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On Some Close-To-Convex Functions Defined By Al-Oboudi Differential Operator

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Abstract

In this paper, we employ the technique of the Al-Oboudi differential operator to study certain subclasses of analytic functions such as $TL_\beta(\alpha), SL_\beta, CCSL_\beta, CCTL_\beta, CCSL_\beta(\alpha)$ ***and*** $CCTL_\beta(\alpha)$. ***These subclasses of analytic functions generalized the concepts of functions with positive and negative coefficients.***

Keywords: Univalent functions, starlike functions, close-to-convex functions with positive and negative coefficients, Salagean differential operator, Al-Oboudi differential operator.

1.0 Introduction

Let A be the class of functions $f(z)$, that are analytic in the unit disk $D = \{z \in C : |z| < 1\}$, with the normalization that $f(0) = f'(0) - 1 = 0$. In other words, the function $f(z)$ in A have the power series representation

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \; z \in D. \qquad (1.1)$$

We denote by S, the class of univalent functions in D. The subclass of univalent functions consisting of convex functions is denoted by S^c while S^* denotes the subclass of starlike functions. Analytically, it is well-known that $f(z) \in S^*$ if and only if $\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0, \; z \in D.$ A function $f(z) \in A$ is said to be close-to-convex if there exists a function $g(z) \in S^*$ such that $\mathrm{Re}\left(\frac{zf'(z)}{g(z)}\right) > 0, \; z \in D.$ The class of close-to-convex functions defined in the unit disk is denoted by $CC.$ One can easily verify that $S^c \subset S^* \subset CC \subset S.$

Lemma 1.1 (Kaplan's Theorem)[1]

Let $f(z)$ be analytic and locally univalent in U, then $f(z)$ is close-to-convex, if and only if

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} d\theta > -\pi, \; z = re^{i\theta}, \qquad (1.2)$$

for each r in (0,1) and every pair θ_1, θ_2 with $0 \le \theta_1 < \theta_2 \le 2\pi.$

In [2] the subfamily T of S consisting of functions f of the form

$$f(z) = z - \sum_{j=2}^{\infty} a_j z^j, \; a_j \ge 0, \;\; j = 2,3,........ \; z \in D \qquad (1.3)$$

was introduced.

The aim of this paper is to define a class of close-to-convex functions with positive and negative coefficients and to give some of its properties using a modified Salagean Operator.

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2.0 Statement of the Problem

Let D^n be the Salagean differential operator. (See [3])

$D^n : A \to A,\ n \in N,$ Defined as:

$$D^0 f(z) = f(z) \qquad (2.1a)$$

$$D^1 f(z) = Df(z) = zf'(z) \qquad (2.1b)$$

$$D^n f(z) = D\left(D^{n-1} f(z)\right) \qquad (2.1c)$$

Consequently, if

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j \qquad (2.2)$$

Application of equations (2.1) to (2.2) results

$$D^0 f(z) = f(z) = z + \sum_{j=2}^{\infty} a_j z^j$$

$$D^1 f(z) = Df(z) = zf'(z) = z + \sum_{j=2}^{\infty} j a_j z^j$$

$$\vdots$$

$$D^n f(z) = D\left(D^{n-1} f(z)\right) = z + \sum_{j=2}^{\infty} j^n a_j z^j \qquad (2.3)$$

It is easy to see that if $f \in T$ and $f(z) = z - \sum_{j=2}^{\infty} a_j z^j,\ a_j \ge 0,\ j = 2,3,\ldots,\ z \in D,$ where T is a function with negative coefficient, then

$$D^n f(z) = z - \sum_{j=2}^{\infty} j^n a_j z^j \qquad (2.4)$$

[4] Let $\beta, \lambda \in \Re,\ \beta \ge 0,\ \lambda \ge 0$ and $f(z) = z - \sum_{j=2}^{\infty} a_j z^j$. We denote by D_λ^β the linear operator defined as follows;

$$D_\lambda^\beta : A \to A$$

$$D_\lambda^\beta f(z) = z + \sum_{j=2}^{\infty} \left(1 + (j-1)\lambda\right)^\beta \qquad (2.5)$$

Given two functions $f(z)$ and $g(z)$ where

$f(z) = z - \sum_{j=2}^{\infty} a_j z^j$ and $g(z) = z - \sum_{j=2}^{\infty} b_j z^j$, the Hadamard product (or convolution) $f * g$ of $f(z)$ and $g(z)$ is defined by

$$(f * g)(z) = z - \sum_{j=2}^{\infty} a_j b_j z^j = (g * f)(z) \qquad (2.6)$$

If $f \in T,\ f(z) = z - \sum_{j=2}^{\infty} a_j z^j,\ a_j \ge 0,\ j = 2,3,\ldots,\ z \in D$

