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Modified Variational Iteration Method for Solving Eight Order Boundary Value Problem using Canonical Polynomials

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Abstract

In this paper, we investigatethe numerical solution of eight order boundary value problem by modified variational iteration method. Canonical polynomials were constructed from the eight-order boundary value problem considered and used as basis function in the approximation employed. The approximate solution of the eight-order boundary value problem is obtained in terms of rapidly convergent series. Several numerical examples are given to verify the reliability and efficiency of the proposed method.All calculations are analyze using maple 18 software

Keywords: Modified Variational iteration method, boundary value problems, canonical polynomials, approximate solutions.

1.0 Introduction

Consider a generalized eight boundary value problem of the form:

$a_8 \frac{d^8}{dx^8} y + a_0 y = f(x),\ a < x < b (1)$

With boundary conditions

$y(a) = k_1\,, y'(a) = k_2\,, y''(a) = k_3\,, y'''(a) = k_3\, y(b) = l_1, y^{\iota}(b) = l_2, y''(b) = l_3.$
$y'''(b) = l_4.$

Where a_8 $and a_0$ are constants, $f(x)$continuous on [a,b].This type of problems are relevant in mathematical modeling of real life situations such as viscoelastic flow, heat transfer, and in other fields of engineering sciences. Over the years, several numerical techniques have been developed for solving problems of this kind. Muhammad Aslam Noor and Mohyud-Din [1] used the variational iteration decomposition method for solving eight order boundary value problem. Muhammad Aslam Noor and Mohyud-Din [2] developed and implemented the homotopy perturbation method and the variational iteration method for solving fifth-order boundary value problems. Also, Mohyud-Din and Ahmet Yildirim [3] implemented the homotopy perturbation method and the variational iteration method for solving Ninth and Tenth-order boundary value problems. Noor and Mohyud-Din [4-5] developed and used the Adomian decomposition method and the variational iteration method for solving fifth-order and other higher-order boundary value problems. Shahid S. Siddiqi and Muzammal Iftikhar [6] used the homotopy perturbation method and the variational iteration method for solving seventh-order boundary value problems. Recently, Njoseh and Mamadu [7] proposed a generalized method to this problem called the power series approximation method (PSAM). Also, the method of Tau and Tau-collocation approximation method was excessively used by Mamadu and Njoseh [8] for the solution of first and second ordinary differential equations. Caglar et. al. [9] also seeks the numerical solution of fifth order boundary value problem with sixth degree B-spline. Also, the method of A domian decomposition method [10] for solving the linear and nonlinear cases of these problems. Similarly, Islam et. al. [11] used the differential transform method (DTM) for twelfth order boundary value problem. In this paper, the variational iteration method using canonical polynomials is implemented to solve linear and nonlinear boundary value of eight order. The method co-joined the variational iteration method and the canonical polynomials.

In this proposed method, the correction functional is corrected for the BVP, and the Lagrange multiplier is computed optimally via the variational theory. For nonlinear and linear BVPs the canonical polynomials are constructed and then used as a basis functions for our approximation and not the use of analytical solution by using the initial conditions. Thereafter, the components are computed recursively. The proposed method work efficiently and the results so far are very encouraging and reliable. The fact that the proposed MVIMCP solves nonlinear problems without using A domian polynomials or He's polynomials can be considered as a clear advantage of this technique over VIDM and VIMHP.

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2.0 Variational Iteration Method

To illustrate the basic concept of the technique, we consider the following general differential equation

$Lu + Nu - g(x) = 0,$ (2.1)

Where L is a linear operator, Na nonlinear operator and $g(x)$ is the inhomogeneous term according to variational iteration method, we can construct a correct functional as follows

$u_{n+1} = u_n(x) + \int_0^x \lambda(t)\big(Lu_n(t) + N\widetilde{u_n(t)} - g(t)\big)dt$ (2.2)

Where$\lambda(t)$ is a Lagrange multiplier, which can be identified optimally via variational iteration method. The subscripts$n$ denote the nth approximation, $\widetilde{u_n}$is considered as a restricted variation. i.e.,$\widetilde{u_n}$= 0. The relation (2.2) is called as a correct functional. The solution of the linear problems can be solved in a single iteration step due to the exact identification of the Lagrange multiplier. In this method, we need to determine the langrange multiplier $\lambda(t)$optimally and hence the successive approximation of solution u will be readily obtained upon using the langrange multiplier and our u_0 .and the solution is given by

