Transactions of the Nigerian Association of Mathematical Physics
***Volume 4, (July, 2017*), pp41 – 44**
© Trans. of NAMP

Variational Iteration Decomposition Method for Solving Seventh Order Boundary Value Problem

Ogeh K.O.[1] and*Ojobor *S. A[2]

[1] Department of Mathematics, University of Ilorin, P.M.B 1515, Ilorin, Nigeria
[2]Department of Mathematics, Delta State University, Abraka, Nigeria

Abstract

In this paper, we considered the variational iteration decomposition method (VIDM) for solving boundary value problems of the seventh order. The VIDM is an elegant combination of the variation iteration method and the A domain decomposition method. The approximate solution of the problem is obtained in terms of rapidly convergent series. Several numerical examples are given to verify the reliability and efficiency of the proposed Method. All computations are implemented using maple18 software.

Keywords: Variational iteration method, A domain decomposition method, boundary value problems, linear and nonlinear problem, approximate solution

1.0 Introduction

Consider the generals eventh-order boundary value problems (BVPs) of the type:

$$u^{(7)}(x) = r(x)y(x) + f(x), \qquad (1)$$

With boundary conditions

$$u(a) = K_1\,, u^{\iota}(a) = k_2\ , u^{\iota\iota}(a) = k_3\,, u^{\iota\iota\iota}(a) = k_4,$$
$$u(b) = L_1, u^{\iota}(b) = L_2, u^{\iota\iota}(b) = L_3\,.$$

Here$r(x) and f(x)$ are continuous functions.This type of problems are relevant in the mathematical modeling of real life situations such as viscoelastic flow, heat transfer, and in other fields of engineering sciences. Over the years, several numerical techniques have been developed for solving problems of this kind. Between the year 2008 and 2010, Noor and Mohyud-Din [1-2] developed and implemented the homotopy perturbation method and the variational iteration method for solving fifth-order and other higher-order boundary value problems. Recently, Njoseh and Mamadu [3] proposed a generalized method to this problem called the power series approximation method (PSAM). Also, the method of tau and tau-collocation approximation method was excessively used by Mamadu and Njoseh [4] for the solution of first and second ordinary differential equations. Caglar et. al. [5] also seek the numerical solution of fifth order boundary value problem with sixth degree B-spline. Also, the method of Adomian decomposition method [6-7] for solving the linear and nonlinear cases of this problems. Similarly, Islam et. al. [8] used the differential transform method (DTM) for twelfth order boundary value problem.
In this paper, the variational iteration decomposition method is implemented to solve linear and nonlinear boundary value of the seventh order. The method co-joined the variational iteration method and the Adomian decomposition method. In this method, the correction functional is corrected for the BVP, and the Lagrange multiplier is computed optimally via the variational theory. For nonlinear BVPs, the A 41omain polynomials are constructed using the algorithms found in [6-7] and are substituted in the correction formula. Thereafter, the components $u_n(x), n \geq 0$, are computed recursively. Finally, the solution is given in an infinite series usually converging to an accurate solution. The method is practically explicit as it requires no perturbation, discretization or linearization.

2.0 Variational Iteration Method

To illustrate the basic concept of the technique, we consider the following general differential equation

$$Lu + Nu = g(x),$$

Correspondence Author: Ogeh K.O., Email: kennethoke503@gmail.com, Tel: +2348105640668, +2348034029170 (OSA)

where$L$ is a linear operator, Na nonlinear operator and $g(x)$ is the inhomogeneous term According to variational iteration method, we can construct a correct functional as follows

$$u_{n+1} = u_n(x) + \int_0^x \lambda\big(Lu_n(t) + N\widetilde{u_n(t)} - g(t)\big)dt, \quad (3)$$

whereλ is a Lagrange multiplier, which can be identified optimally via variational iteration method. The subscripts$n$ denote the nth approximation, $\widetilde{u_n}$is considered as a restricted variation. i.e,$\widetilde{u_n}$= 0. The relation (3) is called as a correction functional. The solution of the linear problems can be solved in a single iteration step due to the exact identification of the Lagrange multiplier. The principles of variational iteration method and its applicability for various kinds of differential equations are given.

