 \qquad (2.10)$$

By applying maple13 software to evaluate α_{js} in equations (2.9) and (2.10), we obtain value of the continuous formula and this continuous formula is evaluated at x= $x_{n+j},\ j=\frac{6}{5},2,3,4$ and derivative is evaluated at $x=x_n$, we have the following discrete block schemes when k=4 with one off-grid point at collocation as follows

$$\left.\begin{aligned}
&y_{n+4}-4y_{n+1}+3y_n=h^2[\frac{137}{720}f_n+\frac{18}{5}f_{n+1}-\frac{3125}{3024}f_{n+\frac{6}{5}}+\frac{173}{80}f_{n+2}+\frac{136}{135}f_{n+3}+\frac{41}{560}f_{n+4}]\\
&y_{n+3}-3y_{n+1}+2y_n=\frac{h^2}{2}[\frac{43}{180}f_n+\frac{79}{15}f_{n+1}-\frac{3125}{1512}f_{n+\frac{6}{5}}+\frac{99}{40}f_{n+2}+\frac{11}{135}f_{n+3}+\frac{1}{210}f_{n+4}]\\
&y_{n+2}-2y_{n+1}+y_n=\frac{h^2}{3}[\frac{89}{480}f_n+\frac{19}{5}f_{n+1}-\frac{3125}{2016}f_{n+\frac{6}{5}}+\frac{103}{160}f_{n+2}-\frac{4}{45}f_{n+3}+\frac{11}{1120}f_{n+4}]\\
&y_{n+\frac{6}{5}}-\frac{6}{5}y_{n+1}+\frac{1}{5}y_n=\frac{h^2}{25}[\frac{268807}{900000}f_n+\frac{65597}{9375}f_{n+1}-\frac{29371}{6048}f_{n+\frac{6}{5}}+\frac{66483}{100000}f_{n+2}-\frac{9932}{84375}f_{n+3}+\frac{28373}{2100000}f_{n+4}]\\
&hz_0-y_{n+1}+y_n=\frac{h^2}{7}[-\frac{12739}{8640}f_n-\frac{181}{18}f_{n+1}+\frac{334375}{36288}f_{n+\frac{6}{5}}-\frac{453}{320}f_{n+2}+\frac{106}{405}f_{n+3}-\frac{617}{20160}f_{n+4}]
\end{aligned}\right\} \qquad (2.11)$$

Equation (2.11) having uniform order six (6) with error constant as follows

$$\left[\frac{209}{50400},\frac{17}{11200},\frac{307}{302400},\frac{155059}{875000000},\frac{101}{67200}\right]^T$$

2.4 Derivation of the fourth hybrid block method when k=4 with one off-grid point at interpolation as follows:

Equation (2.1) was interpolated at x= $x_{n+j},\ j=0,1,\frac{6}{5}$ and equation (2.2) collocated at x= $x_{n+j},\ j=0,1,2,3,4$ which gives the system of non-linear equations of the form

$$\sum_{j=0}^{t+m-1}\alpha_j x_{n+w}^j=y_{n+w},\ w=0,1,\frac{6}{5} \qquad (2.12)$$

$$\sum_{j=2}^{t+m-1}j(j-1)\alpha_j x_{n+z}^{(j-2)}=f_{n+z},\ z=0,1,,2,3,4 \qquad (2.13)$$

Using maple13 software to evaluate α_{js} in equations (2.12) and (2.13), we obtain value of the continuous formula and this continuous formula is evaluated at x= $x_{n+j},\ j=,2,3,4$ and second derivative is evaluated at $x=x_{n+\frac{6}{5}}$ and also first derivative is evaluated at $x=x_n$ we have the following discrete block schemes when k=4 with one off-grid point at interpolation as follows

