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On Coefficient Bounds of Certain Close-To-Convex Functions with Negative Coefficients Using the Modified Salagean Differential Operator

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Abstract

In this paper, we establish the coefficient bounds of a class of close-to-convex functions with negative coefficients using a modified Salagean differential operator.

Keywords: *Univalent functions, Starlike functions, close-to-convex functions, coefficient bounds, Modified Salagean operator.*

1.0 Introduction

Let A denote the class of functions analytic in the unit disk $U = \{z : |z| < 1\}$ and of the form $A := \{f \in H(u) : f(0) = f'(0) - 1 = 0\}$ where $H(u)$ is the set of functions which are analytic in the unit disk $S := \{f \in A : univalent\}$.

We present the definitions of well-known classes of starlike, convex, close-to-convex functions.

$$S^* := \left\{ f \in S : \mathrm{Re}\frac{zf'(z)}{f(z)} > \alpha,\ z \in U \right\},\ \alpha \in [0,1),$$

$$S^c := \left\{ f \in S : \mathrm{Re}\frac{1 + zf'(z)}{f'(z)} > \alpha,\ z \in U \right\},\ \alpha \in [0,1),$$

$$CC := \left\{ f \in S : \exists\ g \in S^*, \ni\ \mathrm{Re}\frac{zf'(z)}{g(z)} > 0,\ z \in U \right\},$$

$$M_\alpha := \left\{ f \in S : \mathrm{Re}\left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(\frac{1 + zf''(z)}{f'(z)}\right)\right] > 0,\ z \in U \right\},\ \alpha \in R$$

The classes S^*, S^c, CC and M_α are known as star like, convex, close-to-convex and $\alpha - convex$ functions respectively. In [1], it was conjectured that if $f \in S$, then the coefficients a_n of f satisfies $|a_n| \le 2.$ That is,

$$f \in S,\ \ f(z) = z + \sum_{j=2}^{\infty} a_n z^n \to |a_2| \le 2 \qquad (1)$$

$$|a_n| \le n,\ \ n \in N,\ \ n \ge 2$$

$$K(z) = \frac{z}{(1-z)^2},\ \ z \in U$$

Proof: See [2].

Definition 1. [3]

We define the operator $D^n : A \to A,\ n \in N = \{0,1,2,...\}$ by

(a) $D^0 f(z) = f(z);$

(b) $D^1 f(z) = Df(z) = zf'(z);$

(c) $D^n f(z) = D(D^{n-1} f(z)) = z(D^{n-1} f(z))',\ z \in U,\ n \ge 1.$

The operator D^n is named the Salagean differential operator. We note that if $f \in A$ is a function of the form

$$f(z) = z - \sum_{j=2}^{\infty} a_j z^j,\ \ z \in U \qquad (2)$$

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And $n \in N$, then

$$D^n f(z) = z - \sum_{j=2}^{\infty} j^n a_j z^j, \; z \in U.$$

Definition 2. [4]

The operator $D_\lambda^n : A \to A, \; n \in N, \; \lambda \geq 0$ is defined by

(a) $D_\lambda^0 f(z) = f(z)$

(b) $D_\lambda^1 f(z) = (1-\lambda) f(z) + \lambda z f'(z) = D_\lambda f(z)$

(c) $D_\lambda^n f(z) = D_\lambda \left(D_\lambda^{n-1} f(z) \right), \quad z \in U.$

If $f \in A$ has the form (2), then

$$D_\lambda^\beta f(z) = z - \sum_{j=2}^{\infty} [1 + (j-1)\lambda]^\beta a_j z^j \qquad (3)$$

(3) is known as the Al-Oboudi differential operator.

