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Global Asymptotic Behaviour of Certain Damped and Unforced Duffing's Oscillator

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Abstract

In this paper, we analyze the global asymptotic behaviour of certain damped and unforced Duffing's oscillator using Lyapunov direct method and constructing an appropriate Lyapunov function. The results show that when the damping coefficient is small the system will still oscillate, but there will be a decrease in amplitude as its energy is converted to heat over time thus, forcing the system to return to its equilibrium point. The main contribution lies in construction an appropriate and suitable Lyapunov function for the cubic Duffing oscillator.

Keywords: Asymptotic Stability, Periodic Solution, Lyapunov Direct Method, Damping
AMS Subject Classification: 58F09, 58F14, 58F22

1.0 Introduction

Many physical phenomena are modelled by nonlinear system of ordinary differential equation. The Duffing oscillator is one of the prototype system of nonlinear dynamics which is used in the study of harmonic oscillation and chaotic nonlinear dynamics in the wake of early studies [1]. The equation has been used to model a variety of physical processes such as signal processing [2], emission characteristics of saw dust particle [3], ultra-wide (UWB) radio system [4] and non-linear spring mass system [5]. Due to its important in Physics, Engineering, Biology and Communication theory, the damped Duffing oscillation has received wide interest. The model is used in the study of fussy modeling and the adaptive control of uncertain chaotic system [6]. The equation is one of the fundamental equations in the study of the motion of a classical particle in a double well potential [7]. In [8-13] different methods have been used to study damped Duffing oscillator producing accurate results. However, various methods for studying global asymptotic behaviour of nonlinear differential equation exist in literature in which solution and a vast number of profound results have been established. For instance see [14-20]. On the use of Lyapunov direct method see [21-24].

Motivated by the above literature and ongoing research in this direction, the objective of this paper is to investigate the global asymptotic behaviour of certain damped and unforced Duffing equation of the form

$$\ddot{x} + \delta\dot{x} + \alpha x + \beta x^3 = 0 \quad (1.1)$$

using the Lypunov direct method. In equation (1.1) δ is the damping coefficient which controls the size of the damping, α is the stiffness constant which controls the linear stiffness and β is the coefficient of nonlinear term which control the amount of non-linearity in the restoring force. The restoring force in equation (1.1) is $\alpha x + \beta x^3$. When $\alpha > 0$ and $\beta > 0$, the spring is called the hardening spring and for $\alpha > 0$ and $\beta < 0$ it is called the softening spring. Other conditions that can be place on $\delta \; and \; \beta \; are$ as follows:

(i) If $\delta \geq 0$ then equation (1.1) will end up at its stable equilibrium points. The equilibrium points, stable and unstable are at $\alpha x + \beta x^3 = 0$

(ii) If $\alpha > 0$, the stable equilibrium is at $x = 0$.

(iii) If $\alpha < 0 \; and \; \beta > 0$, the stable equilibria are at $x = \sqrt{\frac{-\beta}{\alpha}}$ and $x = -\sqrt{\frac{-\beta}{\alpha}}$

(iv) If $\beta < 0$, the phase portrait curves are closed.

2.0. Preliminaries

Definition 2.1. (Stability) The equilibrium point $x = 0$ is stable if for each $\epsilon > 0$ there exist a $\delta > 0$ such that $\|x(0)\| < \delta$ implies that $\|x(t)\| < \epsilon$ for $t \leq 0$.

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Definition 2.2. (Stability in the sense of Lyapunov) The equilibrium point $x = 0$ is stable in the sense of Lyapunov at $t = t_0$ if for any $\epsilon > 0$ there exist a $\delta(t_0, \epsilon) > 0$ such that $\|x(t_0)\| < \delta \Longrightarrow \|x(t)\| < \epsilon, \forall\ t \geq t_0$.
Definition 2.3. (Asymptotically stability) A system is called asymptotically stable around its equilibrium point if it satisfies the following conditions
(i) Given any $\epsilon > 0$ there exist a $\delta_1$such that $\|x(t_0)\| < \delta_1$ then $\|x(t)\| < \epsilon$ for all $t \geq t_0$
(ii) There exist $\delta_2 > 0$ such that if $\|x(t_0)\| < \delta_2$ then $x(t) \to 0$ as $t \to \infty$.
Definition 2.4. (Global asymptotic stability) Let $x = 0$ be an equilibrium of $\dot{x} = f(x)$ and Let $V: D \to R$ be continuously differentiable function such that
(i) $V(0) = 0$
(ii) $\dot{V}(x) > 0$
(iii) $V(x)$ is radially unbounded
(iv) $\dot{V}(x) < 0$,
Then $x = 0$ is globally asymptotically stable.
Definition 2.5. A function $V: \mathbb{R}^+ \to \mathbb{R}$ is positive definite if
(i) $V(x) \leq 0\ \forall\ x$
(ii) $V(x) = 0$ if and only if $x = 0$
(iii) All sublevel sets of V are bounded.