3.0 Adomian Decomposition Method

Consider the differential equation

$$Lu + Ru + Nu = g, \quad (4)$$

where$L$ is the highest-order derivative which is assumed to be invertible,R is a linear differential operator and$Nu$is the nonlinear term. Applying the inverse operator L^{-1} to both sides and using the given conditions, we obtain

$$u = f - L^{-1}(Ru) - L^{-1}(Nu), \quad (5)$$

where the function f represents the terms arising from integrating the source term g and by using the given conditions. Adomian decomposition method defines the solution $u(x)$ by the series

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \quad (6)$$

where the components $u_n(x)$ are usually determined recurrently by using the relation

$$u_0 = f,$$

$$u_{k+1} = f - L^{-1}(Ru_k) - L^{-1}(Nu_k) (7)$$

The nonlinear $N(u)$ operator can be decomposed into an infinite series of polynomials given by

$$N(u) = \sum_{n=0}^{\infty} A_n, \quad (8)$$

where the so-called Adomian polynomials that can be generated for various classes of nonlinearities according to the specific algorithm developed in which yields

$$A_n = \left(\frac{1}{n!}\right)\left(\frac{d^n}{d\lambda^n}\right) N\left(\sum_{i=o}^{n} \lambda^i u_i\right), n = 0,1,2,3 \ldots \quad (9)$$

4.0 Variational Iteration Decomposition Method (VIDM)

To illustrate the basic concept of the variational iteration decomposition method, we consider the general differential equation (1). According to variational iteration method, we can construct a correct functional (2). We define the solution by the series

$$u(x) = \sum_{n=0}^{\infty} u^i(x), \quad (10)$$

and the nonlinear term

$$N(u) = \sum_{n=0}^{\infty} A_n(u_0, u_1, u_2 \ldots u_i), \quad (11)$$

Where $A_n$the so-called Adomian polynomials can be generated for all type of nonlinearities according to the algorithm developed in which we yields the following

$$A_n = \left(\frac{1}{n!}\right)\left(\frac{d^n}{d\lambda^n}\right)\left[\sum_{n=0}^{\infty} N(u(\lambda))\right]_{\lambda=0} = 0. \quad (12)$$

Hence, we obtain the following iterative scheme for finding the approximate solution

$$u_{n+1} = u_n(x) + \int_0^x \lambda\left(Lu_n(t) + \sum_{n=0}^{\infty} A_n - g(t)\right)dt .$$

This method is called as the variational iteration decomposition method (VIDM) and may be viewed as an important and significant improvement as compared with other similar methods.

Numerical Applications

Example 5.1: The following seventh order nonlinear boundary value problem is considered

$$u^{(7)}(x) = e^{-x}u^2(x), 0 < x < 1, 0 \le x \le 1, \quad ` \quad (13)$$

$$u(0) = u'(0) = u''(0) = u'''(0) = 1, u(1) = u'(1) = u''(1) = e, \quad (14)$$

With exact solution $u(x) = e^x$

The correct functional for the boundary value problem (13) and (14) is given as

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(t)\left(\frac{d^7u_n}{dt^7} - e^{-t}u_n^2(t)\right)dt.$$

Making the functional stationary using,$\lambda(t) = (-1)^7\frac{(t-x)^6}{6!}$ as the Lagrange multiplier we get the following iterative method or formula

$$u_{n+1}(x) = u_n(x) + \int_0^x (-1)^7\frac{(t-x)^6}{6!}\left(\frac{d^7u_n}{dt^7} - e^{-t}u_n^2(t)\right)dt,$$