We say that f is in the class $TL_\beta(\alpha)$ if

$$\operatorname{Re}\left(\frac{D_\lambda^{\beta+1} f(z)}{D_\lambda^\beta f(z)}\right) > \alpha,\ \alpha \in [0,1),\ \lambda \ge 0,\ \beta \ge 0,\ z \in D.$$

It was proved in [5] that if $\alpha \in [0,1),\ \lambda \ge 0,\ \beta \ge 0.$ The function $f \in T$ of the form

$f(z) = z - \sum_{j=2}^{\infty} a_j z^j,\ a_j \ge 0,\ j = 2,3,\ldots,\ z \in D$ is in the class $TL_\beta(\alpha)$ if and only if

$$\sum_{j=2}^{\infty} \left[\left(1 + (j-1)\lambda\right)^\beta \left(1 + (j-1)\lambda - \alpha\right)\right] a_j < 1 - \alpha \qquad (2.7)$$

Also, in [6], it was established that if $\alpha \in [0,1),\ \lambda \geq 0,\ \beta \geq 0.$ The function $f \in CCTL_\beta(\alpha)$ with respect to the function $g(z) \in TL_\beta(\alpha)$ if and only if

$$\sum_{j=2}^{\infty}[1+(j-1)\lambda]^\beta\left[(1+(j-1)\lambda)a_j+(2-\alpha)b_j\right]<1-\alpha \qquad (2.8)$$

3.0 Proof of the Problem

We now present some properties of the coefficients of the close-to-convex functions for positive and negative origin and positive and negative coefficients for order α.

Theorem 3.1. Let $\lambda \geq 0,\ \beta \geq 0,$ the function $f \in S$ belongs to the class $CCSL_\beta$ with respect to the function $g(z) \in SL_\beta$ if

$$\sum_{j=2}^{\infty}[1+(j-1)\lambda]^\beta\left[(1+(j-1)\lambda)a_j-b_j\right]<1 \qquad (3.1)$$

Theorem 3.2. Suppose $\lambda \geq 0,\ \beta \geq 0,\ f \in T$ belongs to the class $CCTL_\beta$ with respect to the function $g(z) \in TL_\beta$ if

$$\sum_{j=2}^{\infty}[1+(j-1)\lambda]^\beta\left[b_j-(1+(j-1)\lambda)a_j\right]<1 \qquad (3.2)$$

Theorem 3.3 Let $0 \geq \alpha < 1,\ \lambda \geq 0,\ \beta \geq 0,$ the function $f \in CCSL_\beta(\alpha)$ with respect to the function $g(z) \in SL_\beta(\alpha)$ if

$$\sum_{j=2}^{\infty}[1+(j-1)\lambda]^\beta\left[(1+(j-1)\lambda)a_j-(1-\alpha)b_j+\alpha\right]<1-\alpha \qquad (3.3)$$

Proof: Let $f \in CCSL_\beta(\alpha),\ f(z)=z+\sum_{j=2}^{\infty}a_jz^j,\ a_j \geq 0,\ j \geq 2,$ with respect to the function $g(z)=z+\sum_{j=2}^{\infty}b_jz^j,\ b_j \geq 0,\ j \geq 2,$ $\lambda \geq 0,\ \beta \geq 0,$ then

$$\mathrm{Re}\left(\frac{D_\lambda^{\beta+1}f(z)}{D_\lambda^\beta g(z)}\right)>\alpha$$

It suffices to show that

$$\left|\frac{D_\lambda^{\beta+1}f(z)}{D_\lambda^\beta g(z)}-(1-\alpha)\right|<(1-\alpha)$$

$$=\left|\frac{z+\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta+1}a_jz^j}{z+\sum_{j=2}^{\infty}[1+(j-1)\lambda]^\beta b_jz^j}-(1-\alpha)\right|$$

$$=\left|\frac{\left[\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta+1}a_j-\sum_{j=2}^{\infty}[1+(j-1)\lambda]^\beta b_j+\alpha\left(1+\sum_{j=2}^{\infty}[1+(j-1)\lambda]^\beta b_j\right)\right]z^{j-1}}{1+\sum_{j=2}^{\infty}[1+(j-1)\lambda]^\beta b_jz^{j-1}}\right|$$

$$=\left|\frac{\sum_{j=2}^{\infty}[1+(j-1)\lambda]^\beta\left[(1+(j-1)\lambda)a_j-b_j+\alpha b_j+\alpha\right]z^{j-1}}{1+\sum_{j=2}^{\infty}[1+(j-1)\lambda]^\beta b_jz^{j-1}}\right|$$