$$
\left.\begin{aligned}
& y_{n+4} - \frac{156250}{29371} y_{n+\frac{6}{5}} + \frac{70016}{29371} y_{n+1} + \frac{56863}{29371} y_n = \frac{h^2}{115}\Big[\frac{55829}{3831} f_n + \frac{930064}{3831} f_{n+1} \\
& + \frac{296798}{1277} f_{n+2} + \frac{454864}{3831} f_{n+3} + \frac{30989}{3831} f_{n+4}\Big] \\
& y_{n+3} - \frac{156250}{29371} y_{n+\frac{6}{5}} + \frac{99387}{29371} y_{n+1} + \frac{27492}{29371} y_n = \frac{h^2}{230}\Big[\frac{131319}{10216} f_n + \frac{336123}{1277} f_{n+1} \\
& + \frac{1287657}{5108} f_{n+2} + \frac{19323}{1277} f_{n+3} - \frac{1161}{10216} f_{n+4}\Big] \\
& y_{n+2} - \frac{78125}{29371} y_{n+\frac{6}{5}} + \frac{35008}{29371} y_{n+1} + \frac{13746}{29371} y_n = \frac{h^2}{115}\Big[\frac{13229}{3831} f_n + \frac{76688}{1277} f_{n+1} \\
& + \frac{63374}{38371} f_{n+2} - \frac{2512}{1277} f_{n+3} + \frac{809}{3831} f_{n+4}\Big] \\
& \frac{151200}{29371} y_{n+\frac{6}{5}} - \frac{181440}{29371} y_{n+1} + \frac{30240}{29371} y_n = \frac{h^2}{91784375}\Big[5644947 f_n + 132243552 f_{n+1} - 91784375 f_{n+\frac{6}{5}} \\
& + 12565287 f_{n+2} - 2224768 f_{n+3} + 255357 f_{n+4}\Big] \\
& hz_0 + \frac{8359375}{1233582} y_{n+\frac{6}{5}} - \frac{1877472}{205597} y_{n+1} + \frac{2905457}{1233582} y_n = \frac{h^2}{23}\Big[-\frac{533207}{178780} f_n + \frac{94596}{8939} f_{n+1} \\
& - \frac{45279}{89390} f_{n+2} + \frac{5636}{44695} f_{n+3} - \frac{417}{25540} f_{n+4}\Big]
\end{aligned}\right\} \quad (2.14)
$$

Equation (2.14) having uniform order six (6), with error constant as follows

$$
\left[\frac{148219}{46259325}, \frac{27027}{46993600}, \frac{150947}{277555950}, \frac{4186593}{4589218750}, \frac{43479}{143917900}\right]^T
$$

3.0 The Convergence Analysis

The convergence analysis of all the block hybrid methods are determined using the approach of [4] which state that the block method is presented as a single block r-point multistep method as follows

$$
A^{(0)} Y_m = \sum_{i=1}^{k} A^{(i)} Y_{m-i} + h^2 \sum_{i=0}^{k} B^{(i)} F_{m-i} \qquad (3.1)
$$

where h is fixed mesh size within a block, $A^{(i)}$, $B^{(i)}$, $i = 0(1)k$ are r x r matrix coefficients and $A^{(0)}$ is r by r identity matrix, Y_m, Y_{m-i}, F_m and F_{m-i} are vectors of numerical estimates.

The method in (2.5) is presented in matrix form as

$$
\begin{bmatrix} 0 & -1 & 0 \\ 1 & -\frac{3}{5} & 0 \\ 0 & -2 & 1 \end{bmatrix}
\begin{bmatrix} y_{n+\frac{3}{5}} \\ y_{n+1} \\ y_{n+2} \end{bmatrix}
=
\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & \frac{2}{5} \\ 0 & 0 & -1 \end{bmatrix}
\begin{bmatrix} y_{n-\frac{7}{5}} \\ y_{n-1} \\ y_n \end{bmatrix}
+ h^2
$$

$$
\left[
\begin{bmatrix} -\frac{25}{63} & \frac{1}{12} & -\frac{1}{168} \\ -\frac{163}{1500} & \frac{31}{12500} & -\frac{2}{3125} \\ 0 & \frac{10}{12} & \frac{1}{12} \end{bmatrix}
\begin{bmatrix} f_{n+\frac{3}{5}} \\ f_{n+1} \\ f_n \end{bmatrix}
+
\begin{bmatrix} 0 & 0 & -\frac{13}{72} \\ 0 & 0 & -\frac{247}{18750} \\ 0 & 0 & \frac{1}{12} \end{bmatrix}
\begin{bmatrix} f_{n-\frac{7}{5}} \\ f_{n-1} \\ f_n \end{bmatrix}
\right] \qquad (3.2)
$$