$\lim_{n-\infty} u_n$

3.0 Construction Of Canonical Polynomials

Given the seventh order boundary value problem of the form

$a_8 \frac{d^8}{dx^8} y + a_0 y = f(x), a < x < b$

In operator form, we have

$$L = a_8 \frac{d^8}{dx^8} + a_0$$

Let$LQ_r(x) = x^r$, then

$$Lx^r = \left(a_8 \frac{d^8}{dx^8} + a_0\right)x^r$$

$$Lx^r = a_8 \frac{d^8}{dx^8} x^r + a_0 x^r$$

$$Lx^r = a_7 r(r-1)(r-2)(r-3)(r-4)(r-5)(r-6)(r-7)x^{(r-8)} + a_0 x^r$$

$$Lx^r = a_8 r(r-1)(r-2)(r-3)(r-4)(r-5)(r-6)(r-7)LQ_{r-8}(x) + a_0 LQ_r(x)$$

$$x^r = a_8 r(r-1)(r-2)(r-3)(r-4)(r-5)(r-6)(r-7)Q_{r-8}(x) + a_0 Q_r(x)$$

Hence we obtain the following

$$Q_r(x) = \frac{1}{a_0}(x^r - a_8 r(r-1)(r-2)(r-3)(r-4)(r-5)(r-6)(r-7)Q_{r-8}(x))$$

For $r = 0,1,2,3,3,4,5,6,7,8.$we obtain the following polynomials

$$Q_0(x) = \frac{1}{a_0}, Q_1(x) = \frac{x}{a_0}, Q_2(x) = \frac{x^2}{a_0}, Q_3(x) = \frac{x^3}{a_0}$$

$$Q_4(x) = \frac{x^4}{a_0}, Q_5(x) = \frac{x^5}{a_0}, Q_6(x) = \frac{x^6}{a_0}$$

$$Q_7(x) = \frac{x^7}{a_0}$$

$$Q_8(x) = \frac{x^8 + 40320}{a_0}$$

4.0 Modified Variational Iteration Method Using Canonical Polynomials (Mvimcp)

To illustrate the basic concept of the variational iteration method by canonical polynomials, we consider the general differential equation.

$Lu + Nu - g(x) = 0$ (4.1)

Where L is a linear operator, Na nonlinear operator and $g(x)$ is the inhomogeneous term. According to variational iteration method, we can construct a correct functional

$u_{n+1} = u_n(x) + \int_0^x \lambda(t)\left(Lu_n(t) + N\widetilde{u_n(t)} - g(t)\right)dt$(4.2)

Where$\lambda(t)$ is a Lagrange multiplier, which can be identified optimally via variational iteration method. The subscripts$n$ denote the nth approximation, $\widetilde{u_n}$is considered as a restricted variation. i.e.$\widetilde{u_n}$= 0. The equation (4.2) is called the correct functional.

We assume an approximate solution of the form

$$u(x)=\sum_{i=0}^{N}a_iQ_i(x)$$

Where $Q_i(x)$ are canonical polynomials, a_i constants to be determined, and N the degree of approximant. Hence we obtain the following iterative method

$$u_{n+1}=\sum_{i=0}^{N}a_iQ_i(x)+\int_0^x\lambda(t)\left(L\sum_{i=0}^{N}a_iQ_i(t)+N\sum_{i=0}^{N}a_iQ_i(t)-g(t)\right)dt$$

This method is called the variational iteration method by canonical polynomials (VIMCP) which co-join the variational iteration method with canonical polynomials and may be viewed as an important and significant improvement as compared with other similar methods.

5.0 Numerical Applications

In this section we applied the variational iteration method using canonical polynomials to solve two examples of which are linear and non-linear BVPs. Numerical results also show the accuracy of the proposed method.

