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Coefficient Bounds for Certain Subclass of Multivalent Bessel Functions of Complex Order

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Abstract

The object of this paper is to determine the coefficient bounds $\left|b_{p+1} - ub_{p+1}^2\right| and \left|b_{p+3}\right|$ ***for functions belonging to the class*** $M_{b,p,a}(\varphi)$***. Furthermore, the Fekete-Szego type inequalities is obtained for the class.***

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1.0 Introduction and Preliminaries

Let A_p denote the class of functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \qquad p \in \aleph \tag{1.1}$$

which are analytic and p-valent in the open unit disk $U = \{z : |z| < 1\}$. Similarly let $A_{p,k}(j)$ denote the class of functions of the form.

$$g(z) = \left(\frac{z}{2}\right)^p + \sum_{k=j+p}^{\infty} b_k \left(\frac{z}{2}\right)^k \qquad p, j \in N = \{1,2,3 \dots\} \tag{1.2}$$

which are also analytic and p-valent in the unit disk $U = \{z : |z| < 1\}$. The class of functions defined in (1.2) is known as p valent Bessel function and was established in [1].

Definition 1

Let f and g be analytic in U, then the function f is subordinate to g, if there exist a Schwarz function $w(z)$ analytic in U such that $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$ and $f(z) = g(w(z))$ for all $z \in U$. This is denoted by $f \prec g$. It is also known that if g is univalent in U then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

Definition 2

Let $\varphi(z)$ be an analytic function with positive real part in U such that $\varphi(0) = 1$ and $\varphi'(0) > 0$ and maps U unto a region starlike with respect to 1 and symmetric with respect to the real axis.

A function $f(z) \in A_p$ is said to be in the class $S_{b,p}^*(\varphi)$ if

$$1 + \left(\frac{1}{b}\left(\frac{1}{p}\right)\frac{zf'(z)}{f(z)}\right) - 1 \prec \varphi(z) \quad (p \in N, \quad z \in U) \tag{1.3}$$

A function $f(z) \in A_p$ is said to be in the class $C_{b,p}(\varphi)$ if it satisfies

$$1 - \frac{1}{b} + \frac{1}{bp}\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \varphi(z) \quad (p \in N, \quad z \in U) \tag{1.4}$$

The classes $S_{b,p}^*(\varphi)$ and $C_{b,p}(\varphi)$ were studied in [2].

For b= 1 we have the classes $S_p^*(\varphi)$ and $C_p(\varphi)$ in [3] and for p = b = 1 the classes reduces to the classes $S_p^*(\varphi)$ and $C(\varphi)$ which were earlier introduced and investigated in [4]. These classes becomes the classes of starlike and convex functions of order $\alpha(0 \le \alpha < 1)$ respectively when

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$\varphi(z) = \left(\frac{1+(1-2\alpha)z}{1-\ z} - 1\right)\ (0 \le \alpha < 1)$

Similarly, for the class of functions defined in (1.3) and (1.4), we define equivalent classes for the class of function defined (1.2)

A function $g(z)\ \epsilon\ A_{p,k}(j)$ is said to be in the class $S^*_{b,p}(\varphi)$ if it satisfies

$$\frac{1}{b}\left(\frac{1}{p}\frac{zg'(z)}{g(z)}\right) - 1 \prec \varphi(z) \quad (p\ \epsilon\ N, \qquad z\ \epsilon\ U) \qquad (1.5)$$

and in $C_{b,k,p}(\phi)$ if it satisfies

$$1 - \frac{1}{b} + \frac{1}{bp}\left(1 + \frac{zg''(z)}{g'(z)}\right) \prec \varphi(z) \quad (p\ \epsilon\ N, \qquad z\ \epsilon\ U) \qquad (1.6)$$

