Transactions of the Nigerian Association of Mathematical Physics
Volume 4, (July, 2017), pp79 – 86
© Trans. of NAMP

Single step continuous block method for stiff initial value problems

[1]A. M. Alkali, [2]A. F. Umaru, [1]R. O. Onsachi and [1]A. O. Adesanya

[1]Department of Mathematics, Modibbo Adama University of Technology, Yola.Nigeria
[2]Department of Mathematics, Adamawa State Polytechnic, Yola.Nigeria.

Abstract

This paper considers the development of hybrid one step second derivative multistep block method for stiff initial value problems. We evaluated the first and second derivative of the polynomial basis function at selected grid points, solve for the unknown parameters and substitute the results into the approximate solution to obtain a continuous scheme. Discrete methods which are by-product of the continuous scheme are implemented in block method. The methods are convegent and A-stable, results of numerical examples show that the methods are good for stiff problems.

Keywords: hybrid, convergent, block method, second derivative, zero stability
AMS subject classification: 65L05, 65L06, 65D30

1 Introduction

This paper considers numerical solution to

$$y'(x) = f(x,y), y(x_0) = \zeta, x \in [x_n, x_N] \quad (1)$$

where x_n is the initial point, x_N is the final point, $f : \mathrm{R} \times \mathrm{R}^m \to \mathrm{R}^m$, is a continuously differentiable real valued function. The Jacobian arising from (1) vary slowly and the eigenvalues have negative real part. f is assumed to satisfy the condition of existence and uniqueness of solution in the interval $[a,b]$, that is

(i) $(x,y) \to f(x,y)$ is continous over the region $[a,b] \times D$

(ii) for all the couple (x,y) and (x,y^*): $\| f(x,y) - f(x,y^*) \| \le k \| y - y^* \|$, k being independent of x called Lipschitz constant.

Higher Derivatives Multistep Method (HDMM) which has the general form

$$y_{n+k} = \sum_{j=0}^{r} \alpha_j(t) y_{n+j} + \sum_{i=1}^{l} h^i \sum_{j=0}^{s} \beta_j(t) f_{n+j} \quad (2)$$

where $\alpha_j(t)$ and $\beta_j(t)$ are polynomials of degree $k = r + s - 1$. $k \in [0,r]$, r and s are the number of interpolation and collocation points respectively. The commonest among these methods is the second derivatives method, (when $l = 2$). This method has the advantage of reaching higher order with fewer function evaluation, the method has been reported to have good stability properties for solving stiff problems [1]. Among the authors that developed numerical methods for the solution of stiff problems in the form (1) include [2-9]

2.0 Mathematical Background

2.1 Development of the method

We consider the approximate solution of the form

$$y(x) = \sum_{n=0}^{k} \alpha_n x^n \quad (3)$$

with its i-th derivative of the form

$$y^{(i)}(x) = \sum_{n=i}^{k} n(n-1)(n-2)\cdots(n-i)\alpha_n x^{n-i} \quad (4)$$

with $x \in [a,b]$, where α_n's are constant to be determined. Let the solution of (1) be sought on the partition $\pi_N : a = x_0 < x_1 < \dots x_n < x_{n+1} < \dots < x_N = b$ with a constant stepsize $h = \frac{b-a}{N-1}$. Evaluating the first and second derivative of (3) at $x = x_{n+j}, j = 0,1,\dots s,$ gives a linear system of equation in the form

Correspondence Author: Alkali A.M., Email: oziohumat@gmail.com, Tel: 2348069661098

$$XA = U \tag{5}$$

where

$A = [\alpha_1, \quad \alpha_2 \quad \dots \quad \alpha_{p-1}]^T$

$U = [y_n, \quad y_{n+1} \quad \dots \quad y_{n+r}, \quad f_n \quad f_{n+1} \quad \dots \quad f_{n+s}, \quad g_n \quad g_{n+1} \quad \dots \quad g_{n+m}]^T$

$$X = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & \dots & x_n^k \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_n^4 & \cdots & x_{n+1}^{k-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_{n+r} & x_{n+r}^2 & x_{n+r}^3 & x_{n+r-1}^4 & \cdots & x_{n+r}^{k-1} \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & \cdots & F' x_n^{k-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2x_{n+s} & 3x_{n+s} & 4x_{n+s-1} & \dots & F' x_{n+s}^{k-2} \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & \dots & F'' x_{n+m}^{k-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 2 & 6x_{n+m} & 12x_{n+m}^2 & \cdots & F'' x_{n+m}^{k-3} \end{bmatrix}$$

Where $F' = (K-1)$ and $F'' = (K-1)(K-2)$ represent both the first and second derivatives respectively. Using Crammer's method, we solve for the constant $\alpha' s$ then substitute the results into (5) to give a continuous method in the form

$$y_{n+t} = [y_n + \dots + y_{n+r}] + h[\alpha_1(t) f_n + \dots + \alpha_s(t) f_{n+s}] + h^2 [\beta_1(t) g_n + \dots + \beta_m(t) g_{n+m}] \tag{6}$$

$t = \frac{x - x_n}{h}$, where $\alpha_1(t), \alpha_2(t), \beta_1(t), \beta_2(t)$ are polynomials of degree $r + s + m - 1$.

