Transactions of the Nigerian Association of Mathematical Physics
***Volume 4, (July, 2017*), pp67 – 74**
© Trans. of NAMP

A family of second derivative multistep methods for stiff initial value problems

A.M. Alkali, A. O. Adesanya and *R. O. Onsachi

Department of Mathematics, Modibbo Adama University of Technology, Yola, Nigeria

Abstract

This paper considers the development and implementation of a family of two step second derivative method using interpolation and collocation of polynomial approximate solution. The system of equations obtained is solved using Crammer's rule to give a continuous scheme, evaluating at the selected grid points to give discrete methods which are implemented in block form. The methods are A-stable, numerical examples show that the methods are efficient in handling stiff problems.

Key Words: second derivative, collocation, continuous scheme, discrete scheme, block form.
AMS Subject Classification: 65L05, 65L06

1 Introduction

This paper discusses numerical solution of first order ordinary differential equations (ODEs) which arises frequently in area of engineering, sciences, mechanical systems among other fields in the form

$$y'(x) = f(x, y), \quad y(x_0) = \eta, \quad a \le x \le b \tag{1.1}$$

where $f \in [a,b]$ x $\Re^m \to \Re^m$ is continuously differentiable. An equidistant set of points is defined on the integration interval $a = x_0 < x_{n+1} < \ldots < x_{n+N} = b,$ where $x_n = x_0 + nh, \ n = 0,1,\ldots, N-1, h = \frac{b-a}{N-1}$

Many of the existing numerical integration methods considered for the solution of (1.1) have poor stability properties; this makes it not suitable for large stiff system of ODE. The search for higher order methods with stronger stability properties is carried out in two main directions; use higher derivatives of the solutions and fit in additional stages, off-step points and super-future points. This leads into the research in the field of stiff system of ODE. Equation (1.1) is said to be stiff if it exact solution $y(x)$ include a term that decays exponentially to zero x increases, but whose derivatives are much greater in magnitude than the term itself.

Many authors have developed Second Derivative Multistep Methods (SDMMs) for solving (1.1), among them are; the efficiency of SDMMs for numerical integration of stiff system using four step on grid points was investigated in [1]. The development and implementation of second derivative methods was discussed in [2, 3, 4, 5].

Like traditional Runge Kutta Methods (RKMs), second derivative block multistep collocation integrators admit the addition of extra stages which introduce additional degree of freedom that increase the order of the accuracy and modify the region of absolute stability. Block methods generally preserve the traditional advantage of one step method such as RKM of being self-starting and permits easy change of step size during integration [6]. Their advantage over RKM is that they are less expensive in term of number of function evaluation per step.

Ungraded method have been introduced in [1,7] which considered all points of interpolation against the general assertion that for a method to be implemented in block method, the method increase the dimension of the block. In this paper, we improve on this method by developing methods that do not consider upgraded points and allow evaluation at all points without increasing the dimension of the block.

2.0 Methodology

2.1 Mathematical Background

Let the approximate solution $y(x)$ be in the form

$$y(x) = \sum_{n=0}^{N} a_n x^n \tag{2.1}$$

Correspondence Author: Alkali A.M., Email: oziohumat@gmail.com, Tel: 2348069661098

evaluating the first and second derivatives of (2.1) at $x = x_{n+j}\ j = 0,1,\ldots,N$ gives a system of non-linear system of equation in the form.

$$XA = U \tag{2.2}$$

where

$$A = \begin{bmatrix} a_1 & a_2 & \cdots & a_{2s} \end{bmatrix}^T, \quad U = \begin{bmatrix} y_n' & \cdots & y_{n+s}' & y_n'' & \cdots & y_n'' \end{bmatrix}^T$$

$$X = \begin{bmatrix} 1 & 2x_n & 3x^2 & 4x_n^3 & \cdots & kx_n^{k-1} \\ 1 & 2x_{n+1} & 3x_{n+1}^2 & 4x_{n+1}^3 & \cdots & kx_{n+1}^{k-1} \\ \vdots & & & & & \vdots \\ 1 & 2x_{n+s} & 3x_{n+s}^2 & 4x_{n+s}^3 & \cdots & kx_{n+s}^{k-1} \\ 0 & 2 & 6x_n & 12x_n^2 & \cdots & k(k-1)x_n^{k-2} \\ 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 & \cdots & k(k-1)x_{n+1}^{k-2} \\ \vdots & & & & & \\ 0 & 2 & 6x_{n+s} & 12x_{n+s}^2 & \cdots & k(k-1)x_{n+s}^{k-2} \end{bmatrix}$$

Imposing the following conditions on $y(x)$ in (2.1)

$$\begin{aligned} y(x_{n+j}) &= y_{n+j},\ j = 0, 1, 2, \cdots, s \\ y'(x_{n+j}) &= f_{n+j},\ j = 0, 1, 2, \cdots, s \\ y''(x_{n+j}) &= g_{n+j},\ j = 0, 1, 2, \cdots, s \end{aligned} \tag{2.3}$$