$$\left[1 + \frac{1}{b}\left(\frac{1}{p}\frac{zg''(z)}{g'(z)} - 1\right)\right]^{\alpha}\left[1 - \frac{1}{b} + \frac{1}{bp}\left(1 + \frac{zg''(z)}{g'(z)}\right)\right]^{1-\alpha} \prec \varphi(z)\ \ (b\ \epsilon\ \mathbf{C}\backslash\{0\}) \qquad (1.7)$$

where b is a non zero complex number

Fekete and Szego in 1933 gave the sharp bound for the functional $|a_3 - \mu a^2_2|$ for $f(z)\ \epsilon\ S$ when μ is real. The determination of the sharp bounds for the functional $|a_3 - \mu a^2|$ is known as the Fekete-Szego problem. And this has been investigated by several authors for different subclasses of S in [5 - 14].

In this paper sharp bounds for the Fekete-Szego coefficient functional are obtained for the class $M_{b,p,\alpha}(\phi)$ defined in (1.7),the class is actually the linear combination of the class defined in (1.5) and (1.6) respectively.

Let Ω be the class of analytic functions of the form

$$w(z) = w_1z + w_2z + w_3z + \cdots = \sum_{k=1}^{\infty} w_k z^k \qquad (1.8)$$

in the open unit disk U satisfying $|w(z)| < 1$.

To prove our result, we shall make use of the following lemmas

Lemma 1 [2,4]

If $w\ \epsilon\ \Omega$. Then

$$|w_2 - tw_1^2| \le \begin{cases} -t & if\ t \le -1 \\ 1 & if\ -1 \le t \le 1 \\ t & if\ t > 1 \end{cases}$$

when $t < -1$ or $t > 1$, the equality holds if and only if $w(z) = z$ or one of its rotation. If $-1 < t < 1$, then equality holds only if $w(z) = z^2$ one of its rotations. Equality holds for $t = -1$ if and only if

$$w(z) = z\frac{\lambda + z}{1 + \lambda z} \qquad (0 \le \lambda \le 1)$$

or one of its rotations, while for $t = 1$, the equality holds if and only if

$$w(z) = -z\frac{\lambda + z}{1 + \lambda z} \qquad (0 \le \lambda \le 1)$$

Or one of its rotations,

Although the above upper bound is sharp, it can be improved as follows

when -1< t < 1

$$|w_2 - tw_1^2| + (1+t)|w_1|^2 \le 1 \qquad (-1 < t \le 0)$$

and

$$|w_2 - tw_1^2| + (1-\ t)|w_1|^2 \le 1 \qquad (0 < t < 1)$$

Lemma 2 [2]

If $w\ \epsilon\ \Omega$, then for any complex number t

$$|w_2 - tw_1^2| \le \max\ (1, |t|)$$

The result is sharp for the functions $w(z) = z$ or $w(z) = z^2$

Lemma 3 [13]

If $w\ \ \epsilon\Omega$, then for any real numbers q_1 and q_2, then the following sharp estimate holds

$$|w_3 + q_1w_1w_2 + q_2w_1^2| \le \mathrm{H}\ (q_1, q_2)$$

$$H(q_1q_2) = \begin{Bmatrix} 1\ for\ (q_1, q_2) \in\ D_1\ \cup\ D_2 \\ |q_2|\ for (q_1, q_2) \in \cup_{k=3}^{7}\ D_K \\ \frac{2}{3}(|q_1| + 1)\left(\frac{|q_1|+1}{3|q_1+1+q_2}\right)^{\frac{1}{2}} for\ (q_1, q_2) \in\ D_8 \cup\ D_9 \\ \frac{q_2}{3}\left(\frac{q_1^2-4}{q_1^2-4q_2}\right)\left(\frac{q_1^2-4}{3(q_1-1}\right)^{\frac{1}{2}} for\ (q_1, q_2) \in\ D_{10} \cup\ D_{11} \sim \{\pm 2,\} \\ \frac{2}{3}(|q_1| - 1)\left(\frac{|q_1|-1}{3|q_1-1-q_2}\right)^{\frac{1}{2}} for\ (q_1, q_2) \in\ D_{12} \end{Bmatrix}$$