Evaluating (6) at selected points gives a discrete block method in the form

$$\tau^{(1)} Y_{m+1} = \tau^{(0)} Y_m + h[\mu^{(0)} F_m + \mu^{(1)} F_{m+1}] + h^2 [\gamma^{(0)} G_m + \gamma^{(1)} G_{m+1}] \tag{7}$$

where

$Y_{m+1} = [y_{n+1}, y_{n+2}, \dots y_{n+r}]^T, \quad Y_m = [y_{n-r-1}, y_{n-r-2}, \dots y_n]^T$

$F_{m+1} = [f_{n+1}, f_{n+2}, \dots f_{n+s}]^T, \quad F_m = [f_{n-s-1}, f_{n-s-2}, \dots f_n]^T$

$G_{m+1} = [g_{n+1}, g_{n+2}, \dots g_{n+m}]^T, \quad G_m = [g_{n-m-1}, y_{n-m-2}, \dots g_n]^T$

2.2 Stability Properties

We consider the basic properties of the developed method which include order, local truncation error, consistency, zero-stability, convergent and the region of absolute stability of the method.

2.2.1 Order of the method

Let the linear operator $L\{y(x):h\}$ associated with the block (7) be defined as

$$L\{y(x):h\} = \tau^{(1)} Y_{m+1} - \tau^{(0)} Y_m - h[\mu^{(0)} F_m + \mu^{(1)} F_{m+1}] - h^2 [\gamma^{(0)} G_m + \gamma^{(1)} G_{m+1}] \tag{8}$$

expanding using Taylor series and compairing the coefficients of h we obtain

$$L\{y(x):h\} = c_0 y(x) + c_1 h y'(x) + c_2 h^2 y''(x) + \dots c_p h^p y^p(x) + \cdots \tag{9}$$

The linear operator L and the associated continuous linear multistep method (6) are said to be of order p if where $c_0 = c_1 = c_2 = \dots = c_p = 0$ and $c_{p+1} \neq 0$, c_{p+1} is called the error constant and the Local Truncation Error is given by

$$t_{n+k} = c_{p+1} h^{(p+1)} y^{(p+1)}(x_n) + 0(h^{r+2}) \tag{10}$$

2.2.2 Consistency

A block method is consistent if it has order $p \geq 1$

2.2.3 Zero-stability

A block in (7) is said to be Zero-stable, if the roots of the first characteristics polynomial $\rho(z)$ is defined by $\rho(z) = \det(z\tau^{(1)} - \tau^{(0)})$ satisfies $|z| \leq 1$ and the roots $|z| \leq 1$ have multiplicity not greater than the order of the differential equation.

2.2.4 Convergence

A method is said to be convergent if it is consistent and zero-stable

2.2.5 Linear stability

The linear stability is derived by applying the test equation $y^{(k)} = \lambda^{(k)} y_n$ to yield $y_{w+1} = m(z) y_w$, $z = \lambda h, m(z)$ is the amplification equation given by

$$m(z) = -(\tau^{(1)} - z\mu^{(1)} - z^2 \gamma^{(1)})^{-1} (\tau^{(0)} + z\mu^{(0)} + z^2 \gamma^{(0)})$$

the matrix $m(z)$ has eigenvalues $(0, 0, \dots, \varepsilon_k)$ where ε_k is called the stability function, which is a rational function with real coefficients [8]

2.2.6 Region of absolute stability

A Region of Absolute Stability (RAS) of a Linear Multistep Method (LMM) is the set, $R = \{\bar{h} : \text{for} \bar{h}\}$ where the root of the srability polynomial are absolute less than one. We use boundary locus method to get the region of absolute stability [9].

2.2.7 Specification of the Method

We developed method in the form

$$y_{n+t} = y_n + h\left[\alpha_0(t) f_n + \alpha_u(t) f_{n+u} + \alpha_{\frac{1}{2}}(t) f_{n+\frac{1}{2}} + \alpha_v(t) f_{n+v} + \alpha_1(t) f_{n+1}\right] + h^2\left[\gamma_0(t) g_n + \gamma_{\frac{1}{2}}(t) g_{n+\frac{1}{2}} + \gamma_1(t) g_{n+1}\right]$$