$u_{n+1}(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + Ax^4 + Bx^5 + Cx^6 + \int_0^x (-1)^7 \frac{(t-x)^6}{6!} \left(\frac{d^7u_n}{dt^7} - e^{-t} \sum_{n=0}^{\infty} A_n\right) dt,$
Where A, B, and C are constants to be obtained later.
Where, A_n are adomian polynomials for non linear operator $\mathrm{N(y)} = u_0^2(x)$ and can be generated for all types of non linearity according to the algorithm which yields the following
$A_0 = u_0^2(x), A_1 = 2u_0(x)u_1(x),\ A_2 = 2u_2(x)u_0(x) + \frac{u_1^2}{2}.$
Hence we obtain
$u_0 = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + Ax^4 + Bx^5 + Cx^6,$
$u_1 = \frac{x^7}{5040} + \frac{x^8}{40320} + \frac{x^9}{362880} + \frac{x^{10}}{3628800} + \left(\frac{-1}{39916800} + \frac{A}{831600}\right)x^{11} + \left(\frac{-1}{479001600} + \frac{B}{1995840}\right)x^{12} + O(x)^{13}.$
Using the boundary condition, the values of the unknown constants can be determined as follows
A = 0.041666667529862395, B = 0.008333331197193119, and C = 0.001388890268167299.
Finally the series solution is

$$1 + x + (0.5)x^2 + 0.16667x^3 + 0.0416667x^4 + 0.00833333x^5 + 0.000138889x^6 + 0.000198413x^7 + 0.0000248016x^8 + (2.75573 \times 10^{-6})x^9 + (2.75573 \times 10^{-7})x^{10} + (2.50521 \times 10^{-8})x^{11} + (2.08768 \times 10^{-9})x^{12} + O(x)^{13}$$

Example 5.2: Consider the following seventh order linear boundary value problem
$u^{(7)}(x) = -u(x) - e^x(35 + 12x + 2x^2), 0 \le x \le 1,$ (15)
$u(0) = 0, u'(0) = 1, u''(0) = 0, u'''(0) = -3,$
$u(1) = e, u'(1) = -4e, u''(1) = -3.$(16)
The exact solution of the example is $u(x) = x(1-x)e^x$
The correct functional for the boundary value problem
$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(t)\left(\frac{d^7u_n}{dt^7} - u_n(t) + e^t(35 + 12t + 2t^2)\right) dt.$
Making the correct functional stationary, we get
$u_{n+1}(x) = u_0(x) + \int_0^x (-1)^7 \frac{(t-x)^6}{6!}\left(\frac{d^7u_n}{dt^7} - u_n(t) + e^t(35 + 12t + 2t^2)\right) dt.$
$u_{n+1}(x) = x - \frac{x^3}{2} + Ax^4 + Bx^5 + Cx^6 + \int_0^x (-1)^7 \frac{(t-x)^6}{6!}\left(\frac{d^7u_n}{dt^7} - \sum_{n=0}^{\infty} A_n + e^t(35 + 12t + 2t^2)\right) dt.$
Where A_n are Adomian polynomials. Hence we obtain
$u_0 = x - \frac{x^3}{2} + Ax^4 + Bx^5 + Cx^6,$
$u_1(x) = \frac{x^7}{144} + \frac{x^8}{840} + \frac{x^9}{5760} + \frac{x^{10}}{45360} + \left(\frac{-107}{39916800} - \frac{A}{1663200}\right)x^{11} + \left(\frac{-1}{3548160} - \frac{B}{3991680}\right)x^{12} + O(x)^{13},$
Where A, B and C are unknown, which will be determined later
Using the first two approximations, we obtain the series
$u_0 = x - \frac{x^3}{2} + Ax^4 + Bx^5 + Cx^6 - \frac{x^7}{144} - \frac{x^8}{840} - \frac{x^9}{5760} - \frac{x^{10}}{45360} + \left(\frac{-107}{39916800} - \frac{A}{1663200}\right)x^{11} + \left(\frac{-1}{3548160} - \frac{B}{3991680}\right)x^{12} + O(x)^{13}.$
Using the boundary condition, the value of the unknown constants can be determined as follows
A = -0.3333333170467781, B = -0.12500003614813987 and C = -0.03333331303032349
Finally, the series solution is
$u(x) = x - (0.5)x^3 + 0.333333x^4 - 0.125x^5 - 0.0333333x^6 - 0.00694444x^7 - 0.00119048x^8 - 0.000173611x^9 - 0.0000220459x^{10} - (2.48016 \times 10^{-6})x^{11} - (2.50521 \times 10^{-7})x^{12} + O(x)^{13}.$