2. Statement of Problem and Proofs

[5] Let $f \in T, \; f(z) = z - \sum_{j=2}^{\infty} a_j z^j, \; a_j \geq 0, \; j \geq 2, \; z \in U$ and $g(z) \in TL_\beta(\alpha)$. We say that f is in the class $CCTL_\beta(\alpha)$ if

$$\operatorname{Re} \frac{D_\lambda^{\beta+1} f(z)}{D_\lambda^\beta g(z)} > \alpha. \; \alpha \in [0,1), \; \lambda \geq 0, \; \beta \geq 0, \; z \in U.$$

Theorem 2.1. Suppose $\lambda \geq 0, \; \beta \geq 0, \; f \in T$ belongs to the class $CCTL_\beta$ with respect to the function $g(z) \in TL_\beta$ if

$$\sum_{j=2}^{\infty} [1 + (j-1)\lambda]^\beta \left[b_j - (1 + (j-1)\lambda) a_j \right] < 1 \qquad (4)$$

Proof: Let $f \in CCTL_\beta, \; f(z) = z - \sum_{j=2}^{\infty} a_j z^j, \; a_j \geq 0, \; j \geq 2,$ with respect to the function

$g(z) = z - \sum_{j=2}^{\infty} b_j z^j \in TL_\beta, \; b_j \geq 0, \; j \geq 2, \; \lambda \geq 0$ and $\beta \geq 0$ .we have that $\operatorname{Re}\left\{ \frac{D_\lambda^{\beta+1} f(z)}{g(z)} \right\} > 0$.

It is left to show that $\left| \frac{D_\lambda^{\beta+1} f(z)}{D_\lambda^\beta g(z)} - 1 \right| < 1$

$$= \left| \frac{z - \sum_{j=2}^{\infty} [1 + (j-1)\lambda]^{\beta+1} a_j z^j}{z - \sum_{j=2}^{\infty} [1 + (j-1)\lambda]^\beta b_j z^j} - 1 \right|$$

$$\leq \frac{\sum_{j=2}^{\infty} \left| [1 + (j-1)\lambda]^\beta \left[b_j - (1 + (j-1)\lambda) a_j \right] \right| |z|^{j-1}}{1 - \sum_{j=2}^{\infty} \left| [1 + (j-1)\lambda]^\beta \right| |b_j| |z|^{j-1}}$$

Along the real axis, letting $z \to 1^-$, we have

$$\leq \frac{\sum_{j=2}^{\infty} \left| [1 + (j-1)\lambda]^\beta \left[b_j - (1 + (j-1)\lambda) a_j \right] \right|}{1 - \sum_{j=2}^{\infty} \left| [1 + (j-1)\lambda]^\beta \right| |b_j|} \qquad (5)$$

By hypothesis,

$$\left|\frac{D_\lambda^{\beta+1} f(z)}{D_\lambda^{\beta} g(z)} - 1\right| < \frac{1}{1-\sum_{j=2}^{\infty}\left|[1+(j-1)\lambda]^\beta\right|\left|b_j\right|} < 1$$

Corollary 2.1. Let $\lambda \geq 0,\ \beta \geq 0,\ f \in T$ belongs to the class $CCTL_\beta$ with respect to the function $g(z) \in TL_\beta$; then

$$\left|a_j\right| < \left|\frac{2\sum_{j=2}^{\infty}[1+(j-1)\lambda]^\beta \left|b_j\right| - 1}{\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta+1}}\right|$$

Corollary 2.2. Let $\lambda \geq 0,\ \beta \geq 0,\ f \in T$ belongs to the class $CCTL_\beta$ with respect to the function $g(z) \in TL_\beta$, then

$$\left|b_j\right| < \frac{1+\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta+1}\left|a_j\right|}{2\sum_{j=2}^{\infty}[1+(j-1)\lambda]^\beta}$$

Theorem 2.2. Let $\lambda \geq 0,\ \beta \geq 0,\ \alpha \in [0,1)$, then the function $f \in T$ belongs to the class $CCTL_\beta(\alpha)$ with respect to the function $g(z) \in TL_\beta(\alpha)$ if

$$\frac{\sum_{j=2}^{\infty}[1+(j-1)\lambda]^\beta\left[(1-\alpha)b_j - (1+(j-1)\lambda)a_j + \alpha\right]}{\left|1-\sum_{j=2}^{\infty}[1+(j-1)\lambda]^\beta b_j\right|} < 1-\alpha \qquad (6)$$

Corollary 2.3. Let $\lambda \geq 0,\ \beta \geq 0,\ \alpha \in [0,1)$, then the function $f \in T$ belongs to the class $CCTL_\beta(\alpha)$ with respect to the function $g(z) \in TL_\beta(\alpha)$ if

$$\left|a_j\right| < \frac{(2-\alpha)\sum_{j=2}^{\infty}[1+(j-1)\lambda]^\beta\left|b_j\right| + \alpha\sum_{j=2}^{\infty}[1+(j-1)\lambda]^\beta - 1}{\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta+1}}$$