(7) reduced to

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}y_{n+u}\\y_{n+\frac{1}{2}}\\y_{n+v}\\y_{n+1}\end{bmatrix} = \begin{bmatrix}0&0&0&1\\0&0&0&1\\0&0&0&1\\0&0&0&1\end{bmatrix}\begin{bmatrix}y_{n-u}\\y_{n-\frac{1}{2}}\\y_{n-v}\\y_n\end{bmatrix} + h\begin{bmatrix}0&0&0&\theta_{11}\\0&0&0&\theta_{21}\\0&0&0&\theta_{31}\\0&0&0&\theta_{41}\end{bmatrix}\begin{bmatrix}f_{n-u}\\f_{n-\frac{1}{2}}\\f_{n-v}\\f_n\end{bmatrix} + h\begin{bmatrix}\theta_{12}&\theta_{13}&\theta_{14}&\theta_{15}\\\theta_{22}&\theta_{23}&\theta_{24}&\theta_{25}\\\theta_{32}&\theta_{33}&\theta_{34}&\theta_{35}\\\theta_{42}&\theta_{43}&\theta_{44}&\theta_{45}\end{bmatrix}\begin{bmatrix}f_{n+u}\\f_{n+\frac{1}{2}}\\f_{n+v}\\f_{n+1}\end{bmatrix}$$

$$+ h^2\begin{bmatrix}0&0&0&\theta_{16}\\0&0&0&\theta_{26}\\0&0&0&\theta_{36}\\0&0&0&\theta_{46}\end{bmatrix}\begin{bmatrix}g_{n-u}\\g_{n-\frac{1}{2}}\\g_{n-v}\\g_n\end{bmatrix} + h^2\begin{bmatrix}0&\theta_{17}&0&\theta_{18}\\0&\theta_{27}&0&\theta_{28}\\0&\theta_{37}&0&\theta_{38}\\0&\theta_{47}&0&\theta_{48}\end{bmatrix}\begin{bmatrix}g_{n+u}\\g_{n+\frac{1}{2}}\\g_{n+v}\\g_{n+1}\end{bmatrix}$$

where

$$\theta_{11} = -\frac{1}{420}u\frac{\begin{bmatrix}210uv^2+84u^2v-574u^3v+972u^4v-690u^5v+180u^6v+532u^2v^2\\-1218u^3v^2+912u^4v^2-240u^5v^2+35uv+35u^2-126u^3+182u^4\\-280v^2-120u^5+30u^6\end{bmatrix}}{v^2}$$

$$\theta_{12} = \frac{1}{420}u\frac{\begin{bmatrix}105u-140v-1092u^2v+840u^3v-240u^4v+630uv\\-504u^2+910u^3-720u^4+210u^5\end{bmatrix}}{(2u-1)^2(u-1)^2(u-v)}$$

$$\theta_{13} = -\frac{16}{105}u^4\frac{\begin{bmatrix}63u-105v-392uv^2-154u^2v-24u^3v+30u^4v+336u^2v^2\\-96u^3v^2+266uv-224u^2+278u^3-150u^4+140v^2+30u^5\end{bmatrix}}{(2v-1)^2(2u-1)^2}$$

$$\theta_{14} = \frac{1}{420}u^4\frac{[-126u+182u^2-120u^3+30u^4+35]}{v^2(2v-1)^2(v-1)^2(u-v)}$$

$$\theta_{15} = \frac{1}{420}u^4\frac{\begin{bmatrix}-189u+315v+1064uv^2+1078u^2v-84u^3v-450u^4v\\+180u^5v-1554u^2v^2+1008u^3v^2-240u^4v^2-1036uv\\+784u^2-1226u^3+840u^4-280v^2-210u^5\end{bmatrix}}{(v-1)^2(u-1)^2}$$

$$\theta_{16} = -\frac{1}{420}u^2\frac{\begin{bmatrix}35u-70v-273u^2v+168u^3v-40u^4v\\+210uv-126u^2+182u^3-120u^4+30u^5\end{bmatrix}}{v}$$

$$,\ \theta_{17} = -\frac{8}{105}u^4\frac{[-21u+35v+70u^2v-20u^3v-84uv+56u^2-50u^3+15u^4]}{(2v-1)(2u-1)},$$

$$\theta_{18} = -\frac{1}{420}u^4\frac{[-21u+35v+112u^2v-40u^3v-105uv+70u^2-80u^3+30u^4]}{(v-1)(u-1)}$$

$$\theta_{21} = \frac{1}{53760}\frac{\begin{bmatrix}29u+29v-768uv^2-768u^2v+11312u^2v^2\\+46uv-128u^2-128v^2\end{bmatrix}}{u^2v^2},$$

$$\theta_{22} = -\frac{1}{53760}\frac{[128v-29]}{u^2(2u-1)^2(u-1)^2(u-v)}$$

$$\theta_{23} = \frac{1}{840}\frac{\begin{bmatrix}-739u-739v-3584uv^2-3584u^2v\\+3584u^2v^2+3456uv+768u^2+768v^2+163\end{bmatrix}}{(2v-1)^2(2u-1)^2},$$