5. Conclusion

In this paper, variational iteration Decomposition method has been applied to obtain the numerical solutions of linear and nonlinear seventh order boundary value problems. The method solves nonlinear problems using (VIDM). The method gives rapidly converging series solutions in both linear and nonlinear cases. The numerical results revealed that the present method is a powerful mathematical tool for the solution of seventh order boundary value problems.

Table 1. Comparison of numerical result for example 5.2

x	Exact solution	Approximate solution	Error
0.0	0.0000000	0.0000000	0.0000e+00
0.1	0.0994654	0.0994654	5.0000e-11
0.2	0.1954244	0.1954244	6.0000e-10
0.3	0.2834703	0.2834704	2.7000e-09
0.4	0.3580379	0.3580379	8.7000e-09
0.5	0.4121803	0.4121803	2.1600e-08
0.6	0.4373085	0.4373086	4.6100e-08
0.7	0.4228881	0.4228882	8.7800e-08
0.8	0.3560865	0.3560867	1.5540e-07
0.9	0.2213643	0.2213645	2.6230e-07
1.0	0.0000000	0.0000000	4.3242e-07

Table 2.Comparison of numerical result for example 5.1

x	Exact solution	Approximate solution	Error
0.0	1.0000000	1.0000000	0.0000e+00
0.1	1.1051709	1.1051709	0.0000e+00
0.2	1.2214028	1.2214028	1.0000e-09
0.3	1.3498588	1.3498588	0.0000e+00
0.4	1.4918247	1.4918247	0.0000e+00
0.5	1.6487213	1.6487213	0.0000e+00
0.6	1.8221188	1.8221188	1.0000e-09
0.7	2.0137527	2.0137527	2.0000e-09
0.8	2.2255409	2.2255409	3.0000e-09
0.9	2.4596031	2.4596031	6.0000e-09
1.0	2.7182818	2.7182818	1.0000e-08

References

[1] Muhammad A slam Noor and Syed Tauseef Mohyud-Din. (2010).A new approach to solving fifth order boundary value problem, *International Journal of Nonlinear Science*, **9**, 387-393.

[2] Muhammad A slam Noor and Syed Tauseef Mohyud-Din. (2008). Variational iteration method for fifth order boundary value problem using He's polynomials, *Hindawi Publishing Corporation Article*, ID 954794, 12 pages doi:10.1155/2008/954794.

[3] Njoseh, I.N. and Mamadu, E.J. (2016) Numerical solutions of a generalized Nth Order Boundary Value Problems Using Power Series Approximation Method. *Applied Mathematics,* **7,** 1215-1224. http://dx.doi.org/10.4236/am.2016.711107.

[4] Mamadu, E. J. and Njoseh, I.N. (2016) Tau-Collocation Approximation Approach for Solving First and Second Order Ordinary Differential Equations. *Journal of Applied Mathematics and Physics,* **4:** 383-390.http://dx.doi.org/10.4236/jamp.2016.42045.

[5] Caglar, H.N, Caglar, S.H, and Twizellll E.H (1999) The numerical solution of fifth-order value problems with sixth degree B-spline function, *Applied Mathematics Letters,* **12(5):25-30.**

[6] Yahya Qaid Hasan and Liu Ming Zhu (2008), Modified A domian decomposition method for singular initial value problem in the second order ordinary differential equations, *Surveys in Mathematics and its Applications*, **3,** 183 – 193

[7] G. Adomian, (1990). A review of the decomposition method and some recent results for nonlinear*equation,Math.Comput.Modelling*,**13**(7):1743.MR1071436(92h:00002a).Zbl0713.65051.