Proof: From (6),

Since $\left|1-\sum_{j=2}^{\infty}[1+(j-1)\lambda]^\beta b_j\right| > 0,$ then

$$\Rightarrow \sum_{j=2}^{\infty}(\infty 1+(j-1)\lambda)^\beta\left[(1-\alpha)b_j - (1+(j-1)\lambda)a_j + \alpha\right] < 1-\sum_{j=2}^{\infty}[1+(j-1)\lambda]^\beta b_j$$

$$\Rightarrow \sum_{j=2}^{\infty}(1+(j-1)\lambda)^\beta(1-\alpha)\left|b_j\right| - \sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta+1}\left|a_j\right| + \alpha\sum_{j=2}^{\infty}(1+(j-1)\lambda)^\beta < 1-\sum_{j=2}^{\infty}[1+(j-1)\lambda]^\beta\left|b_j\right|$$

$$\Rightarrow -\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta+1}\left|a_j\right| < 1-(2-\alpha)\sum_{j=2}^{\infty}[1+(j-1)\lambda]^\beta\left|b_j\right| - \alpha\sum_{j=2}^{\infty}[1+(j-1)\lambda]^\beta$$

$$\left|a_j\right| < \frac{(2-\alpha)\sum_{j=2}^{\infty}[1+(j-1)\lambda]^\beta\left|b_j\right| + \alpha\sum_{j=2}^{\infty}[1+(j-1)\lambda]^\beta - 1}{\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta+1}}$$

Corollary 2.4. Let $\lambda \geq 0,\ \beta \geq 0,\ 0 \geq \alpha < 1$, then the function $f \in T$ belongs to the class $CCTL_\beta(\alpha)$ with respect to the function $g(z) \in TL_\beta(\alpha)$ then

$$\left|b_j\right| < \frac{1+\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta+1}\left|a_j\right| - \alpha\sum_{j=2}^{\infty}[1+(j-1)\lambda]^\beta}{(2-\alpha)\sum_{j=2}^{\infty}[1+(j-1)\lambda]^\beta}$$

Proof: From (6)

Since $\left|1-\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta} b_j\right| > 0,$ then

$$\Rightarrow \sum_{j=2}^{\infty}(1+(j-1)\lambda)^{\beta}\left[(1-\alpha)b_j-(1+(j-1)\lambda)a_j+\alpha\right] < 1-\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta} b_j$$

$$\Rightarrow (1-\alpha)\sum_{j=2}^{\infty}(1+(j-1)\lambda)^{\beta}|b_j|-\sum_{j=2}^{\infty}(1+(j-1)\lambda)^{\beta+1}|a_j|+\alpha\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta} < 1-\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta}|b_j|$$

$$\Rightarrow (1-\alpha)\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta}|b_j|+\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta}|b_j| < 1+\sum_{j=2}^{\infty}(1+(j-1)\lambda)^{\beta+1}|a_j|-\alpha\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta}$$

$$|b_j| < \frac{1+\sum_{j=2}^{\infty}(1+(j-1)\lambda)^{\beta+1}|a_j|-\alpha\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta}}{(2-\alpha)\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta}}$$

Remark: When $\alpha = 0,$ we get the same result as obtained in corollary 2.1 and 2.2 respectively.

Corollary 4.5. Using Theorem 2.2, when $g(z) \equiv f(z)$ we have

$$|a_j| < \frac{1-\alpha\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta}}{\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta}[(2-\alpha)-(1+(j-1)\lambda)]}$$

3.0 Results and Conclusion

In this work, we discussed the coefficient bounds for certain analytic functions such as $TL_{\beta}(\alpha),\ SL_{\beta}(\alpha),\ CCSL_{\beta},\ CCTL_{\beta},\ CCSL_{\beta}(\alpha)$ and $CCTL_{\beta}(\alpha)$. These classes generalized the concepts of functions with positive and negative coefficients. We extended the work as in [5] by introducing the class $CCSL_{\beta}$ and $CCTL_{\beta}$. Our study of these classes of functions has thus exposed us to a number of very interesting properties of these classes of functions.

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