We solve for the unknown constants using Crammer's rule to obtain the continuous scheme in the form.

$$y_{n+t} = y_n + \sum_{i=1}^{2} h^i \sum_{j=0}^{s} \beta_j(t) f_{n+j}^{(i-1)} \tag{2.4}$$

where $\alpha_j(x),\ \beta_j(x)$, are polynomials of degree $2s$, $y_{n+j} = y(x_{n+j})$, $f^{(i)} = \frac{\partial^i f}{\partial x^i}$, $f_{n+j} = f(x_{n+j}, y_{n+j})$, $th = x - x_n$. Evaluating (2.4) at the grid points gives the discrete methods which is implemented in block form

$$\zeta^{(1)}Y_{m+1} = \zeta^{(0)}Y_m + h\left(\eta^{(0)}F^{(m)} + \eta^{(0)}F_{m+1}\right) + h^2\left(\gamma^{(0)}G_m + \gamma^{(1)}G_{m+1}\right) \tag{2.5}$$

where

$$Y_{m+1} = \begin{bmatrix} y_{n+1} & y_{n+2} & \cdots & y_{n+s} \end{bmatrix}^T, \quad F_m = \begin{bmatrix} f_{n-1} & f_{n-2} & \cdots & f_n \end{bmatrix}^T$$

$$F_{m+1} = \begin{bmatrix} f_{n+1} & f_{n+2} & \cdots & f_{n+s} \end{bmatrix}^T, \quad Y_m = \begin{bmatrix} y_{n-1} & y_{n-2} & \cdots & y_n \end{bmatrix}^T$$

$$G_m = \begin{bmatrix} g_{n-1} & g_{n-2} & \cdots & g_n \end{bmatrix}^T, \quad G_{m+1} = \begin{bmatrix} g_{n+1} & g_{n+2} & \cdots & g_{n+s} \end{bmatrix}^T$$

2.2 *Stability Properties*

2.2.1. Order of the Method

The operation ℓ is associated with the linear method defined by

$$\ell[y(x):h] = y_{n+t} + y_n - \sum_{i=1}^{2} h^i \sum_{j=0}^{s} \beta_j(t) f_{n+j}^{(i)} y_{n+k} \tag{2.6}$$

where $y(x)$ is an arbitrary function, continuously differentiable on an interval of integration. Equation (2.6) can be written in Taylor expansion about the point x to obtain

$$\ell[y(x):h] = c_0 y(x) + c_1 h y'(x) + c_2 h^2 y''(x) + \cdots + c_p h^p y^{(p)}(x) + \ldots$$

where

$$c_p = \frac{1}{p!}\left[\sum_{j=1}^{r} j^p \theta_j - \frac{1}{(p-1)!}\sum_{j=1}^{r} j^{p-1}\gamma_j\right]$$

Equation (2.6) is of order p if

$$\ell[y(x):h] = 0\left(h^{p+1}\right),\ c_0 = c_1 = \cdots = c_p = 0,\ \ c_{p+1} \neq 0$$

Hence c_{p+1} is called the error constant and $c_{p+1}h^{p+1}y^{(p+1)}(x)$ is called the local truncation error (LTE) [1].

2.2.2. Consistency

Definition 1: (2.5) is said to be consistent if it has order $p \geq 1$.

2.2.3. Zero Stability

Definition 2.1: (2.5) is said to be zero stable if the roots $z_s,\ s = 1, 2, 3, \cdots, n$ of the first characteristics polynomial $\rho(z)$, defined by

$$\rho(z) = \det\left[z\zeta_1^{(1)} - \zeta_2^{(0)}\right] = 0$$

2.2.4 Convergence

Definition 2.2: A method is said to be convergence if it is consistent and zero stable

2.2.5. Linear Stability

Definition 3 The linear stability is derived by applying the test equation $y^{(k)} = \lambda^{(k)} y_n$ to yield $y_{m+1} = \mu(z) y_m,\ z = \lambda h,\ \ \mu(z)$ is the amplification equation given by

$$\mu(z) = -\left(\zeta^{(1)} - z\eta^{(1)} - z^2\gamma^{(1)}\right)^{-1}\left(\zeta^{(0)} + z\eta^{(0)} + z^2\gamma^{(0)}\right)$$

the matrix $\mu(z)$ has eigenvalues $(0, 0, \cdots, \xi_k)$ where ξ_k is called the stability function which is a rational function with real coefficient [8].

2.2.6 Region of Absolute Stability (RAS)

Definition 2.3: A Region of absolute stability (RAS) of a LMM is the set

$R = \{\bar{h} : for\ \bar{h}$ where the root of the stability polynomial are absolute less than one} [8].