The extremal functions up to rotations are of the form $w(z) = z^3,\ w(z) = z$

$$w(z) = w_0(z) = \frac{(z[(1-\lambda)\epsilon_2 + \lambda\epsilon_1] - \epsilon_1\epsilon_2\ z}{1 - [(1-\lambda)\ \epsilon_1 + \lambda\ \epsilon_2]z}$$

$$w(z) = w_1(z) = \frac{z(t_1 - z)}{1 - t_1 z}\quad , \qquad w(z) = w_2(z) = \frac{z(t_2 + Z)}{1 + t_2 z}$$

$$|\ \epsilon_1| = |\ \epsilon_2| = 1,\ \epsilon_1 = t_0 - e^{-\frac{i\theta_0}{2}}(a \pm b),\ t_2 = -e^{-\frac{\theta_0}{2}}(ia \pm b)$$

$$a = t_0 cos\frac{\theta_0}{2},\quad b = \sqrt{i - t_0^2 sin^2\frac{\theta_2}{2},}\qquad \lambda = \frac{b \pm a}{2b}$$

$$t_0 = \left(\frac{2q_2(q_1^2+2) - 3q_2}{3(q_2-1)(q_1^2-4q_2}\right)^{\frac{1}{2}},\quad t_1 = \left(\frac{|q_1|+1}{3(|q_1|+1+q_2)}\right)^{\frac{1}{2}}$$

$$t_2 = \left(\frac{|q_2| - 1}{3(|q_1| - q_2)}\right)^{\frac{1}{2}}$$

$$cos\frac{\theta_0}{2} = \frac{q_1}{2}\left(\frac{q_2(q_1^2 + 8) - 2(q_1^2 + 2)}{2q_2(q_1^2 + 2) - 3q_1^2}\right)$$

The sets D_k $k = 1, 2, \cdots, 12$ are defined as follows

$D_1 = \{ (q_1, q_2) : |q_1| \leq \frac{1}{2}, |q_2| \leq 1\}$

$D_2 = \{ (q_1, q_2) : \frac{1}{2} \leq |q_1| \leq 2, \frac{4}{27}\ (|q_1| + 1)^3 - (|q_1| + 1) \leq q_2 \leq 1\}$

$D_3 = \{ (q_1, q_2) : |q_1| \leq \frac{1}{2}\ ,\ q_2 \leq -1\ \}$

$D_4 = \{ (q_1, q_2) : |q_1| \geq \frac{1}{2},\ q_2 \leq -\frac{2}{3}(|q_1| + 1)\ \}$

$D_5 = \{ (q_1, q_2) : |q_1| \leq 2, q_2 \geq 1\}$

$D_6 = \{ (q_1, q_2) : 2 \leq |q_1| \leq 4,\ q_2 \geq \frac{1}{12}(q_1^2 + 8)\ \}$

$D_7 = \{ (q_1, q_2) : |q_1| \geq 4,\ q_2 \geq \frac{2}{3}(|q_1| - 1)\ \}$

$D_8 = \{ (q_1, q_2) : \frac{1}{2} \leq |q_1| \leq 2,\ -\frac{2}{3}(|q_1| + 1) \leq q_2 \leq \frac{4}{27}(|q_1| + 1)^3 - (|q_1| + 1)\ \}$

$D_9 = \left\{(q_1, q_2):\ |q - 1| \geq 2,\ -\frac{2}{3}(|q_1| + 1) \leq q_2 \leq \frac{2|q_1|(|q_1||+1}{q_1^2+2|q_1|+4}\right\}$

$D_{10} = \left\{(q_1, q_2):\ 2 \leq |q_1| \leq 4, \frac{2|q_1|(|q_1||+1}{q_1^2+2|q_1|+4} \leq q_2 \leq \frac{1}{12}(q_1^2 + 8)\right\}$