We normalize the above block method (3.2) by multiplying matrices $A^{(0)}, A^{(1)}, B^{(0)} \text{ and } B^{(1)}$, with inverse of $A^{(0)}$ to obtain the below method as follow

$$
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} y_{n+\frac{3}{5}} \\ y_{n+1} \\ y_{n+2} \end{bmatrix}
=
\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} y_{n-\frac{7}{5}} \\ y_{n-1} \\ y_n \end{bmatrix}
+ h^2
$$

$$
\left[
\begin{bmatrix} \frac{453}{3500} & -\frac{297}{6250} & \frac{513}{175000} \\ \frac{25}{63} & -\frac{1}{12} & \frac{1}{168} \\ \frac{50}{63} & \frac{2}{3} & \frac{2}{21} \end{bmatrix}
\begin{bmatrix} f_{n+\frac{3}{5}} \\ f_{n+1} \\ f_n \end{bmatrix}
+
\begin{bmatrix} 0 & 0 & \frac{2379}{25000} \\ 0 & 0 & \frac{13}{72} \\ 0 & 0 & \frac{4}{9} \end{bmatrix}
\begin{bmatrix} f_{n-\frac{7}{5}} \\ f_{n-1} \\ f_n \end{bmatrix}
\right] \qquad (3.3)
$$

The block method (3.3) is the normalized form of the above schemes for the block hybrid method (2.5). Considering the first characteristics polynomial of the block method (2.5) is defined as $\rho(\lambda) = \det\left[\lambda I - A_1^{(1)}\right]$ in [1]. Substituting the values of λI and $A_1^{(1)}$ in the function above, gives

$$\left|\lambda\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}-\begin{bmatrix}0&0&1\\0&0&1\\0&0&1\end{bmatrix}\right|=\begin{vmatrix}\lambda&0&-1\\0&\lambda&-1\\0&0&\lambda-1\end{vmatrix}$$

Solving the above determinant yield the following solution

$$\rho(\lambda)=\lambda^2(\lambda-1)=0. \tag{3.4}$$

Since, $\lambda_1=\lambda_2=0, \lambda_3=1$. That is to say the block method is zero-stable and consistent and its order $(4,4,4)^T>1$, as stated in [4,5], hence the block method is convergent. The same analysis holds for block methods (2.8), (2.11) and (2.14). Thus they are zero-stable, consistent and convergent.

3.1 Region of Absolute Stability (RAS)

The regions of absolute stability of all the block methods derived are determine by reformulating them as general linear method expressed as follows

$$\begin{bmatrix}Y\\ Y_{i+1}\end{bmatrix}=\begin{bmatrix}A & U\\ B & V\end{bmatrix}\begin{bmatrix}h^2 f(Y)\\ Y_{i-1}\end{bmatrix} \quad i=1, 2,\ldots,N \tag{3.5}$$

Applying (3.5) to the test equation $y''=\lambda^2 y$. its lead to a recursion of form:

$$M(z):=V+zB(I-zA)^{-1}U, \tag{3.6}$$

where $z=\lambda h$, equation (3.6) is the stability matrix and the stability function is

$$\rho(\eta,z)=\det[\eta I-M(z)] \tag{3.7}$$

Computing the stability function gives the stability polynomial of the methods which is plotted to produce the required absolute stability region of the method. To plot the absolute stability region of equation (2.5) is expressed in the form of equation (3.5) and the values of the matrices A, B, U and V are substituted into equations (3.6) and (3.7), with the aids of maple software, we obtained the characteristics polynomial as follow

$$fz:=\frac{1}{2}\,\frac{1}{-9000+116z^2-978z+9z^3}\big(18\eta^2z^3-21\eta z^3+3451\eta z^2+232\eta^2z^2-161z^2-1956\eta^2 z+13368\eta z-20412z-18000\eta^2-36000+54000\eta\big)$$

and the stability function is obtained as follow

$$fzp:=-\frac{3}{2}\,\frac{1}{\left(-9000+116z^2-978z+9z^3\right)^2}\big(24882000\eta z+66516\eta z^3+1938922\eta z^2+11165z^4\eta+22500000\eta-3750000z-122472z^3-1165750z^2-483z^4-49500000\big)$$

These values of the stability function and characteristics polynomial are used in matlab programme to obtain the region of absolute stability as shown in Figure 1

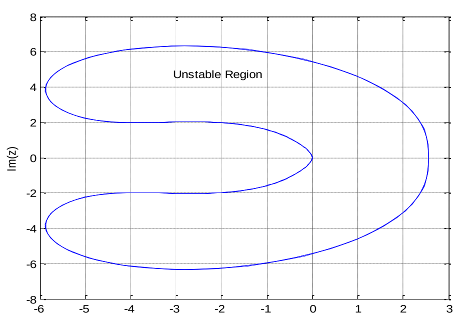

Figure 1: Region of absolute stability when k=2 with one off-grid point at collocation and the block method is A (α)-stable.