2.3 Specification of the Method

We consider method in the form

$$y_{n+t} = y_n + h\left[\alpha_0(t) f_n + \alpha_u(t) f_{n+u} + \alpha_v(t) f_{n+v}\right] + h^2\begin{bmatrix}\beta_0(t) g_n + \beta_u(t) g_{n+u} \\ +\beta_v(t) g_{n+u}\end{bmatrix} \quad (2.7)$$

where u and v are real numbers, $u < v$.

$$\alpha_0(t) = \frac{1}{30} t \frac{\begin{bmatrix}-30t^2v + 24t^4u + 42t^4v + 30t^2v^3 - 60t^3v^2 + 15t^3 - 10t^5 + 30v^3 - 60t^3uv \\ +10t^5uv + 40t^2uv^2 + 15t^3uv^3 - 24t^4uv^2\end{bmatrix}}{v^3}$$

$$\alpha_u(t) = -\frac{1}{30} t^3 \frac{\begin{bmatrix}-30t^3u - 165tv^3 - 50t^3v - 60tv^5 + 120uv^3 + 40uv^5 + 156t^2v^2 + 42t^2v^4 \\ -10t^3v^3 + 75tv - 30t^2 - 50v^2 + 60v^4 + 30v^6 - 210tuv^2 + 120t^2uv - 15tuv^4 \\ +15tuv^6 - 48t^2uv^3 + 30t^3uv^2 - 24t^2uv^5 + 10t^3uv^4\end{bmatrix}}{\left(v^2+1\right)^3}$$

$$\alpha_v(t) = -\frac{1}{30} t^3 \frac{\begin{bmatrix}15t - 30v + 24t^2u - 15tv^2 + 42t^2v - 210tv^4 + 40uv^2 + 120uv^4 + 156t^2v^3 \\ -30t^3v^2 - 30t^2v^5 + 20t^2v^5 + 20t^3v^4 - 10t^3 - 60v^3 + 50v^5 - 165tuv^3 \\ +10t^3uv + 75tuv^5 + 48t^2uv^2 - 120t^2uv^2 - 50t^3uv^3 - 60tuv\end{bmatrix}}{v^3\left(v^2+1\right)^3}$$

$$\beta_0(t) = -\frac{1}{60} t^2 \frac{-24t^3u - 24t^3v + 15t^2v^2 + 40tv - 15t^2 + 10t^4 - 30v^2 - 40tuv^2 + 60t^2uv}{v^2}$$

$$\beta_u(t) = -\frac{1}{60} t^3 \frac{\begin{bmatrix}2t^2u - 75tv^2 + 48t^2v + 15tv^4 + 20uv^2 - 20uv^4 - 24t^2v^3 + 10t^3v^2 \\ -10t^3 + 40v^3 + 60tuv^3 + 20t^3uv - 60t^2uv^2 - 30tuv\end{bmatrix}}{\left(v^2+1\right)^2}$$

$$\beta_v(t) = -\frac{1}{60} t^3 \frac{\begin{bmatrix}15t - 20v + 24t^2u - 75tv^2 + 60t^2v + 40uv^2 - 12t^2v^3 + 10t^3v^2 - 10t^3 \\ +20v^3 + 30tuv^3 + 20t^3uv - 48t^2uv^2 - 60tuv\end{bmatrix}}{v^2\left(v^2+1\right)^2}$$

The order of the method is 6 with

$$LTE = \frac{h^7t^3}{151200}\left(-70t^3u - 70t^3v + 42t^2v^2 + 105tv - 42t^2 + 30t^4 - 70v^2 - 105tuv^2 + 168t^2uv\right)$$

We consider five cases that belong to the family of the method as specify below;

2.3.1 Case One*:* We consider one step equidistant method (i.e. $u = \frac{1}{2},\ v = 1)$, parameters in (2.5) become

$Y_{m+1}=\begin{bmatrix} y_{n+\frac{1}{2}} & y_{n+1}\end{bmatrix}^T,\quad F_m=\begin{bmatrix} f_{n-1} & f_n\end{bmatrix}^T,\quad F_{m+1}=\begin{bmatrix} f_{n+\frac{1}{2}} & f_{n+1}\end{bmatrix}^T$

$Y_m=\begin{bmatrix} y_{n-1} & y_n\end{bmatrix}^T,\ G_m=\begin{bmatrix} g_{n-1} & g_n\end{bmatrix}^T,\ G_{m+1}=\begin{bmatrix} g_{n+\frac{1}{2}} & g_{n+1}\end{bmatrix}^T$