$D_{11} = \left\{(q_1, q_2):\ |q_1| \geq 4, \frac{2|q_1|(|q_1||+1}{q_1^2+2|q_1|+4} \leq q_2 \leq \frac{2|q_1|(|q_1||-1}{q_1^2-2|q_1|+4}\right\}$

$D_{12} = \left\{(q_1, q_2):\ |q_1| \geq 4, \frac{2|q_1|(|q_1||-1}{q_1^2+2|q_1|+4} \leq q_2 \leq \frac{2}{3}(|q_1| - 1)\right\}$

2.0 Main Results

Theorem 2.1:

Let $g(z)$ be given by (1.2) and $\varphi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \cdots$ if $g(z) \in M_{b,p,\alpha}(\varphi)$ then for any real number μ

$$|b_{p+2}-\mu b_{p+1}^2|\leq\begin{cases}\frac{4bp^2B_1}{\tau_\alpha}\left(\frac{B_2}{B_1}-\frac{bpB_1}{\eta_\alpha^2}(\mu p\tau_\alpha+\lambda_\alpha)\right) & if\ \mu\leq\sigma_1\\ \frac{4bp^2B_1}{\tau_\alpha} & if\sigma_1\leq\ \mu\leq\sigma_2,\\ \frac{4bp^2B_1}{\tau_\alpha}\left(\frac{bpB_1}{\eta_\alpha^2}\left(\mu p\tau_\alpha+\lambda_\alpha-\frac{B_2}{B_1}\right)\right) & if\ \mu\geq\sigma_2\end{cases} \quad (2.1)$$

$\sigma_1=\frac{\eta_\alpha^2}{bp^2B_1\tau_\alpha}\left(\frac{B_2}{B_1}-1-\frac{bpB_1\lambda_\alpha}{\eta_\alpha^2}\right),\quad \sigma_2=\frac{\eta_\alpha^2}{bp^2B_1\tau_\alpha}\left(\frac{B_2}{B_1}+1-\frac{bpB_1\lambda_\alpha}{\eta_\alpha^2}\right),$

and

$$\sigma_3=\frac{\eta_\alpha^2}{bp^2B_1\tau_\alpha}\left(\frac{B_2}{B_1}-\frac{bp^2B_1\lambda_\alpha}{\eta_\alpha^2}\right), \quad (2.2)$$

The inequality (2.1) is sharp. Further the result is improved as follows if $\sigma_1 < \mu \leq \sigma_3$, then

$$\left|b_{p+2}-\mu b_{p+1}^2\right|+\frac{\eta_\alpha^2}{bp^2B_1\tau_\alpha}\left(1-\frac{B_2}{B_1}+\frac{bpB_1}{\eta_\alpha^2}(\mu p\tau_\alpha+\lambda_\alpha)\right)|b_{p+1}|^2\leq\frac{4bp^2B_1}{\tau_\alpha} \quad (2.3)$$

and also if $\sigma_3 \leq \mu < \sigma_2$, then

$$\left|b_{p+2}-\mu b_{p+1}^2\right|+\frac{\eta_\alpha^2}{bp^2B_1\tau_\alpha}\left(1+\frac{B_2}{B_1}+\frac{bpB_1}{\eta_\alpha^2}(\mu p\tau_\alpha+\lambda_\alpha)\right)|b_{p+1}|^2\leq\frac{4bp^2B_1}{\tau_\alpha} \quad (2.4)$$

Also by Lemma 2, we can write

$$\left|b_{p+2}-\mu b_{p+1}^2\right|\leq\frac{4bp^2B_1}{\tau_\alpha}\max\left\{1,\left|\frac{bpB_1}{\eta_\alpha^2}(\mu p\tau_\alpha+\lambda_\alpha)-\frac{B_2}{B_1}\right|\right\}, \quad (2.5)$$

for any complex number μ.