$$\theta_{24} = \frac{1}{53760}\frac{[128u-29]}{v^2(2v-1)^2(v-1)^2(u-v)}$$

$$\theta_{25} = \frac{1}{53760}\frac{\begin{bmatrix}-469u-469v-1696uv^2-1696u^2v+1232u^2v^2\\+2322uv+336u^2+336v^2+104\end{bmatrix}}{(v-1)^2(u-1)^2}$$

$$\theta_{26} = \frac{1}{53760}\frac{[-128u-128v+728uv+29]}{uv}$$

$$\theta_{27} = -\frac{1}{3360}\frac{[-152u-152v+560uv+47]}{(2v-1)(2u-1)}$$

$$\theta_{28}=-\frac{1}{53760}\frac{[-40u-40v+168uv+11]}{(v-1)(u-1)}$$

$$\theta_{31}=\frac{1}{420}v\frac{\begin{bmatrix}-84uv^2-210u^2v+574uv^3-972uv^4+690uv^5-180uv^6\\ -532u^2v^2+1218u^2v^3-912u^2v^4+240u^2v^5\\ -35uv+280u^2-35v^2+126v^3-182v^4+120v^5-30v^6\end{bmatrix}}{u^2}$$

$$\theta_{32}=-\frac{1}{420}v^4\frac{[-126v+182v^2-120v^3+30v^4+35]}{u^2(2u-1)^2(u-1)^2(u-v)}$$

$$\theta_{33}=\frac{16}{105}v^4\frac{\begin{bmatrix}105u-63v+154uv^2+392u^2v+24uv^3-30uv^4-336u^2v^2\\ +96u^2v^3-266uv-140u^2+224v^2-278v^3+150v^4-30v^5\end{bmatrix}}{(2v-1)^2(2u-1)^2}$$

$$\theta_{34}=\frac{1}{420}v\frac{\begin{bmatrix}140u-105v+1092uv^2-840uv^3+240uv^4-630uv\\ +504v^2-910v^3+720v^4-210v^5\end{bmatrix}}{(2v-1)^2(v-1)^2(u-v)}$$

$$\theta_{35}=-\frac{1}{420}v^4\frac{\begin{bmatrix}-315u+189v-1078uv^2-1064u^2v+84uv^3\\ +450uv^4-180uv^5+1554u^2v^2-1008u^2v^3\\ +240u^2v^4+1036uv+280u^2-784v^2+1226v^3-840v^4+210v^5\end{bmatrix}}{(v-1)^2(u-1)^2}$$

$$\theta_{36}=\frac{1}{420}v^2\frac{\begin{bmatrix}70u-35v+273uv^2-168uv^3+40uv^4-210uv\\ +126v^2-182v^3+120v^4-30v^5\end{bmatrix}}{u}$$

$$\theta_{37}=\frac{8}{105}v^4\frac{[-35u+21v-70uv^2+20uv^3+84uv-56v^2+50v^3-15v^4]}{(2v-1)(2u-1)}$$

$$\theta_{38}=\frac{1}{420}v^4\frac{[-35u+21v-112uv^2+40uv^3+105uv-70v^2+80v^3-30v^4]}{(v-1)(u-1)}$$

$$\theta_{41}=\frac{1}{420}\frac{[u+v-12uv^2-12u^2v+98u^2v^2+4uv-2u^2-2v^2]}{u^2v^2}$$

$$\theta_{42}=-\frac{1}{420}\frac{[2v-1]}{u^2(2u-1)^2(u-1)^2(u-v)}$$

$$\theta_{43}=\frac{16}{105}\frac{[-11u-11v-56uv^2-56u^2v+56u^2v^2+54uv+12u^2+12v^2+2]}{(2v-1)^2(2u-1)^2}$$

$$\theta_{44}=\frac{1}{420}\left[\frac{2u-1}{v^2(2v-1)^2(v-1)^2(u-v)}\right]$$

$$\theta_{45}=\frac{1}{420}\frac{[-161u-161v-184uv^2-184u^2v+98u^2v^2+348uv+84u^2+84v^2+76]}{(v-1)^2(u-1)^2}$$

$$\theta_{46}=\frac{1}{420}\frac{[-2u-2v+7uv+1]}{uv}$$

$$\theta_{47}=-\frac{8}{105}\frac{[u+v-1]}{(2v-1)(2u-1)}$$

$$\theta_{48}=-\frac{1}{420}\frac{[-5u-5v+7uv+4]}{(v-1)(u-1)}$$

We now consider some case as u and v takes different values

Case 1 $u=\frac{1}{4}$ and $v=\frac{3}{4}$, the block method of the form (7) reduces to

$$\tau(0)=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix},\tau(1)=\begin{bmatrix}0&0&0&1\\0&0&0&1\\0&0&0&1\\0&0&0&1\end{bmatrix}$$