$\zeta^{(1)}=\begin{bmatrix} 1 & 0\\ 0 & 1\end{bmatrix},\quad \zeta^{(0)}=\begin{bmatrix} 0 & 1\\ 0 & 1\end{bmatrix},\ \eta^{(0)}=\begin{bmatrix} 0 & \frac{101}{480}\\ 0 & \frac{7}{30}\end{bmatrix},\ \eta^{(1)}=\begin{bmatrix} \frac{4}{15} & \frac{11}{480}\\ \frac{8}{15} & \frac{7}{30}\end{bmatrix}$

$\gamma^{(0)}=\begin{bmatrix} 0 & \frac{13}{960}\\ 0 & \frac{1}{60}\end{bmatrix},\quad \gamma^{(1)}=\begin{bmatrix} -\frac{1}{24} & -\frac{1}{320}\\ 0 & -\frac{1}{60}\end{bmatrix}$

For zero stability, $\rho(z)=\left|z\begin{bmatrix} 1 & 0\\ 0 & 1\end{bmatrix}-\begin{bmatrix} 0 & 1\\ 0 & 1\end{bmatrix}\right|=z^2-z=0,$ solving for z gives $z_1=0,\ z_2=1.$ The block method is consistent and zero stable, therefore, it is convergent. The stability function $\xi=-\dfrac{720z+156z^2+18z^3+z^4+1440}{-720z+156z^2-18z^3+z^4+1440}$ with LTE= $\begin{bmatrix} -\frac{17}{604800} & -\frac{17}{302400}\end{bmatrix}^T$ The region of absolute stability is shown in Fig. 1

2.3.2 Case Two*:* We consider two step equidistant points $(i.e.\ u=1,\ v=2)$, parameters in (2.5) become

$Y_{m+1}=\begin{bmatrix} y_{n+1} & y_{n+2}\end{bmatrix}^T,\ F_m=\begin{bmatrix} f_{n-1} & f_n\end{bmatrix}^T,\ F_{m+1}=\begin{bmatrix} f_{n+1} & f_{n+2}\end{bmatrix}^T$

$Y_m=\begin{bmatrix} y_{n-1} & y_n\end{bmatrix}^T,\ G_m=\begin{bmatrix} g_{n-1} & g_n\end{bmatrix}^T,\ G_{m+1}=\begin{bmatrix} g_{n+1} & g_{n+2}\end{bmatrix}^T$

$\zeta^{(1)}=\begin{bmatrix} 1 & 0\\ 0 & 1\end{bmatrix},\quad \zeta^{(0)}=\begin{bmatrix} 0 & 1\\ 0 & 1\end{bmatrix},\ \eta^{(0)}=\begin{bmatrix} 0 & \frac{101}{240}\\ 0 & \frac{7}{15}\end{bmatrix},\ \eta^{(1)}=\begin{bmatrix} \frac{8}{15} & \frac{11}{240}\\ \frac{16}{15} & \frac{7}{15}\end{bmatrix}$

$\gamma^{(0)}=\begin{bmatrix} 0 & \frac{13}{240}\\ 0 & \frac{1}{15}\end{bmatrix},\quad \gamma^{(1)}=\begin{bmatrix} -\frac{1}{6} & -\frac{1}{80}\\ 0 & -\frac{1}{15}\end{bmatrix}$

For zero stability, $\rho(z)=\left|z\begin{bmatrix} 1 & 0\\ 0 & 1\end{bmatrix}-\begin{bmatrix} 0 & 1\\ 0 & 1\end{bmatrix}\right|=z^2-z=0,$ solving for z gives $z_1=0,\ z_2=1.$ The block method is consistent and zero stable, therefore, it is convergent. The stability function $\xi=-\dfrac{90z+39z^2+9z^3+z^4+90}{-90z+39z^2-9z^3+z^4+90}$ with LTE= $\begin{bmatrix} -\frac{13}{9450} & -\frac{13}{4725}\end{bmatrix}^T$. The region of absolute stability is shown in Fig. 2

2.3.3. Case Three*:* We consider a case when u and v are even numbers $(i.e. u=2, v=4)$, parameters in (2.5) reduced to

$Y_{m+1}=\begin{bmatrix} y_{n+2} & y_{n+4}\end{bmatrix}^T,\ F_m=\begin{bmatrix} f_{n-1} & f_n\end{bmatrix}^T,\ F_{m+1}=\begin{bmatrix} f_{n+2} & f_{n+4}\end{bmatrix}^T$

$Y_m=\begin{bmatrix} y_{n-1} & y_n\end{bmatrix}^T,\ G_m=\begin{bmatrix} g_{n-1} & g_n\end{bmatrix}^T,\ G_{m+1}=\begin{bmatrix} g_{n+2} & g_{n+4}\end{bmatrix}^T$