Along the real axis, letting $z \to 1^-$, we have

$$\leq\frac{\sum_{j=2}^{\infty}[1+(j-1)\lambda]^\beta\left[(1+(j-1)\lambda)a_j-(1-\alpha)b_j+\alpha\right]}{1+\sum_{j=2}^{\infty}[1+(j-1)\lambda]^\beta|b_j|}$$

By hypothesis,

$$\left|\frac{D_\lambda^\beta f(z)}{D_\lambda^\beta g(z)}-(1-\alpha)\right|<\frac{1-\alpha}{1+\sum_{j=2}^{\infty}[1+(j-1)\lambda]^\beta|b_j|}<1-\alpha.$$

Theorem 3.4. Let $\lambda \geq 0,\ \beta \geq 0,\ \ 0 \geq \alpha < 1,$ then the function $f \in T$ belongs to the class $CCTL_\beta(\alpha)$ with respect to the function $g(z) \in TL_\beta(\alpha)$ if

$$\sum_{j=2}^{\infty}\left[1+(j-1)\lambda\right]^{\beta}\left[(1-\alpha)b_j-(1+(j-1)\lambda)a_j+\alpha\right]<1-\alpha \tag{3.4}$$

Proof: Let $f \in CCTL_{\beta}(\alpha),\ f(z)=z-\sum_{j=2}^{\infty}a_j z^j,\ a_j \geq 0,\ j\geq 2,$ with respect to the function

$g(z)=z-\sum_{j=2}^{\infty}b_j z^j,\ b_j \geq 0,\ j\geq 2,\ \lambda \geq 0,\ \beta \geq 0,$ then

$$\operatorname{Re}\left(\frac{D_{\lambda}^{\beta+1}f(z)}{D_{\lambda}^{\beta}g(z)}\right)>\alpha$$

We show that

$$\left|\frac{D_{\lambda}^{\beta+1}f(z)}{D_{\lambda}^{\beta}g(z)}-(1-\alpha)\right|<(1-\alpha) \qquad 0\geq \alpha<1$$

$$=\left|\frac{z-\sum_{j=2}^{\infty}(1+(j-1)\lambda)^{\beta+1}a_j z^j}{z-\sum_{j=2}^{\infty}(1+(j-1)\lambda)^{\beta}b_j z^j}-(1-\alpha)\right|$$

$$=\left|\frac{z-\sum_{j=2}^{\infty}(1+(j-1)\lambda)^{\beta+1}a_j z^j-(1-\alpha)\left[z-\sum_{j=2}^{\infty}(1+(j-1)\lambda)^{\beta}b_j z^j\right]}{z-\sum_{j=2}^{\infty}(1+(j-1)\lambda)^{\beta}b_j z^j}\right|$$

$$\leq\frac{\left|\sum_{j=2}^{\infty}(1+(j-1)\lambda)^{\beta}\left[-(1+(j-1)\lambda)a_j+b_j+\alpha z-\alpha b_j\right]\right|\left|z^{j-1}\right|}{\left|1-\sum_{j=2}^{\infty}(1+(j-1)\lambda)^{\beta}b_j\right|\left|z^{j-1}\right|}$$

Along the real axis, letting $z\to 1^{-}$, we have

$$\leq\frac{\sum_{j=2}^{\infty}\left|(1+(j-1)\lambda)^{\beta}\left[(1-\alpha)b_j-(1+(j-1)\lambda)a_j+\alpha\right]\right.}{\left|1-\sum_{j=2}^{\infty}\left[1+(j-1)\lambda\right]^{\beta}b_j\right|}$$

By hypothesis,

$$\left|\frac{D_{\lambda}^{\beta+1}f(z)}{g(z)}-(1-\alpha)\right|<\frac{1-\alpha}{\left|1-\sum_{j=2}^{\infty}\left[1+(j-1)\lambda\right]^{\beta}b_j\right|}<1-\alpha.$$

4.0. Conclusion

This work is concerned with analytic functions defined in an open unit disk. The classes defined generalized the concepts of functions with positive and negative coefficients. We use the technique of Al-Oboudi differential operator.

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