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Three step explicit non-linear method for stiff and singular first order initial value problems

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Abstract

This paper considers the algorithm for the construction of explicit three step non-linear method for the solution of stiff and singular first order initial value problems using interpolation and collocation technique to give a system of non-linear equations, solving for the unknown constants using Crammer's rule and substituting the results into the rational approximate solution gives a non-linear method, implemented in predictor-corrector method. The properties of the developed method viz; convergency and stability region are investigated, the method is tested on some numerical examples and results show that it is efficient in handling stiff and singular initial value problems.

Keywords: interpolation, collocation, rational approximation, predictor-corrector, stiff problems, singular problems
AMS Subject Classification: 65L05, 65L06

1.0 Introduction

Most of the physical problems are modeled into first order ordinary differential equations, the few that are modeled into higher orders can be solved by reducing it to system of first order. Our interest is to develop a non-linear method for solving first order initial value problems in the form

$$y' = f(x,y),\ y(x_n) = \eta_0 \quad x_n \le x \le b \tag{1}$$

where $f: \mathrm{R} \times \mathrm{R}^m \to \mathrm{R}^m$ is a real value piecewise continuous function in the interval $x \in [x_n, b]$ and satisfies the existence and uniqueness theorem. Moreover the solutions to (1) include terms that decay exponentially to zero as x increases but whose derivatives are much greater in magnitude than the term itself. Problems modeled using (1) are commonly found in thermodynamics, electrostatics, theory of stellar structure, thermal behaviour of a spherical cloud of gas, isothermal gas sphere and theory of thermionic current among other fields.

Most numerical methods developed on local representation of the theoretical solution by polynomial generally perform pooly when the problem is stiff or posses singularities. Rational approximation has been effective when the solution shows poles or rapid growth on the neighborhood of points. Moreover the efficiency lies in the fact that rational function of degree (n/n) usually produce a better approximation than the Taylor polynomial of degree $2n$.

A method in [1] was based on a local representation of the theoretical solution by

$$y(x) = \frac{p_n(x)}{b+x} \tag{2}$$

where $p_n(x)$ is a polynomial of degree n. The developed method was reported to handle special singular initial value problems. [2] Using the solution in the form

$$y(x) = \frac{\sum_{k=0}^{s} a_k x^k}{1 + \sum_{k=1}^{r} b_k x^k} \tag{3}$$

Luke et al. [2] developed method for problems possessing singularities. Others scholar that have developed various method using (3) based on Taylor series expansion include [3, 4, 5].

Method of interpolation and collocation of the approximate solution is well established in developing method for linear multistep method but the application to non-linear methods has not been well established in literature. In this paper, we apply the method of collocation and interpolation of (3) in constructing non-linear method for the solution of stiff and singular first order initial value problems.

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2.0 Methodology

2.1 Mathematical Background

We consider approximation solution in the form (3) where a_k and b_k are constants to be determined. We seek approximation within the interval $a = x_n < x_1 < x_2 < \cdots < b$, $h = \frac{b - x_n}{N-1}$ is the step size. Interpolating and collocating (3) at x_{n+j}, $j = 0,1,2,...k$, gives a system of non linear equations of dimension $(k-1)\times(k-1)$, $k = r + s - 1, r$ and s are the numbers of interpolation and collocation points respectively. Writing in matrix form gives

$$AX = U \tag{4}$$

where

$$A = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_m & b_1 & b_2 & \cdots & b_N \end{bmatrix}^T$$

$$U = \begin{bmatrix} y_n & y_{n+1} & y_{n+2} & \cdots & y_{m+n} & y'_{n+1} & y'_{n+2} & \cdots & y'_{N+n} \end{bmatrix}^T$$