$\zeta^{(1)}=\begin{bmatrix} 1 & 0\\ 0 & 1\end{bmatrix},\quad \zeta^{(0)}=\begin{bmatrix} 0 & 1\\ 0 & 1\end{bmatrix},\ \eta^{(0)}=\begin{bmatrix} 0 & \frac{101}{120}\\ 0 & \frac{14}{15}\end{bmatrix},\ \eta^{(1)}=\begin{bmatrix} \frac{16}{15} & \frac{11}{120}\\ \frac{32}{15} & \frac{14}{15}\end{bmatrix}$

$\gamma^{(0)}=\begin{bmatrix} 0 & \frac{13}{60}\\ 0 & \frac{4}{15}\end{bmatrix},\quad \gamma^{(1)}=\begin{bmatrix} -\frac{2}{3} & -\frac{1}{20}\\ 0 & -\frac{4}{15}\end{bmatrix}$

For zero stability, $\rho(z)=\left|z\begin{bmatrix} 1 & 0\\ 0 & 1\end{bmatrix}-\begin{bmatrix} 0 & 1\\ 0 & 1\end{bmatrix}\right|=z^2-z=0,$ solving for z gives $z_1=0,\ z_2=1.$ The block method is consistent and zero stable, therefore, it is convergent. The stability function $\xi=-\dfrac{90z+39z^2+9z^3+z^4+90}{-90z+39z^2-9z^3+z^4+90}$ with LTE= $\begin{bmatrix} -\frac{496}{4725} & -\frac{992}{4725}\end{bmatrix}^T$. The region of absolute stability is shown in Fig. 3.

2.3.4 Case Four: We consider a case when u and v are odd numbers $(i.e. u=1, v=3)$, parameters in (2.5) reduced to

$Y_{m+1}=\begin{bmatrix} y_{n+1} & y_{n+3}\end{bmatrix}^T,\ F_m=\begin{bmatrix} f_{n-1} & f_n\end{bmatrix}^T,\ F_{m+1}=\begin{bmatrix} f_{n+1} & f_{n+3}\end{bmatrix}^T$

$Y_m=\begin{bmatrix} y_{n-1} & y_n\end{bmatrix}^T,\ G_m=\begin{bmatrix} g_{n-1} & g_n\end{bmatrix}^T,\ G_{m+1}=\begin{bmatrix} g_{n+1} & g_{n+3}\end{bmatrix}^T$

$\zeta^{(1)}=\begin{bmatrix} 1 & 0\\ 0 & 1\end{bmatrix},\quad \zeta^{(0)}=\begin{bmatrix} 0 & 1\\ 0 & 1\end{bmatrix},\ \eta^{(0)}=\begin{bmatrix} 0 & \frac{182}{405}\\ 0 & \frac{6}{5}\end{bmatrix},\ \eta^{(1)}=\begin{bmatrix} \frac{131}{240} & \frac{31}{6480}\\ \frac{81}{80} & \frac{63}{80}\end{bmatrix}$

$$\gamma^{(0)}=\begin{bmatrix}0 & \frac{17}{240}\\ 0 & \frac{3}{10}\end{bmatrix},\quad \gamma^{(1)}=\begin{bmatrix}-\frac{29}{240} & -\frac{1}{432}\\ \frac{81}{80} & -\frac{3}{16}\end{bmatrix}$$

For zero stability, $\rho(z)=\left|z\begin{bmatrix}1&0\\0&1\end{bmatrix}-\begin{bmatrix}0&1\\0&1\end{bmatrix}\right|=z^2-z=0,$ solving for z gives $z_1=0,\ z_2=1$. The block method is consistent and zero stable, therefore, it is convergent. The stability function $\xi=-\frac{4\left(50z+37z^2+15z^3+3z^4+30\right)}{-160z+88z^2-24z^3+3z^4+120}$ with LTE= $\begin{bmatrix}-\frac{67}{15120} & -\frac{9}{560}\end{bmatrix}^T$. The region of absolute stability is shown in Fig. 4.

2.3.5 Case Five*:* We consider a case when u and v are rational numbers $\left(half\ \ step\ \ method\ \ u=\frac{1}{4},v=\frac{1}{2}\right)$, parameters in (2.5) reduced to

$$Y_{m+1}=\left[y_{n+\frac{1}{4}}\quad y_{n+\frac{1}{2}}\right]^T,\ \ F_m=\left[f_{n-1}\quad f_n\right]^T,\ \ F_{m+1}=\left[f_{n+\frac{1}{4}}\quad f_{n+\frac{1}{2}}\right]^T$$

$$Y_m=\left[y_{n-1}\quad y_n\right]^T,\ \ G_m=\left[g_{n-1}\quad g_n\right]^T,\ \ G_{m+1}=\left[g_{n+\frac{1}{4}}\quad g_{n+\frac{1}{2}}\right]^T$$

$$\zeta^{(1)}=\begin{bmatrix}1&0\\0&1\end{bmatrix},\quad \zeta^{(0)}=\begin{bmatrix}0&1\\0&1\end{bmatrix},\quad \eta^{(0)}=\begin{bmatrix}0&\frac{101}{960}\\0&\frac{7}{60}\end{bmatrix},\quad \eta^{(1)}=\begin{bmatrix}\frac{2}{15}&\frac{11}{960}\\\frac{4}{15}&\frac{7}{60}\end{bmatrix}$$

$$\gamma^{(0)}=\begin{bmatrix}0&\frac{13}{3840}\\0&\frac{1}{240}\end{bmatrix},\quad \gamma^{(1)}=\begin{bmatrix}-\frac{1}{96}&-\frac{1}{1280}\\0&-\frac{1}{240}\end{bmatrix}$$