$$\mu(0)=\begin{bmatrix}0&0&0&\frac{31}{270}\\0&0&0&\frac{65027}{552960}\\0&0&0&\frac{161}{1440}\\0&0&0&\frac{2233}{20480}\end{bmatrix},\mu(1)=\begin{bmatrix}\frac{31}{270}&\frac{256}{945}&\frac{8}{35}&\frac{256}{945}\\ \frac{3197}{552960}&\frac{11551}{60480}&-\frac{41}{1120}&-\frac{1679}{60480}\\ \frac{13}{4320}&\frac{268}{945}&\frac{4}{35}&-\frac{4}{315}\\ -\frac{57}{20480}&\frac{669}{2240}&\frac{297}{1120}&\frac{179}{2240}\end{bmatrix}$$

$$\gamma(0)=\begin{bmatrix}0&0&0&\frac{1}{252}\\0&0&0&\frac{2129}{516096}\\0&0&0&\frac{5}{1344}\\0&0&0&\frac{201}{57344}\end{bmatrix},\gamma(1)=\begin{bmatrix}-\frac{1}{252}&0&0&0\\ -\frac{239}{516096}&0&\frac{27}{2048}&0\\ -\frac{1}{4032}&0&0&0\\ \frac{9}{57344}&0&\frac{27}{2048}&0\end{bmatrix}$$

$$Y_{m+1}=\left[y_{n+1},y_{n+\frac{1}{4}},y_{n+\frac{1}{2}},y_{n+\frac{3}{4}}\right]^T,Y_m=\left[y_{n-1},y_{n-\frac{1}{4}},y_{n-\frac{1}{2}},y_n\right]^T$$

$$F_{m+1}=\left[f_{n+1},f_{n+\frac{1}{4}},f_{n+\frac{1}{2}},f_{n+\frac{3}{4}}\right]^T,\ F_m=\left[f_{n-1},f_{n-\frac{1}{4}},f_{n-\frac{1}{2}},f_n\right]^T$$

$$G_{m+1}=\left[g_{n+1},g_{n+\frac{1}{4}},g_{n+\frac{1}{2}},g_{n+\frac{3}{4}}\right]^T,\ G_m=\left[g_{n-1},g_{n-\frac{1}{4}},g_{n-\frac{1}{2}},g_n\right]^T$$

The method is of order 9. The amplification matrix $\mu(z)$ has eigenvalues:

$$\mu(z)=\left[\frac{\begin{array}{c}25z^8+295z^7-8874z^6-247536z^5\\+1127520z^4+121772160z^3+2057529600z^2\\+15989944320z+50791587840\end{array}}{\begin{array}{c}5760z^7-262656z^6+6155136z^5\\-92430720z^4+939651840z^3-6368544000z^2\\+26336378880z-50791587840\end{array}}\right]$$

With LTE =

$$\left[-\frac{1}{59256852480},-\frac{92249}{110587508372275520},-\frac{221}{215991227289600},-\frac{1}{52732233225},-\frac{45125}{2211750167445504}\right]^T.$$

The region of absolute stabilty is shown in figure 1.

Case 2: $u=\frac{3}{7}$ and $v=\frac{5}{7}$, (7) reduced to

$$\tau(0)=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix},\tau(1)=\begin{bmatrix}0&0&0&1\\0&0&0&1\\0&0&0&1\\0&0&0&1\end{bmatrix}$$

$$\mu(0)=\begin{bmatrix}0&0&0&\frac{341}{2250}\\0&0&0&\frac{30643883}{205885750}\\0&0&0&\frac{128587}{864000}\\0&0&0&\frac{6599245}{44471322}\end{bmatrix},\mu(1)=\begin{bmatrix}\frac{93}{640}&\frac{16807}{5760}&-\frac{64}{27}&\frac{16807}{108000}\\\frac{12537531}{527067520}&\frac{103423}{31360}&-\frac{47424}{16807}&-\frac{43179}{196000}\\\frac{1947}{81920}&\frac{7344659}{2211840}&-\frac{599}{216}&-\frac{3042067}{13824000}\\\frac{2319375}{105413504}&\frac{580625}{169344}&-\frac{1240000}{453789}&-\frac{6445}{42336}\end{bmatrix}$$

$$\gamma(0)=\begin{bmatrix}0&0&0&\frac{1}{150}\\0&0&0&\frac{264189}{41177150}\\0&0&0&\frac{739}{115200}\\0&0&0&\frac{94525}{14823774}\end{bmatrix},\gamma(1)=\begin{bmatrix}-\frac{1}{160}&0&\frac{8}{45}&0\\-\frac{251127}{131766880}&0&\frac{899064}{4117715}&0\\-\frac{39}{20480}&0&\frac{313}{1440}&0\\-\frac{46875}{26353376}&0&\frac{1735000}{7411887}&0\end{bmatrix}$$

