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Variation Iteration Decomposition Method for Analytic Solution of Gas Dynamic Equation

Ebimene J, Mamadu[1] and Joshua S. Apanapudor[2]

[1] Department of Mathematics, University of Ilorin, P.M.B. 1515, Ilorin, Nigeria
[2]Department of Mathematics and Computer Science, Delta State University, Abraka, Nigeria.

Abstract

This paper consider the mechanism of the variation iteration decomposition method (VIDM) for obtaining the exact solution of gas dynamic. The proposed method is an elegant mixture of the variation iteration method and the decomposition method. The method is highly effective and reliable for both homogeneous and inhomogeneous cases of the gas dynamic equation since discretization, linearization or perturbation are not recognized.

Keywords: Variation iteration method, Adomian polynomials, Gas dynamic equation, Decomposition method.

1.0 Introduction.

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In this paper, we apply the variation iteration decomposition method for obtaining the exact solution of the gas dynamics equation of the form

$$\frac{\partial u}{\partial t}+\frac{1}{2}\frac{\partial(u^2)}{\partial x}-u(1-u)=f(x,t), \quad 0\le x\le 1,\ t>0. \qquad (1)$$

Most conventional analytic methods for (1) are relatively restricted and do not have solutions in compact form. In recent years, several numerical algorithms have been developed for (1) by various researchers to explore its analytic solution; the homotopy perturbation (HPM) [1-2], the variation iteration method (VIM) [3-4], the decomposition method [5] etc. The motivation of this paper is to apply the variation iteration decomposition method (VIDM) for obtaining the exact solution of (1). The method is an elegant combination of the variation iteration method and the decomposition method [6]. We first formulate the correction functional for (1) and determine the Lagrange multiplier optimally via variation theory [7]. The Adomian polynomials, $A_n,\ n\ge 0,$ are introduced in the correction functional and determined using the specified algorithm [5,8-11]. The variation iteration decomposition method gives the solution in a compact series which converges rapidly to the exact solution after few iterations. The method requires no discretization, linearization or perturbation. Also, it requires less computational effort with no rounding-off and computational errors. The variation iteration decomposition method is compared with the homotopy perturbation method available in the literature for efficiency and accurancy.

This paper is organized as follows. Section 2 present the concept of variation iteration method. Section 3 present the basis of Adomian decomposition method. Section 4 is devoted to the analysis of the variation iteration decomposition method. Illustrations are given in section 5. Finally, the conclusion is presented in section 6.

2.0 Variation Iteration Method

Consider the general differential equation

$$Ly(x,t)+Ny(x,t)=f(x,t) \qquad (2)$$

with prescribed auxiliary conditions, where $y(x,t)$ is unknown function, L is a linear operator and N a non linear operator, and $f(x,t)$ is the source term. According to [3-4,7], we can construct correction functional for equation (2) as:

$$y_{n+1}(x,t)=y_n(x)+\int_0^x \lambda(\beta)\left[Ly_n(x,\beta)+N\tilde{y}_n(x,\beta)-f(x,\beta)\right]d\beta, \qquad n\ge 0 \qquad (3)$$

Correspondence Author: Ebimene J.M., Email: mamaduebimene@hotmail.com[1] Tel: +2348062650313, +2348035390744 (JSA)

where $\lambda(\beta)$ is a general Lagrange multiplier, and $\tilde{y}_n(x,\psi)$ is a restricted variable. Abbasbandy and Sivanian [7] obtained the generalized value of the Lagrange multiplier via variation theory as

$$\lambda(\beta)=\frac{(-1)^{\alpha}}{(\alpha-1)!}(\beta-x)^{(\alpha-1)} \qquad (4)$$

where α is the order of the derivatives.