For zero stability, $\rho(z)=\left|z\begin{bmatrix}1&0\\0&1\end{bmatrix}-\begin{bmatrix}0&1\\0&1\end{bmatrix}\right|=z^2-z=0,$ solving for z gives $z_1=0,\ z_2=1$. The block method is consistent and zero stable, therefore, it is convergent. The stability function $\xi=-\frac{5760z+624z^2+36z^3+z^4+23040}{-5760z+624z^2-36z^3+z^4+23040}$ with LTE= $\begin{bmatrix}-\frac{59}{77414400} & -\frac{59}{38707200}\end{bmatrix}^T$. The region of absolute stability is shown in Fig. 5.

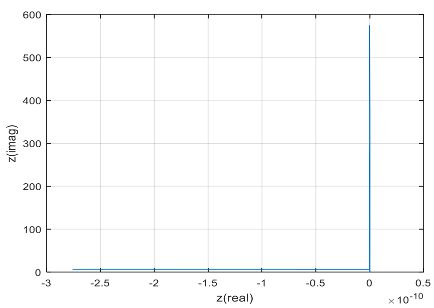

Figure 1: RAS for Case 1

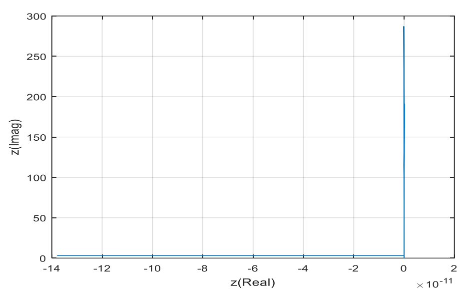

Figure 2: RAS for Case 2

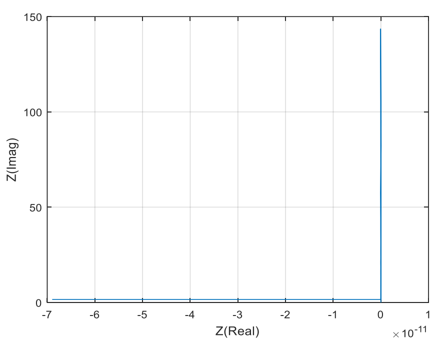

Figure 3: RAS for Case 3

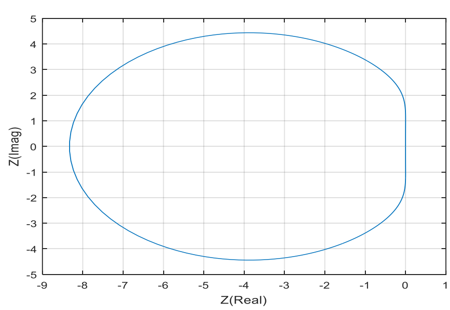

Figure 4: RAS for Case 4

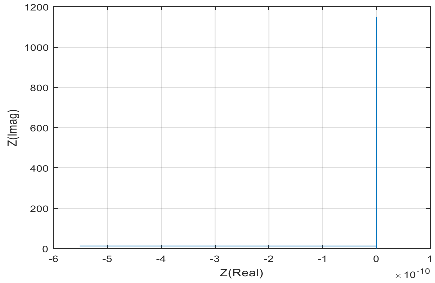

Figure 5: RAS for Case 5

3.0 Numerical Experiments

The following notations are used in the results $abs(y-y_n)_i$ means the absolute difference between exact solution and computed solution for case i. The results of case 2,3 and 4 gave the same result, hence results of case 4 is given.