$$X = \begin{bmatrix} 1 & x_n & \cdots & x_n^m & y_n x_n & \cdots & y_n x_n^N \\ 1 & x_{n+1} & \cdots & x_{n+1}^m & y_{n+1}x_{n+1} & \cdots & y_{n+1}x_{n+1}^N \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 1 & x_{n+r} & \cdots & x_{n+r}^m & y_{n+r}x_{n+r} & \cdots & y_{n+r}x_{n+r}^N \\ 0 & 1 & \cdots & mx_n^{m-1} & -\left(y'_n x_n + y_n\right) & \cdots & -\left(y'_n x_n^N + Ny_n x_n^{N-1}\right) \\ 0 & 1 & \cdots & mx_{n+1}^{m-1} & -\left(y'_{n+1}x_{n+1} + y_{n+1}\right) & \cdots & -\left(y'_{n+1}x_{n+1}^N + Ny_{n+1}x_{n+1}^{N-1}\right) \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 1 & \cdots & mx_{n+s}^{m-1} & -\left(y'_{n+s}x_{n+s} + y_{n+s}\right) & \cdots & -\left(y'_{n+s}x_{n+s}^N + Ny_{n+s}x_{n+s}^{N-1}\right) \end{bmatrix}$$

Solving (4) for the constants using Crammer's rule and substitute the result into (3) after some algebraic manipulation gives the required non-linear method.

2.2 Stability Properties

Order: We associate the operator ℓ with the non-linear method defined by

$$\ell[y(x):h] = y_{n+t} - y(x_{n+t}) = 0 \tag{5}$$

where $y(x)$ is an arbitrary function continuously differentiable on $[a,b]$. Following [6], we can write terms in (5) as a Taylor series expansion about the point x to obtain the expansion

$$\ell[y(x):h] = c_0 y(x) + c_1 h y'(x) + ... + c_p h^p y^p(x) + ...$$

where the constant coefficients, c_p, $p = 0,1,2,...$ are given as

$$c_p = \frac{1}{p!}\left[\sum_{j=1}^{r} j^p \Phi_j - \frac{1}{(p-1)!}\sum_{j=1}^{r} j^{p-1}\Psi_j\right]$$

(5) has order p if

$$\ell[y(x):h] = 0\left(h^{p+1}\right), c_0 = c_1 = ... = c_p = 0, c_{p+1} \neq 0$$

Therefore c_{p+1} is the error constant and $c_{p+1}h^{p+1}y^{p+1}$ is the local truncation error (LTE).

Consistency**:** A method is to be consistent if;

(i) it has order $p \geq 1$ (ii) $\lim_{h\to 0}\left(\frac{y_{n+j} - y_n}{h}\right) = jy'_n$, $j = 1,2,3,...$

Zero stability: A method is said to be zero stable if $\lim_{h\to 0} y_{n+j} = y_n$

Convergence: A method is said to be convergence if;

(i) $\lim_{h\to 0}\left(y_{n+j} - y_n\right) \to 0$ (ii) it is consistent and zero stable

A-Stability: A method is said to be A-stable if:

$\lim_{z\to\infty}(R(z)) \leq 1$, this implies that the method is bounded. $R(z)$ is the stability function, $z = \lambda h$.

L-Stability:A method is L-stable if it is;

(i) A-stable (ii) $\lim_{z\to\infty}(R(z)) \to 0$

2.3 Specification of the Method

The following points are considered in the development of the method as shown in Table 1.

Table 1: Selected grid points

MD	IP	CP	EP	AS
y_{n+1}	0	0	1	$\frac{a}{1+bx}$
y_{n+2}	0,1	0,1	2	$\frac{a_0 + a_1 x}{1 + b_1 x + b_2 x^2}$
y_{n+3}	0,1,2	0,1,2	3	$\frac{a_0 + a_1 x + a_2 x^2}{1 + b_1 x + b_2 x^2 + b_3 x^3}$

where, $y_{n+1}-$ explicit one step method, $y_{n+2}-$ explicit two steps method, $y_{n+3}-$ explicit three steps method, MD − Method, IP − Interpolation Points, CP − Collocation Points, EP- Evaluation points, AS − Approximate Solution.
We solved for the unknown constants and substituted the results into the approximate solution, After some algebraic manipulation, the results obtained are:

$$y_{n+1}=\frac{y_n^2}{y_n-hy_n'} \tag{6}$$

$$y_{n+2}=\frac{\left[3y_ny_{n+1}^2-3y_n^2y_{n+1}-2hy_n'y_{n+1}^2-hy_n^2y_{n+1}'\right]}{\left[5y_ny_{n+1}-4y_n^2-y_{n+1}^2-hy_ny_{n+1}'+2h^2y_n'y_{n+1}'-2hy_n'y_{n+1}\right]} \tag{7}$$