Example 5.1: Considers the following eight order nonlinear boundary value problem **[1]**

$$u^{(viii)}(x)=e^{-x}u^2(x),\ 0\le x\le 1 \qquad (5.1)$$

$$u(0)=u''(0)=u^{iv}(0)=u^{vi}(0)=1, u(1)=u''(1)=u^{iv}(1)=u^{vi}(1)=e, \qquad (5.2)$$

With exact solution $u(x)=e^x$

The correct functional for the boundary value problem (5.1) and (5.2) is given as

$$u_{n+1}(x)=u_n(x)+\int_0^x\lambda(t)\left(\frac{d^8u_n}{dt^8}-e^{-t}u_n^2(t)\right)dt$$

Making the correct functional stationary using, $\lambda(t)=\frac{(-1)^8(t-x)^6}{7!}$ as the Lagrange multiplier, hencewe get the following iterative method

$$u_{n+1}(x)=u_n(x)+\int_0^x\frac{(-1)^8(t-x)^8}{7!}\left(\frac{d^8u_n}{dt^8}-e^{-t}u_n^2(t)\right)dt$$

Applying the modified variational iteration Method by canonical polynomials, We assume an approximate solution of the form

$$u(x)=\sum_{i=0}^{8}a_iQ_i(x)$$

Therefore we have the following iterative formula

$$u_{n+1}(x)=\sum_{i=0}^{8}a_iQ_i(x)+\left(\frac{d^8u_n}{dt^8}-e^{-t}\left(\sum_{i=0}^{8}a_iQ_i(t)\right)^2\right)dt$$

$$u_{n+1}(x)=-a_0-a_1x-a_2x^2-a_3x^3-a_4x^4-a_5x^5-a_6x^6-a_7x^7-a_8x^8-40320a_8$$
$$+\left(\frac{d^8u_n}{dt^8}-e^{-t}\left(\sum_{i=0}^{8}a_iQ_i(t)\right)^2\right)dt$$

Hence we obtain

$$u_0(x)=-a_0-a_1x-a_2x^2-a_3x^3-a_4x^4-a_5x^5-a_6x^6-a_7x^7-a_8x^8-40320a_8$$

$$u_1(x)=\left(2a_0a_8+\frac{1}{40320}a_0^2+40320a_8^2\right)x^8+\left(\frac{1}{181440}a_0a_1-\frac{2}{9}a_0a_8+\frac{2}{9}a_1a_8-\frac{1}{362880}a_0^2-4480a_8^2\right)x^9$$
$$+\left(\frac{1}{907200}a_0a_2-\frac{1}{907200}a_0a_1+\frac{1}{45}a_0a_8-\frac{2}{45}a_1a_8+\frac{2}{45}a_2a_8+\frac{1}{1814400}a_1^2+\frac{1}{362880}a_0^2\right.$$
$$\left.+448a_8^2\right)x^{10}+O(x)^{11}$$

The series solution is given as

$$u(x)=-a_0-a_1x-a_2x^2-a_3x^3-a_4x^4-a_5x^5-a_6x^6-a_7x^7-40320a_8+\left(2a_0a_8+\frac{1}{40320}a_0^2+40320a_8^2\right)x^8$$
$$+\left(\frac{1}{181440}a_0a_1-\frac{2}{9}a_0a_8+\frac{2}{9}a_1a_8-\frac{1}{362880}a_0^2-4480a_8^2\right)x^9$$
$$+\left(\frac{1}{907200}a_0a_2-\frac{1}{907200}a_0a_1+\frac{1}{45}a_0a_8-\frac{2}{45}a_1a_8+\frac{2}{45}a_2a_8+\frac{1}{1814400}a_1^2+\frac{1}{362880}a_0^2\right.$$
$$\left.+448a_8^2\right)x^{10}+O(x)^{11}$$

Applying the boundary condition (5.2) the values of the unknown constants can be determined as follows

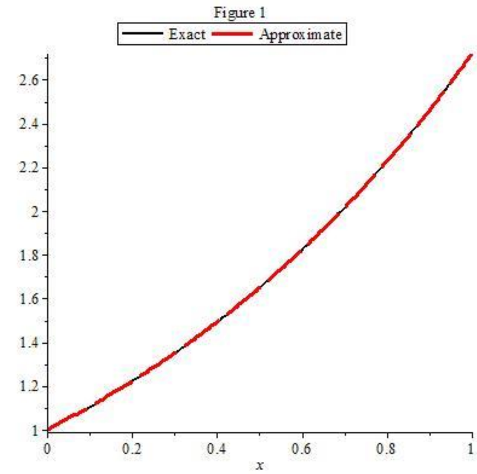
$$a_0 = 1.298791972, \quad a_1 = -0.9999974177, \quad a_2 = -0.5000000$$
$$a_3 = -0.166670493, \quad a_4 = -0.04166667, \quad a_5 = -0.00833333$$
$$a_6 = -0.001386288, \quad a_7 = -0.0001997970, \quad a_8 = -0.00005701368233$$