Example 3.1: Four dimensional problems in [8]

$$\begin{pmatrix} y_1'(x) \\ y_2'(x) \\ y_3'(x) \\ y_4'(x) \end{pmatrix} = \begin{pmatrix} -10^4 y_1(x)+100y_2(x)-10y_3(x)+y_4(x) \\ -100y_2(x)+10y_3(x)-10y_4(x) \\ -y_3(x)+10y_4(x) \\ -0.1y_4(x) \end{pmatrix}, \quad \begin{pmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \\ y_4(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

within the range $0 \le x \le 1.$ The eigenvalues of the Jacobian matrix

$\lambda_1 = -0.1,\ \lambda_2 = -1.0, \lambda_3 = -1000\ \ and\ \ \lambda_{41} = -10000$. The exact solution is given as

$$y_1(x) = -\frac{89990090}{89990100}e^{-0.1x} + \frac{818090}{89901009}e^{-x} + \frac{9989911}{899010090}e^{-1000x} + \frac{8907111917\,9}{8999010009\,0}e^{-10000x}$$

$$y_2(x) = \frac{9100}{89991}e^{-0.1x} - \frac{910}{8991}e^{-x} + \frac{9989911}{9989001}e^{-1000x}$$

$$y_3(x) = \frac{100}{9}e^{-0.1x} - \frac{91}{9}e^{-x}$$

$$y_4(x) = e^{-0.1x}$$

Table 1 shows the comparison with the existing results. The following notations are used in Table 1. SDEBDF represent the method of [8]

Table 1: Results of Example 3.1

h	y_i	$abs(y-yn)_1$	$abs(y-yn)_4$	$abs(y-yn)_5$	SDEBDF
0.1	y_1	7.3900e – 04	1.3358e – 10	1.3265e – 10	2.25e – 10
	y_2	3.4694e – 18	3.4694e – 18	1.3010e – 16	2.29e – 09
	y_3	6.6613e – 16	2.2204e – 16	1.4433e – 14	2.50e – 07
	y_4	5.5511e – 17	5.5511e – 17	1.3045e – 15	2.06e – 08
0.05	y_1	1.0735e – 10	1.3188e – 10	1.3005e – 10	5.31e – 12
	y_2	1.7347e – 17	5.2042e – 18	8.2399e – 16	7.27e – 11
	y_3	1.7764e – 15	4.4409e – 16	9.0150e – 14	5.90e – 09
	y_4	1.9429e – 16	8.3267e – 17	7.9936e – 15	1.34e – 09

Example 3.2 Consider nonlinear system on the range $0 \le x \le 2$

$$\begin{pmatrix} y_1'(x) \\ y_2'(x) \\ y_3'(x) \end{pmatrix} = \begin{pmatrix} -0.013y_2 - 1000y_1y_2 - 2500y_1y_3 \\ -0.013y_2 - 1000y_1y_2 \\ -2500y_1y_3 \end{pmatrix}, \quad \begin{pmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

with the solution at $x = 2$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} -0.3616933169e-5 \\ 0.9815029948230 \\ 1.018493388244 \end{pmatrix}$$

This problem was solved using block hybrid second derivative method of order six developed $(\mathrm{BHSDM})_6$ in [9]. Results in Table 3 show that our methods compete favorably with the existing method.

Table 2: Results of Example 3.2

h	x	y_i	$abs(y-yn)_1$	$abs(y-yn)_4$	$abs(y-yn)_5$	$BMSDM_6$
$\frac{1}{8}$	2	y_1	9.8387e – 07	3.7679e – 10	6.7037e – 12	9.850e – 07
		y_2	5.4917e – 05	7.4198e – 05	1.3292e – 06	4.939e – 05
		y_3	5.3933e – 05	7.4199e – 05	1.3292e – 06	4.840e – 05
$\frac{1}{16}$	2	y_1	1.9223e – 08	4.7569e – 11	8.1220e – 13	1.927e – 08
		y_2	5.1836e – 06	9.3516e – 06	1.6660e – 07	4.198e – 06
		y_3	5.1645e – 06	9.3516e – 06	1.6660e – 07	4.179e – 06

Example 3.3 Nonlinear Kap's problem within the interval $0 \le x \le 10$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} -1002y_1 + 1000y_2^2 \\ y_1 - y_2(1+y_2) \end{bmatrix}, \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

with the exact solution

$$\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} e^{-2x} \\ e^{-x} \end{bmatrix}$$

The result of the problem is shown in Table 3. The following notations are used; SDMM_{10} and SDMM_{14} are the second derivative multistep method of order 10 and 14 respectively developed in [1]. It must be noted that the value of h in case five is higher than other cases.

Table 3: Results of Example 3.3

h	y_i	$abs(y-yn)_1$	$abs(y-yn)_4$	$abs(y-yn)_5$	$CDMM_{10}$	$CDMM_{14}$
	y_2	5.905e – 03	5.083e – 02	7.951e – 02	1.941e – 02	2.618e – 02
150	y_1	1.087e – 06	8.878e – 06	1.571e – 05	6.388e – 04	2.468e – 04
	y_2	1.258e – 04	1.178e – 03	1.962e – 03	6.113e – 03	5.360e – 04
250	y_1	3.596e – 10	2.945e – 09	5.225e – 09	1.789e – 05	8.163e – 10
	y_2	2.292e – 06	2.150e – 05	3.586e – 05	1.227e – 03	9.759e – 06
500	y_1	7.122e – 19	5.833e – 18	1.034e – 17	1.601e – 09	1.616e – 18
	y_2	1.020e – 10	9.571e – 10	1.596e – 09	1.526e – 05	4.343e – 10