3.0 The Adomian Decomposition Method

Consider the standard operation [5,8-11]

$$Ly(x,t)+Ry(x,t)+Ny(x,t)=G(x,t), \qquad (5)$$

with prescribed auxiliary conditions, where $y(x,t)$ is the unknown function, L is the highest order derivative which is assumed to be invertible, $Ny(x,t)$ is the nonlinear term, and $G(x,t)$ is the source term. Applying the inverse operator L^{-1} to both sides of equation (5), and using the prescribed conditions,
we obtain,

$$y=L^{-1}(G(x,t))-L^{-1}(Ry(x,t))-L^{-1}(Ny(x,t)), \qquad (6)$$

The standard adomian defines the solution $y(x,t)$ as

$$y(x,t)=\sum_{n=0}^{\infty}y_n(x,t) \qquad (7)$$

and the nonlinear term as

$$Ny(x,t)=\sum_{n=0}^{\infty}A_n, \qquad (8)$$

where A_n are the adomian polynomial determined normally from the relation [5,8-11]

$$A_n=\frac{1}{n!}\left[\frac{d^n}{d\lambda^n}N\left(\sum_{i=0}^{n}\lambda^i y_i(x,t)\right)\right]_{\lambda=0} \qquad (9)$$

If the nonlinear term is a non linear function $F(y(x,t))$, the adomian polynomians are arranged into the form [5,8-11]

$$A_0=F(y_0(x,t))$$
$$A_1=y_1(x,t)F'(y_0(x,t)) \qquad (10)$$
$$A_2=y_2(x,t)F'(y_0(x,t))+\frac{y^2{}_1}{2!}F''(y_0(x,t))$$
$$A_3=y_3(x,t)F'(y_0(x,t))+y_1(x,t)y_2(x,t)F''(y_0(x,t))+\frac{y_1{}^3(x,t)}{3!}F'''(y_0(x,t))$$

4. 0 Variation Iteration Decompositin Method

We consider the equation

$$Ly(x,t)+Ry(x,t)+Ny(x,t)=G(x,t), \qquad (11)$$

with prescribed auxiliary conditions, where $y(x,t)$ is the unknown function, L is the highest order derivative which is assumed to be invertible, $Ny(x,t)$ is the nonlinear term, and $G(x,t)$ is the source term. Also, let the correction functional be uniquely as in equation (3).
We unknown function $y(x)$ be define as

$$y(x,t)=\sum_{n=0}^{\infty}y_n(x,t),$$

The decomposition method [5,8-11] involves finding the components $y_n(x,t),\ n\geq 0,$ respectively.
Also, we define the nonlinear term as

$$Ny(x,t)=\sum_{n=0}^{\infty}A_n,$$

where A_n are the adomian polynomial. Also, the linear terms are defined uniquely in this text as $\sum_{n=0}^{\infty}y_n(x,t), n\geq 0$.

Hence, the analytic solution for the gas dynamic equation is obtained using the relation

$$y_{n+1}(x,t) = y_n(x,t) + \int_0^t \lambda(\beta)\left[\frac{\partial}{\partial t}\left(\sum_{n=0}^{\infty} y_n(x,\beta)\right) + \sum_{n=0}^{\infty} A_n(x,\beta) - f(x,\beta)\right]d\beta,\ n \geq 0 \qquad (12)$$

which is the variation iteration decomposition method (VIDM).

5.0 Illustrations

In this section, we apply the variation decomposition method to solve the nonlinear gas dynamics to illustrate the effectiveness and reliability of the method. Results obtained are compared with variation iteration homotopy method [1].

5.1. Homogeneous gas Dynamic equation [1]:

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial(u^2)}{\partial x} - u(1-u) = 0 \quad 0 \leq x \leq 1,\ t > 0, \qquad (13)$$

where $f(x,t) = 0.$

The exact solution is $u(x) = e^{t-x}.$

The correction functional for equation (13) is given as

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\beta)\left[\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial(u^2)}{\partial x} - u(1-u)\right]d\beta,$$

where $\lambda(\beta) = -1$ from using equation (3).