$$Y_{m+1}=\left[y_{n+1},y_{n+\frac{3}{7}},y_{n+\frac{1}{2}},y_{n+\frac{5}{7}}\right]^T,Y_m=\left[y_{n-1},y_{n-\frac{3}{7}},y_{n-\frac{1}{2}},y_n\right]^T$$

$$F_{m+1}=\left[f_{n+1},f_{n+\frac{3}{7}},f_{n+\frac{1}{2}},f_{n+\frac{5}{7}}\right]^T,F_m=\left[f_{n-1},f_{n-\frac{3}{7}},f_{n-\frac{1}{2}},f_n\right]^T$$

$$G_{m+1}=\left[g_{n+1},g_{n+\frac{3}{7}},g_{n+\frac{1}{2}},g_{n+\frac{5}{7}}\right]^T,G_m=\left[g_{n-1},g_{n-\frac{3}{7}},g_{n-\frac{1}{2}},g_n\right]^T$$

The method is of order 9. The amplification matrix $\mu(z)$ has eigenvalues:

$$\mu(z)=\left[\frac{\begin{array}{c}147z^8+6832z^7+90560z^6-2386944z^5\\-61931520z^4+174489600z^3+19941949440z^2\\+249707888640z+1070176665600\end{array}}{\begin{array}{c}131072z^7-5832704z^6+133300224z^5\\-1967063040z^4+19798425600z^3-133648220160z^2\\+552924610560z-1070176665600\end{array}}\right]$$

with LTE

$$\left[\frac{1}{109734912000},-\frac{580753}{61437504651264000},\frac{10399}{1674421875000000},\frac{1}{1874923848000},\frac{121}{266355081216000}\right]^T$$

Case 3: $u=\frac{2}{5}$ and $v=\frac{3}{5}$, (7) reduced to

$$\tau(0)=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix},\tau(1)=\begin{bmatrix}0&0&0&1\\0&0&0&1\\0&0&0&1\\0&0&0&1\end{bmatrix}$$

$$\mu(0)=\begin{bmatrix}0&0&0&\frac{329}{2160}\\0&0&0&\frac{60061}{421875}\\0&0&0&\frac{13117}{92160}\\0&0&0&\frac{1423}{10000}\end{bmatrix},\mu(1)=\begin{bmatrix}\frac{329}{2160}&\frac{3125}{3024}&-\frac{48}{35}&\frac{3125}{3024}\\\frac{169}{16875}&\frac{359}{189}&-\frac{82176}{109375}&-\frac{851}{945}\\\frac{2761}{276480}&\frac{746875}{387072}&-\frac{24}{35}&-\frac{115625}{129024}\\\frac{2487}{250000}&\frac{1083}{560}&-\frac{67824}{109375}&-\frac{97}{112}\end{bmatrix}$$

$$\gamma(0)=\begin{bmatrix}0&0&0&\frac{17}{2520}\\0&0&0&\frac{1156}{196875}\\0&0&0&\frac{631}{107520}\\0&0&0&\frac{5133}{875000}\end{bmatrix},\gamma(1)=\begin{bmatrix}-\frac{17}{2520}&0&0&0\\-\frac{866}{984375}&0&\frac{3456}{15625}&0\\-\frac{283}{322560}&0&\frac{7}{32}&0\\-\frac{153}{175000}&0&\frac{3456}{15625}&0\end{bmatrix}$$

$$Y_{m+1}=\left[y_{n+1},y_{n+\frac{2}{5}},y_{n+\frac{1}{2}},y_{n+\frac{3}{5}}\right]^T,Y_m=\left[y_{n-1},y_{n-\frac{2}{5}},y_{n-\frac{1}{2}},y_n\right]^T$$

$$F_{m+1}=\left[f_{n+1},f_{n+\frac{2}{5}},f_{n+\frac{1}{2}},f_{n+\frac{3}{5}}\right]^T,F_m=\left[f_{n-1},f_{n-\frac{2}{5}},f_{n-\frac{1}{2}},f_n\right]^T$$

$$G_{m+1}=\left[g_{n+1},g_{n+\frac{2}{5}},g_{n+\frac{1}{2}},g_{n+\frac{3}{5}}\right]^T,G_m=\left[g_{n-1},g_{n-\frac{2}{5}},g_{n-\frac{1}{2}},g_n\right]^T$$

The method is of order 9. The amplification matrix $\mu(z)$ has eigenvalues:

$$\mu(z)=\left[-\frac{2584z^8+59080z^7-333075z^6-33409500z^5-234234375z^4+9637875000z^3+233798906250z^2+2147512500000z+7654500000000}{-937500z^7+41953125z^6-963750000z^5+142535515625z^4-143381250000z^3+965228906250z^2-3976087500000z+7654500000000}\right]$$

with LTE

$$\left[-\frac{1}{658409472000},-\frac{3444247}{368625027907584000},\frac{59333}{17046725197824000},-\frac{71}{11249543088000},-\frac{4309}{627908203125000}\right]^T.$$