Consequently, the series solution is given as

$$1 + 0.9999974177x + (0.5)x^2 + 0.166670493x^3 + 0.04166667x^4 + 0.00833333x^5 + 0.001386288x^6 + -0.0001997970x^7 + 0.0000248015873x^8 - (2.75571 \times 10^{-6})x^9 + (2.75573 \times 10^{-7})x^{10} + O(x)^{11}$$

Table 5.1(Error estimates)

x	Exact solution	Approximate solution	Error
0.0	1.0000000	1.0000000	0.0000e+00
0.1	1.1051709	1.1051707	2.5500e-07
0.2	1.2214028	1.2214023	4.8500e-07
0.3	1.3498588	1.3498581	6.7300e-07
0.4	1.4918247	1.4918239	7.9700e-07
0.5	1.6487213	1.6487204	8.4200e-07
0.6	1.8221188	1.8221180	8.0400e-07
0.7	2.0137527	2.0137520	6.8800e-07
0.8	2.2255409	2.2255403	5.0000e-07
0.9	2.4596031	2.4596028	2.6400e-07
1.0	2.7182818	2.7182818	1.0000e-09



Error = Exact Solution- Approximate solution

Example 5.2: Consider the following eight order linear boundary value problem

$$u^{(viii)}(x) = u(x) - 8xe^x, 0 \leq x \leq 1 \qquad (5.3)$$

With boundary conditions

$$u(0) = 0, u'(0) = 1, u''(0) = 0, u'''(0) = -3, u(1) = 0, u'(1) = -e, u''(1) = -4e.$$
$$u'''(1) = -6e \qquad (5.4)$$

The exact solution of the example is $u(x) = (1-x)e^x$

The correct functional for the boundary value problem

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(t)\left(\frac{d^8 u_n}{dt^8} - u_n(t) + 8te^t\right)dt$$

Making the correct functional stationary, using $\frac{(-1)^8(t-x)^7}{7!}$ as the Lagrange multiplier, we have the following

$$u_{n+1}(x) = u_n(x) + \int_0^x \frac{(-1)^8(t-x)^7}{7!}\left(\frac{d^8 u_n}{dt^8} - u_n(t) + 8te^t\right)dt$$

Applying the modified variational iteration Method by canonical polynomials, we assume an approximate solution of the form

$$u(x) = \sum_{i=0}^{8} a_i Q_i(x)$$

Therefore we have the following iterative formula

$$u_{n+1}(x) = \sum_{i=0}^{8} a_i Q_i(x) + \int_0^x \frac{(-1)^8(t-x)^7}{7!}\left(\frac{d^8 u_n}{dt^8} - \sum_{i=0}^{8} a_i Q_i(t) + 8te^t\right)dt$$

$$u_{n+1}(x) = -a_0 - a_1 x - a_2 x^2 - a_3 x^3 - a_4 x^4 - a_5 x^5 - a_6 x^6 - a_7 x^7 - a_8 x^8 - 40320 a_8 + \int_0^x \frac{(-1)^8(t-x)^7}{7!}\left(\frac{d^7 u_n}{dt^7} - \sum_{i=0}^{8} a_i Q_i(t) + 8te^t\right)dt$$

Hence we obtain

$$u_0(x) = -a_0 - a_1 x - a_2 x^2 - a_3 x^3 - a_4 x^4 - a_5 x^5 - a_6 x^6 - a_7 x^7 - a_8 x^8 - 40320 a_8$$

$$u_1(x) = \left(-a_8 - \frac{1}{40320}a_0\right)x^8 + \left(-\frac{1}{362880}a_1 - \frac{1}{45360}\right)x^9 + \left(-\frac{1}{181440}a_2 - \frac{1}{226800}\right)x^{10} + \left(-\frac{1}{604800}a_3 - \frac{1}{1663200}\right)x^{11} + O(x)^{12}$$