Example 3.4 Linear problem

$$y'(x) = \begin{bmatrix} -1 & 95 \\ -1 & -97 \end{bmatrix} y, \quad y(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

within the range $0 \le x \le 10$, with the initial condition

$$y(0) = \frac{1}{47}\begin{bmatrix} 95e^{-2x} - 48e^{-95x} \\ 48e^{-96x} - e^{-2x} \end{bmatrix}$$

The eigenvalues of the Jacobian at x =0 are -2 and -96. The results are shown in Table 4. The following notations are used, BHSDM_6 is the block hybrid second derivative method of order 6 developed in [9], SDEBDF is second derivative extended backward differentiation formula developed in [8]. F^6 is the result of exponentially fitted scheme of order six developed in [10]. The dash (-) in Table 4 indicate that the result are not given for those step size. Results of Table 4 show that our method competes favourably with the existing methods.

Table 4: Results of Example 3.4

h	y_i	$abs(y-yn)_1$	$abs(y-yn)_4$	$abs(y-yn)_5$
0.125	y_1	9.0497e – 11	5.4647e – 07	3.3359e – 12
	y_2	1.2884e – 10	5.7780e – 07	3.5115e – 14
0.0625	y_1	3.4540e – 12	5.4678e – 10	5.2736e – 14
	y_2	3.6358e – 14	5.7556e – 12	5.5555e – 16
0.03125	y_1	5.3901e – 14	8.9981e – 12	6.6613e – 16
	y_2	5.6812e – 16	9.4715e – 14	6.9389e – 18

Table 4: Results of Example 4 (continuation)

h	y_i	$BHSDM_6$	F^6	$SDEBDF$
0.125	y_1	9e – 11	-	-
	y_2	1e – 06	-	-
0.0625	y_1	3e – 12	1.3e – 09	3.4e – 09
	y_2	3e – 10	4.7e – 07	3.6e – 07
0.03125	y_1	5e – 14	6.0e – 11	3.4e – 09
	y_2	5e – 12	5.0e – 08	3.5e – 07

4 Conclusion

We have developed a family of second derivative methods for the solution of stiff initial value problems. Five cases are considered, the results shows that the family can be divided into two lineages namely:

(i) equal interval as shown in cases 1,2,3 and 5. This is equally shown in graph of RAS

(ii) non equal interval as shown in case 4.

The results show that despite the low order of the method, it gives better stability properties and approximation than the upgraded method of [1,7], moreover, it is observed that as the step length is decreasing, the better the accuracy, this supports the claim of [11]

References

[1] D. G. Yakubu and S. Marcus, Second derivative of high order accuracy methods for the numerical integration of stiff initial value problems, *Afr. Mat.* (2016) doi:10.1007/513370/513370-015-0389-5

[2] C. E. Abhulimen and S. A. Okunuga (2008), Exponentially fitted second derivative multistep method stiff initial value problem for ODE's *J. Eng. Sci. Appl*. **5**, (2008), 36-47

[3] W. H. Enright, Second derivative multistep method for stiff ordinary differential equations. *SIAM J. Numer Anal.*, **11**(2), (1974), 321-331

[4] G. K. Gupta, Implementing second derivative multistep methods using the Nordsieck polynomial representation. *J. Maths Comput* **,** **32**(141), (1978), 13-18

[5] J. C. Butcher and G. Hojjati, Second derivative methods with Runge Kutta stability. Numer. Algorithms, **40**, (2005), 415-29

[6] J. D. Lambert, *Computational methods in ordinary differential equations*. New York: John Willy and Sons, 1973

[7] D. G. Yakubu and S. Marcus, The efficiency of second derivative methods for the numerical integration of stiff systems. *Journal of the Nigerian Mathematical Society*. *doi.org/10.1016/j.jnnms.*2016.02.002

[8] J. O. Ehigie, S. A. Okunuga and A. B. Sofoluwe, A class of exponentially fitted second derivative extended backward differentiation formula for solving stiff problems. Fasciculi Mathematici, **Nr 51**, 71-83

[9] F. F. Ngwane and S. N. Jator, Block hybrid second derivative multistep methods for stiff systems, *International* Journal *of Pure and Applied Mathematics*, **80**(4), (2012), 543-559

[10] C. E. Abhulimen and G. E. Omeike., A Sixth – order exponentially fitted scheme for the numerical solution of systems of ODEs. *J. App. Math. Bioinfo.* **1**(1), (20110, 175-186.

[11] T. A. Anake, D. O. Awoyemi and A. O. Adesanya, One step implicit hybrid block method for the direct solution of general second order ODEs, *International Journal of Applied Mathematics*, **42**(2), (2012), IJAM_42_4_04