Transactions of the Nigerian Association of Mathematical Physics
***Volume 4*, (*July, 2017*), pp31 – 36**

On The Fundamental Properties Of Incidence Matrix Of Symmetric Balanced Incomplete Block Designs

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Abstract

This paper considers the incidence matrix of symmetric balanced incomplete block designs (BIBD). The theorem due to Bruck – Ryser – Chowla on the existence of symmetric balanced incomplete block designs is presented. In particular, we obtain a set of conditions for existence which can be applied to the incidence matrix of a symmetric block design. It is established that under suitable hypothesis, if the incidence matrix of such block designs satisfies all conditions, then the design would be symmetric.

Keywords: Incidence matrix, symmetric balanced incomplete block design, Bruck – Ryser – Chowla theorem, relations.

Introduction:

One major concern of mathematicians in combinatorial analysis is the arrangement of objects according to specified rules and the problem of ascertaining the number of ways in which this may be done. Over the past few decades, articles by many authors and attention in combinatorial design theory had been focused on how to provide answers to fundamental questions such as: Is it possible to arrange elements (called points) of a finite set into subsets, so that it satisfies a certain balance relation? What are the conditions for existence block designs? What steps are involved in the construction of BIBD? In some cases authors have constructed algorithms to check the balance properties of design and applied block designs to various areas including error analysis [1]. The questions mentioned above form the basis for several combinatorial works [2-6].

Block designs are of various forms which include the following: t- design, orthogonal Latin squares, pair-wise balanced designs, balanced incomplete block designs (BIBD), etc. The study of block design theory can be traced to early 20th century ignited by its application in the analysis and design of statistical experiments, analysis of variance (ANOVA), software testing, tournament scheduling, mathematical biology, cryptography, lotteries and algorithm designs, quantum computing, etc [7-9].

The most commonly researched of all types of block designs is the balanced incomplete block design (BIBD) and its foundation work can be traced to the efforts of notable statisticians which include Yates and Fisher who used BIBD in the computation of ANOVA and other statistical experiments. In most authors' efforts, balanced incomplete block designs are often presented by means of square matrices called incidence matrices. For example, [10] established the conditions for the existence of singular incidence matrices of block designs.

In this article, a set of conditions is obtained for the incidence matrix of a symmetric block design and it is established that each of these conditions imply the other and as such, they form the criteria for any given block design to be a symmetric balanced incomplete block design.

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2.0 Preliminaries

Generally speaking, by a block design we mean a pair (S,Σ), where S is a set of elements usually referred to as points and Σ a collection of non-empty subsets or multi-sets which are usually referred to as 'blocks'. Sometimes, some of such blocks may be the same in which case we call them repeated blocks. A design is usually called simple if it contains no repeated blocks.

Rather than being a collection of subsets, a block design is a combinatorial relation which can be represented by a matrix equation of the form;

$$AA^T = B = \begin{pmatrix} r & \alpha & . & . & \alpha \\ \alpha & r & . & . & . \\ . & . & . & . & .. \\ . & . & . & . & . \\ \alpha & . & . & . & r \end{pmatrix} = (r-\alpha)I_V + \alpha J_v, \; W_v A = kW_b \qquad (1)$$

with $\det B = (r-\alpha)^{(v-1)}(v\alpha - \alpha + r) \qquad (2)$

where A is called its incidence matrix, A^T is the transpose of A, J_v is the $v \times v$ matrix of '1^s', I_v is a $v \times v$ identity matrix W_v, W_b are the vectors of *v* and *b* respectively. The equation $AA^T = B$ is a relation of quadratic forms. In the block design (S,Σ) the arrangement of points and blocks occur with a relation of incidence or tactical configuration that explains which objects belong to which block. Moreover, the block designs discussed in this article are termed incomplete due to the fact that *k*< *v*. By complete block design we refer to the set of all ${}^vC_k$ points, that is, the combination of *v* points taken *k* at a time as blocks.

Definition 1:

Given the positive integers $v, k, \alpha \in Z$, where $v > k \geq 2$, by a balanced incomplete block design (BIBD), we mean an arrangement of v distinct objects into b blocks, where each block contains exactly k distinct objects, each object occurs in exactly r different blocks, and every pair of distinct objects s_i, s_j occurs together in exactly α blocks such that the following conditions are satisfied

$$\left.\begin{array}{l} (i)\, bk = vr \\ (ii)\, r(k-1) = \alpha(v-1) \quad (Balanced \;\; property) \end{array}\right\} \qquad (3)$$

Definition 2:

A balanced incomplete block design is said to be symmetric if the total number of v distinct points in its arrangement is equal to the number of blocks, that is if *v* = *b*.

Naturally, the specifications of block designs are generally represented special *0 – 1* matrix called the incidence matrices.