The region of absolute stability is as shown below

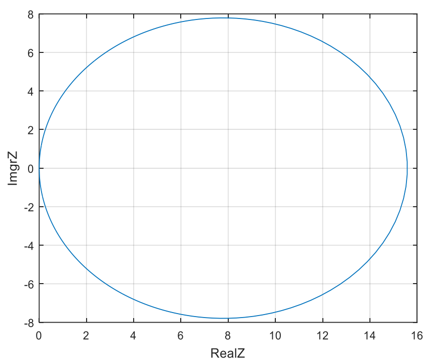

Figure 1: RAS for Case 1

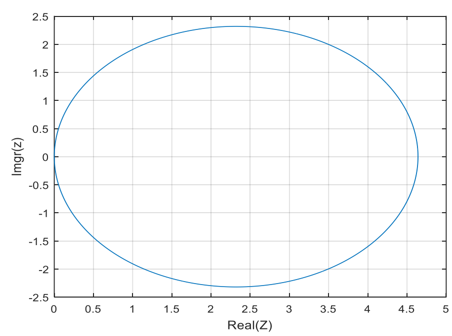

Figure 2: RAS for Case 2

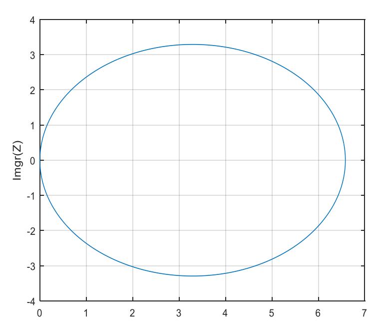

Figure 3: RAS for Case 3

3.0 Numerical Examples

We present three numerical examples to check the efficiency of the developed method. The following notations were used in the Tables below: $SDBM_1$ is Second Derivatives Block Method for case I; $SDBM_2$ is Second Derivative Block Method for case II; $SDBM_3$ is Second Derivative Block Method for case III

Example I: We consider a stiff nonlinear system of two dimensional with corresponding initial conditions

$$\begin{bmatrix} y_1'(x) \\ y_2'(x) \end{bmatrix} = \begin{bmatrix} -1002y_1(x)+1000y_2(x)^2 \\ y_1(x)-y_2(x)(1+y_2(x)) \end{bmatrix}, \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

the exact solution is

$$\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} \exp(-2x) \\ \exp(-x) \end{bmatrix}$$

The computed results are shown in Table 1. The method was found to be convergent and A-stable. Table 1 shows that the results of the new method and that of the existing methods.

Example II: Consider a stiff system of equation

$$\begin{bmatrix} y_1'(x) \\ y_2'(x) \\ y_3'(x) \end{bmatrix} = \begin{bmatrix} 998 & 1998 \\ -999 & -1999 \end{bmatrix} \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix}, \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

the exact solution is

$$\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} 4e^{-x}-3e^{-1000x} \\ -2e^{-x}+3e^{-1000x} \end{bmatrix}$$

we solve the problem in the range [0,40]. Table 2 compares the absolute error of Example II using the developed method it was found to satisfy the basic stability properties compete favourably with the existing methods.

Example III: We consider the linear problem considered in [10, 11]

$$y'(x) = -y+95z,\ y(0)=1, \quad z'(x) = -y-97z,\ z(0)=1,\ x\in[0,1]$$

The eigenvalues of the Jacobian matrix at $(x)=0$ and $\lambda_1=-2$ and $\lambda_2=-96.$ The analytical solution of the problem is given as

$$y(x)=\frac{1}{47}\left(95e^{-2x}-48e^{-96x}\right), \quad z(x)=\frac{1}{47}\left(48e^{-96x}-e^{-2x}\right).$$

The results obtained from the solution of Example 3 using stepsizes 0.03125 and 0.1, the numerical results were compared and shown in Table 3. From the results obtained it show that the new method compete favourably with the existing methods.

Table 1: Absolute errors for Example 1

x	y_i	N_1SDBM_1	N_1SDBM_2	N_1SDBM_3	$SDMM$
100	y_1	$3.1411e-(03)$	$6.0609e-(03)$	$3.5748e-(03)$	$6.7359e-(03)$
	y_2	$6.6010e-(03)$	$1.0213e-(02)$	$6.5495e-(03)$	$2.6182e-(02)$
150	y_1	$1.1609e-(06)$	$2.2161e-(06)$	$1.3049e-(06)$	$2.4686e-(06)$
	y_2	$1.4105e-(04)$	$2.1862e-(04)$	$1.3973e-(04)$	$5.3609e-(04)$
250	y_1	$3.8401e-(10)$	$7.3291e(-10)$	$4.3154e-(10)$	$8.1636e-(10)$
	y_2	$2.5698e-(06)$	$3.9833e-(06)$	$2.5457e-(06)$	$9.7597e-(06)$
500	y_1	$7.6043e-(19)$	$1.4513e-(18)$	$8.5456e-(19)$	$1.6166e-(18)$
	y_2	$1.1436e-(10)$	$1.7726e-(10)$	$1.1329e-(10)$	$4.3432e-(10)$