$$y_{n+3}=\frac{\begin{bmatrix} y_n^2y_{n+1}^2+64y_n^2y_{n+2}^2+9y_{n+1}^2y_{n+2}^2-72y_ny_{n+1}y_{n+2}^2 \\ +54y_ny_{n+1}^2y_{n+2}-56y_n^2y_{n+1}y_{n+2}-8h^2y_n^2y_{n+1}'y_{n+2}' \\ +36hy_n'y_{n+1}y_{n+2}^2-36hy_n'y_{n+1}^2y_{n+2}+20hy_ny_{n+1}^2y_{n+2}' \\ +16hy_ny_{n+2}^2y_{n+1}'-20hy_n^2y_{n+1}y_{n+2}'-16hy_n^2y_{n+2}y_{n+1}' \\ -12h^2y_n'y_{n+1}^2y_{n+2}'-24h^2y_n'y_{n+2}^2y_{n+1}' \end{bmatrix}}{\begin{bmatrix} 9y_{n+1}y_{n+2}^2+64y_ny_{n+1}^2-63y_n^2y_{n+1}-8y_ny_{n+2}^2+72y_n^2y_{n+2} \\ -16hy_{n+1}^2y_{n+2}'-2hy_{n+2}^2y_{n+1}'-48hy_n'y_{n+1}^2-12hy_n'y_{n+2}^2 \\ -18hy_n^2y_{n+1}'-36hy_n^2y_{n+2}'-74y_ny_{n+1}y_{n+2}+52hy_ny_{n+1}y_{n+2}' \\ +20hy_ny_{n+2}y_{n+1}'-12h^2y_n'y_{n+1}y_{n+2}'-24h^2y_n'y_{n+2}y_{n+1}' \\ +24h^3y_n'y_{n+1}'y_{n+2}'-8h^2y_ny_{n+1}'y_{n+2}'+60hy_n'y_{n+1}y_{n+2} \end{bmatrix}} \tag{8}$$

The analysis of the results are as shown in Table 2.

Table 2: Stability Analysis of the Methods

MTD	Order	LTE	$R(z)$
y_{n+1}	1	$-\frac{h^2\left(5040(y_n)^2-2520y_n''y_n\right)}{5040y_n}$	$\frac{1}{1-z}$
y_{n+2}	3	$\frac{h^4\left(18(y_n'')^3+6(y_n')^2y_n^4+4(y_n''')^2y_n-3y_n''y_n^5-24y_n'y_n''y_n'''\right)}{18\left((2y_n')^2-y_n''y_n\right)}$	$-\frac{1}{2z-1}$
y_{n+3}	5	$-\frac{h^6\begin{pmatrix}800(y_n''')^4-75y_n^{13}-180(y_n'')^3y_n^6+450(y_n'')^2y_n^8-40(y_n''')^2y_n^7 \\ -12(y_n')^2y_n^{10}-6y_n''y_n^{11}+120y_n'''y_n^{10}-360y_n'y_n''y_n^9+600y_n'y_n'''y_n^8 \\ -480y_n'(y_n''')^2y_n^5-1800y_n''(y_n''')^2y_n^4+720(y_n'')^2y_n'''y_n^5+240y_n'y_n''y_n'''y_n^6\end{pmatrix}}{200\left(18(y_n'')^3+6(y_n')^2y_n^4+4(y_n''')^2y_n-3y_n''y_n^5-24y_n'y_n''y_n'''\right)}$	$-\frac{1}{3z-1}$

Mtd − method;R(z)− stability function; LTE − results of the local truncation error. Table 2 shows clearly that the methods are convergent and L-stable.

3.0 Numerical Examples

We consider the following problems to test the efficiency of the developed method.