Definition 3:

Given any block design (S,Σ), such that $S = \{s_1, s_2, \ldots s_v\}$, and $\Sigma = \{\Sigma_1, \Sigma_2, \ldots, \Sigma_b\}$, then the incidence matrix of the design is a $v \times b$, $0-1$ matrix $M = \{s_{i.j}\}$, $i = 1,2,..v$ and $j = 1,2,..b$, where $s_1, s_2, \ldots s_v$ are the objects or points and $\Sigma_1, \Sigma_2, \ldots, \Sigma_b$ are blocks, define by the rule

$$S_{i,j} = \begin{cases} 1, \; if \;\; s_i \in \Sigma_j \\ 0, \; if \;\; s_i \notin \Sigma_j \end{cases} \qquad (4)$$

Example 1:

An example of a balanced incomplete block design is the $(7,3,1) - BIBD$, where $S = \{1, 2, 3, 4, 5, 6, 7\}$ *and* $\Sigma = \{123, 145, 167, 246, 256, 347, 356\}$. The blocks of the balanced incomplete block design are the six lines and the circle in the diagram below usually called the Fano Plane.

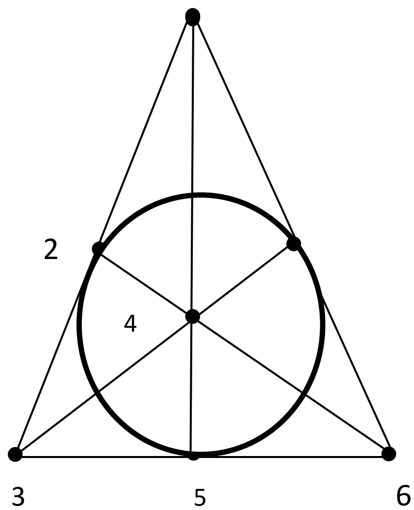

Figure 1: Fano Plane for a (7,3,1) BIBD

The incidence matrix of the (7, 3, 1) – BIBD represented by *Figure 1* is given as;

$$M = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The following are some results which dealt with the properties of the incidence matrices of block designs as well as the necessary and sufficient condition on the existence of symmetric block designs.

Theorem 1[11]

Assume that (S,Σ) is a balanced incomplete block design with parameters v, k, α, then,

(i) Every point appears in exactly $r = \frac{\alpha(v-1)}{k-1}$ blocks.

(ii) (S,Σ) has exactly $b = \frac{vr}{k} = \frac{\alpha(v^2-v)}{k^2-k}$ blocks.

The result below gives the necessary as well as the sufficient condition for the existence of a symmetric block design. It is called 'The Bruck – Ryser – Chowla theorem.

Theorem 2

If (S,Σ) is a symmetric block design with parameters v, k, α,

(i) If v is even, then $k-\alpha$ is a square (ii) if v is odd, then $z^2 = (k-\alpha)x^2 + (-1)^{(v-1)/2}\alpha y^2$ has a solution in integers $x, y, z,$ not all zero.

The following lemma is a special case of the result from the property of incidence matrix as found in [10].

Lemma 1

Assume that (S,Σ) is a self – dual block (that is, $M = M^T$ is isomorphic), where *M* is its incidence matrix. If the left and right kernel of M are obviously rationally equivalent, then the product of the positive eigen values of NN^T is an integral square.

3.0 The main Result

Consider the incidence matrix *M* of a symmetric block design (S,Σ) with parameters v, k, α. Clearly $v = b \text{ and } k = r$ and hence $k-\alpha$ is a square if v is even by Theorem 2.

The following assumptions are necessary for the main result;

Let *M* be an incidence matrix of the block design (S,Σ) which is symmetric, Then

$C_1 : MJ = JM$

$C_2 : JM = kJ$

$C_3 : MM^T = B = (K-\alpha)I + \alpha J$

$C_4 : M^T M = B = (K-\alpha)I + \alpha J$

where $J = J_V$ *and* $I = I_v$ are the v by v matrix and identity matrix of order *v* respectively. The following result gives a set of conditions for the incidence matrix of a symmetric block design.

Theorem 3

Suppose that *M* is $v \times v$ nonsingular incidence matrix of a $(v, k, \alpha) - BIBD\ (S, \sum)$. Then the parameters v, k, α satisfy the equation $k^2 - k = \alpha(v-1)$. Suppose also that either condition C_3 or C_4 and also either condition C_1 or C_2 are satisfied by *M*. Then *M* satisfies the entire conditions $C_1 - C_4$ and as such, $(S, \sum)$ is a symmetric block design.

Proof:

Claim that C_3 and C_1 hold. By *Equation (2),* we see that $\det B = (K-1)^{(V-1)}(V\alpha - \alpha + K)$.