The series solution is given as

$$u(x) = -a_0 - a_1 x - a_2 x^2 - a_3 x^3 - a_4 x^4 - a_5 x^5 - a_6 x^6 - a_7 x^7 - 40320 a_8 + \left(-a_8 - \frac{1}{40320} a_0\right) x^8 + \left(-\frac{1}{362880} a_1 - \frac{1}{45360}\right) x^9 + \left(-\frac{1}{181440} a_2 - \frac{1}{226800}\right) x^{10} + \left(-\frac{1}{604800} a_3 - \frac{1}{1663200}\right) x^{11} + O(x)^{12}$$

Applying the boundary condition (5.4) the values of the unknown constants can be determined as follows

$a_0 = -15.95250807,\quad a_1 = 0.0025854167,\quad a_2 = 0.5000000$

$a_3 = 0.3376196834,\quad a_4 = 0.12500000,\quad a_5 = 0.03113511420$

$a_6 = 0.0069444444,\quad a_7 = 0.001883519946,\quad a_8 = 0.0003708459343$

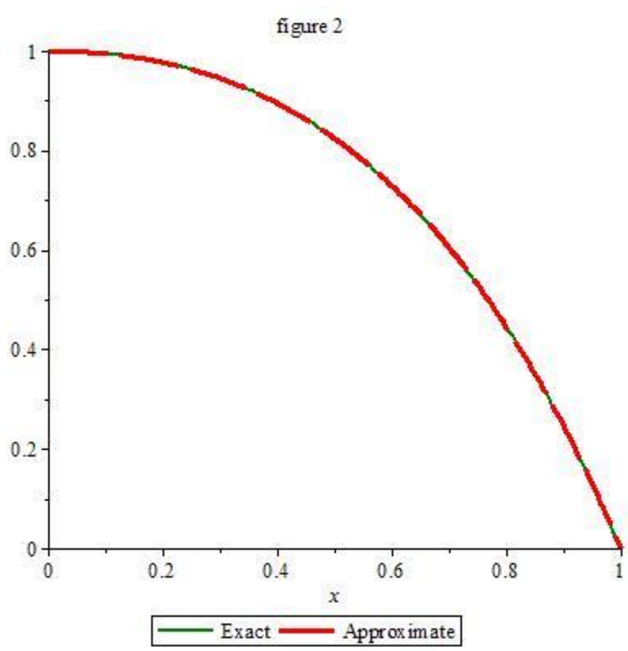
Consequently, the series solution is given as

$$u(x) = 1 + 0.0025854167x + 0.5000000x^2 - 0.3376196834x^3 - 0.12500000x^4 - 0.03113511420x^5 - 0.00694444444x^6 - 0.001883519946x^7 - 0.0000248015873x^8 - 0.00002203873066x^9 - 0.000004133598x^{10} - (6.5199 \times 10^{-7})x^{11} + O(x)^{12}$$

Table5.2 (Error estimates)

x	Exact solution	Approximate solution	Error
0.0	0.0000000	0.0000000	0.0000e+00
0.1	0.9946538	0.9949081	2.5428e-04
0.2	0.9771222	0.9776057	4.8349e-04
0.3	0.9449012	0.9455663	6.6510e-04
0.4	0.8950948	0.8958762	7.8134e-04
0.5	0.8243606	0.8251816	8.2097e-04
0.6	0.7288475	0.7296278	7.8025e-04
0.7	0.6041258	0.6047891	6.6332e-04
0.8	0.4451082	0.4455899	4.8171e-04
0.9	0.2459603	0.2462135	4.8171e-04
1.0	0.0000000	0.0000000	0.0000e+00

Error = Exact Solution- Approximate solution



6.0 Conclusion

In this paper, the modifiedvariational iteration method using canonical polynomials has been appliedto obtain the numerical solutions of linear and nonlinear eight order boundary value problems. The modification involves the construction of canonical polynomials coupled withvariational iterationmethod.The method gives rapidly converging series solutions in both linear andnonlinear cases which occur in physical problem. The numerical results revealed that the present method isalso a powerful mathematical tool for the solution of eight order boundaryvalue problems.

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