Problem 1: Linear system

$$y'=\begin{pmatrix}-100 & 9.901\\ 0.1 & -1\end{pmatrix}y,\ y(0)=\begin{pmatrix}1\\ 10\end{pmatrix}$$

within the limit $x\in[0,10]$, $N=500$ with the exact solution

$$y(x)=\begin{pmatrix}\exp(-0.99x)\\ 10\exp(-0.99x)\end{pmatrix}$$

The eigenvalues of the problem are $\lambda_1=0.99$ and $\lambda_2=-100.01$. The following notations are used in Table 3; 3BEBDF is the 3point block extended backward differentiation formula developed in [7], 2PBOSM is the two point block one step method in [8], NMTD is the results of the new method, MaxE = $\max_{1\le i\le T}\left(\max_{1\le i\le N}(error_i)_t\right)$ where T is the total steplengh, N is the number of iterations. Time is in micro-second.

Table 3: Comparison of Results of Problem 1

h	3BEBDF MaxE	Time	2PBOSM MaxE	Time	NMTD MaxE	Time
1.0e-02	8.35e-02	4188	2.56e-03	2117	1.7960e-03	12
1.0e-03	9.10e-03	40190	2.55e-04	18134	1.8182e-04	34
1.0e-04	9.18e-04	398665	2.55e-05	79488	1.8207e-05	259
1.0e-05	9.19e-05	3969950	2.55e-06	414915	1.8198e-06	2464

Problem 2: The system in the range $0\le x\le 10$, N = 500

$$y'=\begin{pmatrix}0 & 1\\ -100 & -101\end{pmatrix}y,\ y(0)=\begin{pmatrix}1.01\\ -2\end{pmatrix}$$

with the exact solution

$$y(x) = \begin{pmatrix} 0.01e^{-100x} + e^{-x} \\ -e^{-100x} - e^{-x} \end{pmatrix}$$

The eigenvalues of this problems are $\lambda_1 = -1$ and $\lambda_2 = -100.$ Table 4 shows the comparison of results with existing results. The following notations are used in the table. RMM, RMM2 and RMM3 are the maximum error in one, 2 and 3 step methods respectively developed in [9] and NMTD is the results of the new method.

Table 4: Comparison of Resilts of Problem 2

N	RMM	RMM2	RMM3	NMTD
2560	2.59963e-03	1.18777e-03	1.21408e-03	1.5164e-05
5160	6.90425e-04	3.33834e-04	3.36962e-04	1.8558e-04
10240	1.79972e-04	8.86875e-05	8.90599e-05	9.5458e-06

Problem 3: Singular problem in the interval $0 \le x \le 1$, N = 500

$$y' = y^2,\ y(0) = 1$$

with the exact solution

$$y(x) = \frac{1}{1-x}$$

Table 5 shows the comparison of results with existing results. The following notations are used in the table ADU order 6 results of[5], NMTD is the results of the new method.

Table 5: Comparison of Resilts of Problem 3

x	0	0.2	0.4	0.6	0.8	1.0
ADU	0	6.2e-12	1.0e-10	2.0e-10	6.1e-07	8.9e-05
NMTD	0	5.5e-16	1.3e-15	2.4e-15	3.4e-14	1.6e-13

Problem 4: Consider a singular problem in the range $0.1 \le x \le 0.75$, N = 500

$$y' = 1 + y^2,\ y(0) = 1$$

with the exact solution

$$y(x) = \tan(x + \pi/4).$$

x values for $x = -\frac{\pi}{4}$ is the singular point. Table 6 shows the comparison of results with existing results. The notations are used in the table FAT is the results order 3 method of [10], NMTD is the results of the new method.

Table 6: Comparison of Resilts of Problem 4

x	0.1	0.3	0.5	0.75
FAT	-	5.5066e-04	1.7543e-03	1.1808e-01
NMTD	6.1166e-05	2.3952e-04	6.1220e-04	1.0404e-01

4.0 Conclusion

We have derived a explicit three steps method for stiff and singular problems of the form (1). We adopted the method of collocation and interpolation of rational approximate solution as the basis function to generate the multi-step non-linear method. The method developed was found to be convergent and L-stable and that explains why it performed well on this class of problems. Numerical results obtained shows that the method developed perform better than the existing ones with which we compared our results. We concluded that the newly developed method is computationally reliable.

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