Clearly, $K - \alpha \neq 0,\ v\alpha - \alpha + k \neq 0$ since *M* is a nonsingular matrix. Thus

$$MM^T = (K-\alpha)I + \alpha J, \quad MJ = KJ \tag{5}$$

By the right side of *Equation (5),* multiplying through by M^{-1} one finds out that

$$M^{-1}(MJ) = M^{-1}(KJ) \tag{6}$$

and so $k \neq 0$ *and* $M^{-1}J = k^{-1}J$. Similarly, $(MJ)^T = (KJ)^T$ *or* $JM^T = KJ$ since $J^T = J$ Note also that $J^2 = vJ$ and we obtain

$$\begin{aligned} M^T = M^{-1}(MM^T) &= (k-\alpha)M^{-1} + \alpha M^{-1}J \\ &= (k-\alpha)M^{-1} + \alpha k^{-1}J \\ kJ &= JM^T \\ &= (k-\alpha)JM^{-1} + \alpha k^{-1}J^2 \\ &= (k-\alpha)JM^{-1} + \alpha k^{-1}vJ \end{aligned}$$

It follows that

$$JM^{-1} = \frac{k - \alpha k^{-1}v}{k-\alpha} J = mJ \tag{7}$$

where *m* is a constant and as such *Equation (7)* results in $J = mJM$

$$\begin{aligned} vJ = J^2 &= (mJM)J \\ &= mJ(MJ) \\ &= mJ(kJ) \\ &= mkJ^2 \\ &= mkvJ \end{aligned} \tag{8}$$

$$\Rightarrow v = mkv \ \Rightarrow mk = 1, \quad m = k^{-1}$$

However, one had

$$m(k-\alpha) = k - \alpha k^{-1}v \tag{9}$$

Substituting $m = k^{-1}$ in *Equation (9)* yields

$$k^{-1}(k-\alpha) = k - \alpha k^{-1}v \tag{10}$$

Multiplying *Equation (10)* by k gives

$k - \alpha = k^2 - \alpha v$ which is the relation

$$k^2 - k = \alpha(v-1) \tag{11}$$

as required to be proved. In a similar way, $JM^{-1} = mJ = k^{-1}J$ yields

$$J = k^{-1}JM$$

i.e.

$$JM = kJ \tag{12}$$

which is C_2. Using the result in *Equation (7)*, we see that

$$M^T = (k-\alpha)M^{-1} + \alpha k^{-1}J \tag{13}$$

Multiplying both sides by M yields

$$\begin{aligned} M^T M &= (k-\alpha)I + \alpha k^{-1}JM \\ &= (k-\alpha)I + \alpha k^{-1}(kJ) \\ &= (k-\alpha)I + \alpha J \end{aligned} \tag{14}$$

which is C_4 and hence C_3 and C_1 imply C_4 and C_2 . Moreover, $k^2 - k = \alpha(v-1)$ holds true.

Claim that C_3 and C_2 hold and that M is nonsingular. Then we have

$$MM^T = (k-\alpha)I + \alpha$$

$$JM = kJ$$

Multiplying both sides of C_3 by J yields

$$J(MM^T) = (k-\alpha)J + \alpha J^2$$

$$kJM^T = (k-\alpha)J + \alpha vJ = mJ, \quad m = k-\alpha+\alpha v$$

$$kJ(M^TJ) = mJ^2$$

$$kJ(JM)^T = mJ^2$$

$$kJ(kJ)^T = mJ^2$$

$$k^2J^2 = mJ^2$$

It follows that $k^2 = m = k - \alpha + \alpha v$ which is again $k^2 - k = \alpha(v-1)$ which is *Equation (11)* that we wish to establish.

Similarly,

$kJM^T = mJ = k^2J \ \ with\ k \neq 0$ by the fact that M is nonsingular. As such, $JM^T = kJ$ which results

$$AJ = (JM^T)^T = (kJ)^T = kJ \tag{15}$$

Equation (13) is C_1 as we require. Now,

$$\begin{aligned} M^T M &= M^{-1}(MM^T)M \\ &= (k-\alpha)I + \alpha M^{-1}JM \end{aligned}$$

Since $MJ = kJ = JM$, we see that $M^{-1}JM = J$ and as such

$$M^T M = (k-\alpha)I + \alpha J \tag{16}$$

Equation (14) is C_4. Therefore, C_3 and C_2 imply conditions C_1 and C_4. By substituting M^T for M , we see that C_4 and either C_1 or C_2 imply the other relations and our main result is established.

4. Conclusion

We see that M being an incidence matrix of a block design (S, Σ) amounts to *Equation (14)* stating that any two distinct block designs have exactly α objects in common and as such M^T is also an incidence matrix of the block design and by implication of the conditions $C_1 - C_4$, (S, Σ) is a symmetric block design.

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