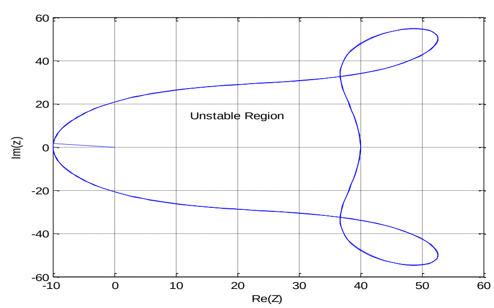

Figure 2: The block method is A (α)-stable.

The same analysis holds for block methods (2.8), (2.11) and (2.14) as shown in figures: 2, 3 and 4 respectively.

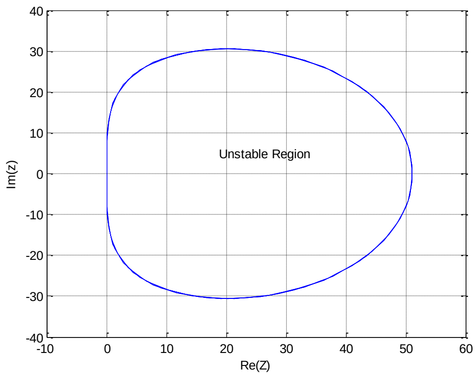

Figure 3: The block method is A-stable.

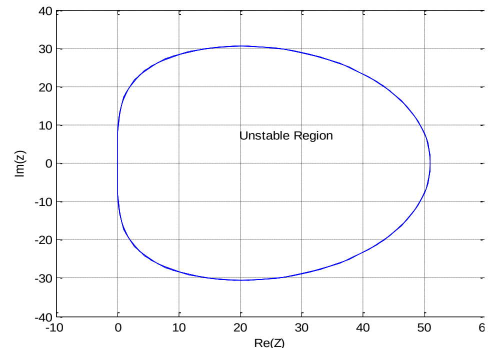

Figure 4: The block method is A-stable.

4. Numerical experiment

This section deals with numerical experiment by considering the derived discrete schemes in block form for solution of stiff and non-stiff differential equations of second order initial value problems for cases when k=2 and 4.

Problem 1

Consider the problem solved by Yahaya and Muhammad (2016)

$y'' = -y; y(0) = 1, y'(0) = 1$

h=0.1, $0.1 \le x \le 0.4$

Exact solution: $y(x) = Sinx + Cosx$

Problem 2

We consider the stiff differential equation

$y'' = 2y^3; y(1) = 1, y'(1) = -1$

h=0.1, $0.1 \le x \le 0.4$

Exact solution: y(x) = $\frac{1}{x}$

Table 1: Comparison of Errors when k=2 at Collocation for Problem 1

x	Exact Solution	Yahaya and Muhammad (2016)	Error of Proposed Method
0.1	1.094837582	7.782E-06	2.000000000E-09
0.2	1.178735909	2.1E-05	7.000000000E-09
0.3	1.250856696	5.1550E-05	1.500000000E-08
0.4	1.310479336	8.7836E-05	2.700000000E-08

Table 2: Comparison of Errors when k=2 at Interpolation for Problem 1

x	Exact Solution	Yahaya and Muhammad (2016)	Error of Proposed Method
0.1	1.094837582	7.782E-06	2.000000000E-09
0.2	1.178735909	2.1E-05	8.000000000E-09
0.3	1.250856696	5.1550E-05	1.700000000E-08
0.4	1.310479336	8.7836E-05	3.00000000E-08

Table 3: Comparison of Errors when k=4 at Collocation for Problem 1

x	Exact Solution	Yahaya and Muhammad (2016)	Error of Proposed Method
0.1	1.094837582	2.1E-08	0.000000000E+00
0.2	1.178735909	2.8E-08	1.000000000E-09
0.3	1.250856696	2.2E-08	3.000000000E-09
0.4	1.310479336	4.0E-08	2.000000000E-09