We take initial approximation as $u_0(x,t) = e^{-x}$.

Now applying the variation iteration decomposition method to have

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left[\frac{\partial}{\partial t}\left(\sum_{n=0}^{\infty} u_n\right) + \frac{1}{2}\frac{\partial}{\partial x}\left(\sum_{n=0}^{\infty} A_n\right) - \sum_{n=0}^{\infty} u_n + \sum_{n=0}^{\infty} A_n\right]dt,$$

where A_n are the adomian polynomials for $Ny(x,t) = \frac{1}{2}\frac{\partial(u^2)}{\partial x} + u^2$. A similar choice of the nonlinear term in equation (13) is $Ny(x,t) = u^2$, which yields same results as $Ny(x,t) = \frac{1}{2}\frac{\partial(u^2)}{\partial x} + u^2$. Thus, using the algorithm in (10), we have

$$A_0 = \frac{1}{2}\frac{\partial({u_0}^2)}{\partial x} + {u_0}^2,$$

$$A_1 = \frac{d}{dx}\left(\frac{1}{2}\frac{\partial({u_0}^2)}{\partial x} + {u_0}^2\right)u_1,$$

$$A_2 = \frac{d}{dx}\left(\frac{1}{2}\frac{\partial({u_0}^2)}{\partial x} + {u_0}^2\right)u_2 + {u_1}^2\cdot$$

Using the above relations for $n \geq 0,$ we obtain

$$y(x) = e^{-x} + te^{-x} + \frac{1}{2}e^{-x}t^2 + \frac{1}{6}e^{-x}t^3 + \frac{1}{24}e^{-x}t^4 + \frac{1}{120}e^{-x}t^5 + \cdots = e^{-x}(1 + t + \frac{1}{2}t^2 + \ldots) = e^{t-x},$$

which is exactly the same result obtained by homotopy perturbation method [1].

5.2. Inhomogeneous gas Dynamic equation [1]:

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial(u^2)}{\partial x} - u(1-u) = -e^{(t-x)} \quad 0 \leq x \leq 1,\ t > 0, \qquad (14)$$

where $f(x,t) = -e^{(t-x)}$.

The exact solution is $u(x) = 1 - e^{(t-x)}.$

The correction functional for equation (14) is given as

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\beta)\left[\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial(u^2)}{\partial x} - u(1-u) + e^{(t-x)}\right]d\beta,$$

where $\lambda(\beta) = -1$ from using equation (3).

We take initial approximation as $u_0(x,t) = 1 - e^{-x}$.

Now applying the variation iteration decomposition method to have

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left[\frac{\partial}{\partial t}\left(\sum_{n=0}^{\infty} u_n\right) + \frac{1}{2}\frac{\partial}{\partial x}\left(\sum_{n=0}^{\infty} A_n\right) - \sum_{n=0}^{\infty} u_n + \sum_{n=0}^{\infty} A_n \right] dt,$$

where A_n are the adomian polynomials for $Ny(x,t) = u^2$. Thus, using the algorithm in (10), we have

$A_0 = u_0{}^2,$

$A_1 = 2u_0 u_1,$

$A_2 = 2u_0 u_2 + u_1{}^2.$

Using the above relations for $n = 0\ and\ 1$, we obtain

$$y(x) = 1 - e^{-x} + e^{-x} - e^{t-x} + \cdots = 1 - e^{t-x}.$$

which is exactly the same result obtained by homotopy perturbation method [1].

6.0 Conclusion

We have successively implemented the variation iteration decomposition method for finding the exact solution of gas dynamic equation without discretization, linearization or perturbation. It is clear that VIDM gives the solution in a compact series which converges rapidly to the exact solution wth less computational effort. Also, rounding-off and computational errors are avoided. Hence, the method is efficient and accurate as compared to HPM [1] in finding the analytic solution of the gas dynamic equation.

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