Furthermore, by Lemma 3 we have

$$b_{p+3}=\frac{8bB_1p^2}{[(1-\alpha)Y+3\alpha p]}\{w_3+q_1w_1w_2+q_2w_1^3\}$$

Which becomes

$$|b_{p+3}|\leq\frac{8bB_1p^2}{[(1-\alpha)Y+3\alpha p]}H(q_1q_2), \quad (2.6)$$

such that $H(q_1,q_2)$ is as defined by Lemma 3,

$$q_1=\frac{2B_2\tau_\alpha\eta_\alpha-bpB_1^2\gamma_k^\alpha)\eta_\alpha^2}{\tau\alpha\eta_\alpha^3B_1}$$

And

$$q_2=\frac{B_3\tau\alpha\eta_\alpha^3+[\lambda_\alpha\gamma_k^\alpha-([1-\alpha]k+\alpha p^3)\tau_\alpha]b^2p^2B_1^3-bpB_1B_2\,\eta_\alpha^2\gamma_k^\alpha}{\tau_\alpha\eta_\alpha^3B_1}$$

Where

$\lambda_\alpha=[p^3(1-\alpha)+4p(1-\alpha)+3(\alpha-1)-\alpha p^2]$, $\eta_\alpha=[p[1-p(1-\alpha)]+3(1-\alpha)]$,
$\tau_\alpha=2p+4-4\alpha,\qquad \gamma_k^\alpha=[(1-\alpha)k-3\alpha p^2]$
$k=[(p+1)^2(p^2+p+1)],\quad k=p^3-8p-8,\qquad Y=2p^2+7p+3$

Proof:

If $g(z)\ \epsilon\ M_{b,p,\alpha}(\phi)$, then there exist an analytic function $w(z)=w_1z+w_2z^2+w_3z^3+\cdots\in\Omega$ such that

$$\left[1+\frac{1}{b}\left(\frac{1}{p}\frac{zg'(z)}{g(z)}-1\right)\right]^\alpha\left[1-\frac{1}{b}+\frac{1}{bp}\left(1+\frac{zg''(z)}{g'(z)}-1\right)\right]^{1-\alpha}=\varphi(w(z)). \quad (2.7)$$

Expanding the left hand side of (2.7) binomially we have

$$1+(1-\alpha)\left[\left(\frac{b_{p+1}[(p+2)-(p+1)(p-1)]}{2bp^2}\right)z+\left(\frac{b_{p+1}^3[(p+1)(P-1)-(p+2)(P+1)}{4bp^2}+\frac{b_{p+2}[(p+2(p+1)-(p+2)(p-1)}{4bp^2}\right)z^2+\right]$$

$$\left(\frac{b_{p+1}^3[(p+1)^3(P-1)+(p+1)^2(P+2)}{8bp^4}+\frac{b_{p+1}b_{p+2}[2(p+1)(p-1)-(p+1)^2(p+2)]}{8bp^3}-\frac{b_{p+1}b_{p+2}[(p+2)^2]}{8bp^2}\right.$$
$$+\frac{b_{p+3}[(p+3)(p-1)+(p+3)(p+2)]}{8bp^2}z^3+\cdots$$
$$+\frac{(1-\alpha)(-\alpha)}{2!}\left[+\alpha\left[\left(\frac{b_{p+1}(p+1)}{2bp}-\frac{b_{p+1}}{2b}\right)z+\left(\frac{b_{p+1}^2-b_{p+2}}{4b}-\frac{b_{p+1}^2(p+1)}{4bp}+\frac{b_{p+2}(p+2)}{4bp}\right)z^2\right]\right]$$
$$\left.+\left(\frac{b_{p+1}b_{p+2}}{4b}-\frac{b_{p+3}+b_{p+1}^3}{8b}+\frac{b_{p+1}^2-b_{p+1}b_{p+2}(p+1)}{8bp}\frac{b_{p+1}b_{p+2}}{8bp}+\frac{b_{p+3}(p+3)}{8bp}\right)z^3+\cdots\right)$$