Theorem 2.6. Consider the autonomous differential equation
$$\dot{x} = f(x) \qquad (2.1)$$
Suppose there exists a function $V: R^n \to R$ which is continuously differentiable and satisfies the following conditions
(i) $V(x)$ is positive definite i.e. $\left||x|\right| \leq V(x)$
(ii) The time derivative $\dot{V}$ of $V(x)$ along the solution path of equation (2.1) is negative semi-definite i.e. $\dot{V} \leq 0$. Then the trivial solution $x = 0$ of equation (2.1) is locally stable (stable in the sense of Lyapunov)
Proof. Since $V(x)$ is positive definite then $V(0) = 0$ and $V(x) > 0 \ \Rightarrow \parallel V(x) \parallel = V(x)$ and
$$\parallel x \parallel \leq V(x) \qquad (2.2)$$
$V(x)$ is continuously differentiable implies $V(x)$ is continuous and so continuous at the origin. So that given any $\varepsilon > 0\ there\ exists\ \delta > 0\ s.t\ \parallel x_0 - 0 \parallel < \delta\ implies\ \parallel V(x_0) - V(0) \parallel < \varepsilon$ that is
$$\parallel x_0 \parallel < \delta\ implies\ \parallel V(x_0) \parallel < \varepsilon \qquad (2.3)$$
$Let\ x(t)$ be any solution of equation (2.1) s.t $\parallel x_0 \parallel < \delta$. Since $\dot{V}$ is negative semi-definite i.e. $\dot{V} \leq 0$ then V is non-increasing. This means that if $x \geq x_0$ then
$$V(x) \leq V(x_0) \qquad (2.4)$$
Combining equation (2.2), (2.3) and (2.4) we have that $\parallel x_0 \parallel < \delta$ implies $\parallel x \parallel \leq V(x) \leq V(x_0) < \varepsilon \Rightarrow \parallel x \parallel < \delta$. Thus given $\epsilon > 0\ there\ exists\ \delta > 0\ s.t\ \parallel x_0 \parallel < \delta$ implies $\parallel x(t) \parallel < \varepsilon$. This shows that the trivial solution of $x = 0$ of equation (2.1) is stable.
Definition 2.7. Lyapunov functions are positive definite function that is used to establish stability. Intuitively, the word lyapunov function physically means energy. In lyapunov function, a continuously differentiable function V can be expressed as a function of the equilibrium point of a differential system or as a function of an independent variable x. Given the gradient of V defined by
$$\nabla V(x) = \frac{\partial V(x)}{\partial x_1}, \frac{\partial V(x)}{\partial x_2}, \ . \ . \ . \frac{\partial V(x)}{\partial x_n} \qquad (2.5)$$
and the Lie derivative of V defined by the function
$\dot{V} = R^n \to R$
then
$$\dot{V}(x) = \nabla V(x) \,.\, \dot{x} = \nabla V(x) \,.\, f(x) \qquad (2.6)$$
where $\dot{x} = f(x(t))$ is a time-invariant nonlinear system.
$\dot{V}$ is a function of the state $\dot{V}(x)$ evaluated at a certain state $\bar{x}$ to give the rate of increase of V. $\dot{V}(x)$ is the derivative of V with respect to time along the trajectory of the system passing through $\bar{x}$.
Remark: Lyapunov functions arise naturally for linear systems but in general the construction of lyapunov function for nonlinear differential equation with higher exponent is an open problem.

3.0 Variable Gradient Method

Variable gradient method is a logical and systematic method of generating lyapunov function for determining stability of nonlinear autonomous systems. The method is based on the assumption of a variable gradient function from which both V and $\dot{V}$ may be determined, The n unknown elements of each of the n component of the variable gradient are determined from the constraints V and the curl equation. The type of V function produced include those involving higher order terms, one or more integrals and terms with more than two state variables as factors.

However, construction of lyapunov function using this method involves obtaining a scalar V and time derivative $\dot{V}$ in which the state variables are implicit function of time [25].
Assume that the gradient of the lyapunov function $V(x)$ is known up to some parameters then

$$\nabla V(x) = [\nabla V_1, \nabla V_2, . . . \ \nabla V_n]^T = \left[\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2} . . . \frac{\partial V}{\partial x_n}\right]^T \quad (3.1)$$

The curl condition which simplify the coefficient in the i and j component gives

$$\frac{\partial^2 V}{\partial x_i \partial x_j} = \frac{\partial \nabla V_i}{\partial x_j} = \frac{\partial \nabla V_j}{\partial x_i} \quad (3.2)$$

The lyapunov function $V(x)$ is given by

$$V(x) = \int_0^x \nabla V(x) ds \quad (3.3)$$

The integral in equation (3.3) is independent of the integration path. This shows that the integral does not depend on the end point but traced from the path taken.