Table 4: Comparison of Errors when k=4 at Interpolation for Problem 1

x	Exact Solution	Yahaya and Muhammad (2016)	Error of Proposed Method
0.1	1.094837582	2.1E-08	2.000000000E-09
0.2	1.178735909	2.8E-08	7.000000000E-09
0.3	1.250856696	2.2E-08	1.500000000E-08
0.4	1.310479336	4.0E-08	2.700000000E-08

Table 5: Results for the derived method (2.5) at Collocation with One Point Off-Grid when k=2 for Problem 2

x	Exact Solution	Computed Solution	Error of Proposed Method
0.1	0.909090909	0.909091462	5.53E-07
0.2	0.833333333	0.833336042	2.709E-06
0.3	0.769230769	0.769500339	2.6957E-04
0.4	0.714285714	0.715257479	9.65173E-04

Table 6: Results for the derived method (2.8) at Interpolation with One Point Off-Grid when k=2 for Problem 2

x	Exact Solution	Computed Solution	Error of Proposed Method
0.1	0.909090909	0.909091462	5.53E-07
0.2	0.833333333	0.833336042	2.709E-06
0.3	0.769230769	0.774275169	5.0444E-03
0.4	0.714285714	0.707477991	6.807723E-03

Table 7: Results for the derived method (2.11) at Collocation with One Point Off-Grid when k=4 for Problem 2

x	Exact Solution	Computed Solution	Error of Proposed Method
0.1	0.909090909	0.909091075	1.66E-07
0.2	0.833333333	0.833333783	4.5E-07
0.3	0.769230769	0.769231477	7.08E-07
0.4	0.714285714	0.714286867	1.153E-06

Table 8: Results for the derived method (2.14) at Interpolation with One Point Off-Grid when k=4 for Problem 2

x	Exact Solution	Computed Solution	Error of Proposed Method
0.1	0.909090909	0.909091076	1.67E-07
0.2	0.833333333	0.833333784	4.51E-07
0.3	0.769230769	0.769231480	7.11E-07
0.4	0.714285714	0.714286870	1.156E-06

5. Conclusion

All newly derived hybrid block methods developed for the step numbers k=2 and 4 have one off-grid point at both collocation and interpolation respectively. The method can be used for the solution of special second order ordinary differential equation of type (1.1). The derived methods were implemented in block mode which have the advantages of being self-starting, uniformly of order four (4) at k=2 and order six (6) at k=4 respectively, and do not need predictors. The stability domains of the methods are presented in Figures 1 to 4. Maple13 and Matlab 2013 software packages were employed to generate the schemes and results. Also, each of the new block methods displayed its superiority over work of [7].

References:

[1] J. D. Lambert, Computational methods in ordinary differential equations. New York, John Willey and Sons. 1973

[2] Y. A. Yahaya, and Z. A. Adegboye, A family of 4-step block methods for special second order in ordinary differential equations. Proceedings Mathematical Association of Nigerian, 23-32. (2008)

[3] D. O. Awoyemi, A Class of continuous stomer-cowell type methods for special second order ordinary differential equations. Journal of Nigerian Mathematical Society, 5, (1), 100-108. (1998)

[4] S. O. Fatunla, Block methods for second order initial value problems. International Journal of Computer Mathematics, 4, 55-63. (1991)

[5] P. Henrici, Discrete variable methods for ordinary differential equations, New York, John Willey and Sons, 265. (1962)

[6] J. P. Chollom, J. N. Ndam, and G. M. Kumleg, Some properties of the block linear multistep methods. Science World Journal,2(3). 2007

[7] R. Muhammad, Some reformulated block implicit linear multistep methods into Runge-kutta type methods for solution of ordinary differential equations. Doctoral thesis (unpublished) Federal University of Technology Minna, Nigeria. (2016)

[8] J. D. Lambert, Numerical methods in ordinary differential systems: The initial value problems, New York, John Willey & Sons. 1991

[9] S. D. Yakubu, Block hybrid linear multistep methods for solution of special second order ordinary differential equations. Mtech thesis (unpublished) Federal University of Technology Minna, Nigeria. (2017)