Table 2. Absolute error for Example 2

x	y_i	N_1SDBM_1	N_1SDBM_2	N_1SDBM_3	$SDMY(8)$	$SDMY(11)$
40	y_1	$5.347e-(16)$	$6.145e-(12)$	$3.730e-(15)$	$3.812e-(07)$	$1.022e-(07)$
	y_2	$5.352e-(16)$	$6.145e-(12)$	$3.730e-(15)$	$1.906e-(07)$	$5.111e-(08)$
70	y_1	$1.046e-(23)$	$6.317e-(21)$	$4.500e-(23)$	$8.909e-(12)$	$9.160e-(13)$
	y_2	$5.663e-(24)$	$6.293e-(21)$	$2.250e-(23)$	$4.454e-(12)$	$4.580e-(13)$
100	y_1	$6.431e-(29)$	$3.743e-(28)$	$2.586e-(28)$	$2.082e-(18)$	$6.668e-(18)$
	y_2	$3.452e-(29)$	$1.905e-(28)$	$1.293e-(28)$	$1.041e-(18)$	$3.334e-(18)$

Table 3.Absolute error for Example 3

	Method	$y_1\|error\|$	$z_1(e-02)\|error\|e-(02)$
	JK	$0.2735e-(08)$	$-0.2879e-(08)$
0.03125	$H4$	$0.2735e-(08)$	$-0.2879e-(08)$
	$AB7$	$0.2735e-(07)$	$-0.2879e-(05)$
	$SDEBDF$	$0.2735e-(07)$	$-0.2879e-(09)$
	N_1SDBM_1	$1.1102e-(16)$	$8.6736e-(19)$
	N_1SDBM_2	$1.6653e-(16)$	$1.3010e-(18)$
	N_1SDBM_3	$5.5511e-(17)$	$4.3368e-(19)$

4. Conclusion

An A-stable hybrid one step second derivative block method has been derived for the solution of stiff initial value problems, the method was found to be consistent, zero-stable and convergent. The results from the numerical example show that the developed method performed better and compete fevourably with the existing method

REFERENCES

[1] T. A. Anake, *Continuous implicit hybrid one step method for the solution of initial value problems of general second - order ordinary differential equations*. Unpublished doctoral thesis, Covenant University, Ota, Nigeria, 2011.

[2] A. O. Adesanya, B. B. Abubakar, and A. M. Alkali, Starting one step, five off step stomer cowell method by Adams Bashforth method for the solution of first order initial value problems of ordinary differential equations. *International Journal of Mathematical Sciences and Engineering*, **7**(*5*), (2013), 367-375

[3] A. O. Adesanya, B. Abdulqadir and Y. S. Ibrahim, Hybrid one-step block method for the solution of third order initial value problems of ordinary differential equations. *International Journal of Applied Mathematics and Computation*, **6**(1), (2014), 10-16.

[4] A. O. Adesanya, A.U., Fotta and R. O. Onsachi, Runge kutta collocation method for the solution of first order ordinary differential equations. *Nonlinear Analysis and Differential Equations.* **4**(1), (2016), 17-26.

[5] A. U. Fotta, T. J. Alabi and B. Abdulqadir, Block method with one hybrid point for the Solution of first order initial value problems of ordinary differential equations. *International Journal of Pure and Applied Mathematics*. **103**(3), (2015), 511-521.

[6] A. U. Fotta, A, Bello and T. I. Shelleng, Hybrid block method for the solution of first order IVPs of ODEs. *International Journal of Mathematics*. **11**(6), (2015), 60-66.

[7] Y. A. Yahaya and A. M. Badmus, A class of collocation methods for general second order ordinary differential equations. *Journal of the Nigerian Mathematical Society.* **3**(2), (2009), 119-130

[8] D. G. Yakubu and S. Markus, The efficiency of second derivative multistep methods for the numerical integration of stiff systems. *Journal of Nigerian Mathematical Society*.(2016) doi.org/10.1016/j.jnnms.2016.02.002.

[9] J. O. Ehigie, S. A. Okunuga and A. B. Sofoluwe, A class of exponentially fitted second derivative extended backward differentiation formula for solving stiff problems. Fasciculi Mathematici, **Nr 51**, (2013), 71-83

[10] L. W. Jackson and S. K. Kenue, A fourth order exponentially fitted method,SIAM Journal of Numerical Analysis, 11,(1974), 965-978.

[11] J. R. Cash, On the exponential fitted of composite multiderivative linear multistep method. SIAM Journal of Numerical Analysis, 18(5), (1981), 808-821.