4.0 Results

4.1 Procedure for Variable Gradient Method Of Construction Lyapunov Function

(1) Write down the gradient of the scalar function.
(2) Simplify the terms in the gradient function to get their coefficient using the curl condition.
(3) Compute and integrate $\dot{V}(x)$.
(4) Choose value for k_{ij} for $i, j = 1 \ . . . \ n$ so that $\dot{V}(x)$ is negative definite and $V(x)$ is positive definite.

4.2 Variable Gradient Method of Construction Lyapunov Function

Employing the method used in [26] and using the equivalent systems of equation (1.1) where $\delta > 0, \alpha > 0$ and $\beta > 0$, the gradient is in the form $\nabla V(x) = [\nabla V_1 \ . \ . \ . \nabla V_n]^T$
where $\nabla V_i = \sum_{j=1}^{n} k_{ij} x_j$, $i = 1, \ . . . \ n$ and $\nabla V = \frac{\partial V}{\partial x} = g(x)$

$$\nabla V(x) = \begin{bmatrix} k_{11}x_1 + k_{12}x_2 \\ k_{21}x_1 + k_{22}x_2 \end{bmatrix} \quad (4.1)$$

Simplifying the coefficient r_{ij} $i, j = 1, \ . . . \ n$ using the curl condition $\frac{\partial^2 V}{\partial x_i \partial x_j} = \frac{\partial \nabla V_i}{\partial x_j} = \frac{\partial \nabla V_j}{\partial x_i}$
we have

$$\frac{\partial \nabla V_1}{\partial x_2} = x_1 \frac{\partial k_{11}}{\partial x_2} + x_2 \frac{\partial k_{12}}{\partial x_2} + k_{12} \quad (4.2)$$

and

$$\frac{\partial \nabla V_2}{\partial x_1} = x_1 \frac{\partial k_{21}}{\partial x_1} + x_2 \frac{\partial k_{22}}{\partial x_1} + k_{12} \quad (4.3)$$

Since $\frac{\partial \nabla V_1}{\partial x_2} = \frac{\partial \nabla V_2}{\partial x_1}$ we have

$$x_1 \frac{\partial k_{11}}{\partial x_2} + x_2 \frac{\partial k_{12}}{\partial x_2} = x_1 \frac{\partial k_{21}}{\partial x_1} + x_2 \frac{\partial k_{22}}{\partial x_1} \quad (4.4)$$

Obtaining $\dot{V}(x)$ such that $\frac{\partial k_{ir}}{\partial x_j} = 0, i \neq j, r = 1, 2, \ldots$ and r_{ii} constant, $i = 1, 2, \ . . .$ we have

$$\frac{\partial \nabla V_1}{\partial x_2} = x_1 \frac{\partial k_{11}}{\partial x_2} \text{ and } \frac{\partial \nabla V_2}{\partial x_1} = x_2 \frac{\partial k_{22}}{\partial x_1}$$

$$\dot{V}(x) = \left[\frac{\partial V}{\partial x}\right]^T f(x) = [k_{11}x_1 \quad k_{22}x_2] \begin{bmatrix} x_2 \\ -\alpha x_2 - \delta x_1 - k(x_1) \end{bmatrix}$$

$$= k_{11}x_1x_2 - k_{22}(\alpha x_2^2 + \delta x_1 x_2 + k(x_1)x_2) \quad (4.5)$$

Integrating $\dot{V}(x)$ and choosing r_{ij} $i, j = 1, \ . . . \ n$ so that $V(x)$ is positive definite and $\dot{V}(x)$ is negative definite gives

$$V(x) = \int_0^{x_1} \nabla V_1 (s_1, 0) ds_1 + \int_0^{x_2} \nabla V_2 (x_1, s_2) ds_2$$

Since $x_2 = 0$ the first term in equation (4.5) vanishes and we have

$$V(x) = -k_{22}\left[\frac{\alpha x_2^3}{3} + \frac{\delta x_1 x_2^2}{2} + \frac{k(x_1)x_2^2}{2}\right]$$

For $V(x)$ to be positive definite we let $k_{22} = -1$ so that

$$V(x) = \frac{1}{6}[2\alpha x_2^3 + 3\delta x_1 x_2^2 + 3k(x_1)x_2^2] > 0 \quad (4.6)$$

$$\dot{V}(x) = -[k_{22}(\alpha x_2^2 + \delta x_1 x_2 + k(x_1)x_2^2) - k_{11}x_1x_2] < 0 \quad (4.7)$$

Hence (4.6) and (4.7) shows that the equilibrium point is asymptotically stable.
Since $V(x) \geq 0, x = 0$ is global asymptotic stable because $V(x) \to \infty$ as $\|x\| \to \infty$.

5.0 Conclusion

From our result, we were able to obtain the global asymptotic stability of an unforced Duffing equation. The stability analysis was analysed using the variable gradient method to achieve global asymptotic stability. This method of Lyapunov construction has an advantage over other methods in literature because it allows us to determine the stability of a differential system without explicitly integrating the differential equation. This approach invariably cause the damping coefficient to decrease hence forcing the system to return to equilibrium as fast as possible.

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