and on the right hand side

$$\varphi(w(z)=1+B_1(w_1z+w_2z^2+w_3z^3+\cdots)+B_2(w_1z+w_2z^2+w_3z^3+\cdots)^2+B_3(w_1z+w_2z^2+w_3z^3+\cdots)^3$$
$$=1+B_1w_1z+(B_1w_2+B_1w_1^2)z^2+(B_1w_3+2B_2w_1w_{2+}B_1w_1^3)z^3+\cdots, \quad (2.9)$$

From (2.9) and (2.8), we have

$$\frac{b_{p+1}}{2bp^2}(1-\alpha)[(p+2)-(p+1)(p-1)]+\alpha[p(p+1)-p^2])=B_1w_1$$

Which implies

$$b_{p+1}=\frac{2bp^2B_1w_1}{[p[1-p(1-\alpha)]+3(1-\alpha)]} \quad (2.10)$$

From (2.9) and (2.8) we also have

$$b_{p+2}=\frac{2p+4-4\alpha}{4bp^2}\left[\frac{B_1w_2R_\alpha^2+B_1w_1^2\eta_\alpha^2-bpB_1^2w_1^2[3p(\alpha-1)\lambda_\alpha+3(\alpha-1)-\alpha p^2}{\eta_\alpha^2}\right] \quad (2.11)$$

And for the third coefficient we have

$$b_{p+3}=\frac{[(1-\alpha)Y+3\alpha p]}{8bp^2}+\frac{b_{p+1}^2[(1-\alpha)k+\alpha p^3}{8bp^4}+\frac{b_{p+1}b_{p+2}[(1-\alpha)k-3\alpha p^2}{8bp^3}=B_1w_3+2B_2w_1w_2+B_3$$

Further simplification gives

$$b_{p+3}=\frac{8bp^2}{[(1-\alpha)Y+3\alpha p]}\left[B_1w_3+\frac{2B_2\tau_\alpha\eta_\alpha-bpB_1^2\gamma_k^\alpha)\eta_\alpha^2w_1w_2}{\tau_\alpha\eta_\alpha^3}\right.$$
$$\left.+\frac{B_3\tau_\alpha\eta_\alpha^3+[\lambda_\alpha\gamma_k^\alpha-[(1-\alpha)k+\alpha p^3]\tau_\alpha]b^2p^2B_1^3-bpB_1B_2\eta_\alpha^2\gamma_k^\alpha}{\tau_\alpha\eta_\alpha^3}\right] \quad (2.12)$$

From (2.10) and (2.11) we have that

$$b_{p+2}-\mu b_{p+1}^2=\frac{4bp^2}{\tau_\alpha}\left[\frac{B_1w_2\eta_\alpha^2+B_2w_1^2\eta_\alpha^2-bpB_1^2w_1^2\lambda_\alpha}{\eta_\alpha^2}\right]-\frac{\mu 4b^2p^4B_1^2w_1^2}{\eta_\alpha^2}-$$
$$=\frac{4bp^2B_1}{\tau_\alpha}\left[w_2+\left(\frac{B_2}{B_1}-\frac{bpB_1\lambda_\alpha}{\eta_\alpha^2}-\frac{\mu bp^2B_1\tau_\alpha}{\eta_\alpha^2}\right)w_1^2\right],$$

Thus

$$b_{p+2}-\mu b_{p+1}^2=\frac{4bp^2B_1}{\tau_\alpha}\{w_2-vw_1^2\}, \quad (2.13)$$

Where

$$v=\frac{bpB_1}{\eta_\alpha^2}(\mu p\tau_\alpha+\lambda_\alpha)-\frac{B_2}{B_1} \quad (2.14)$$

If $v\le -1, then$

$$\frac{bpB_1}{\eta_\alpha^2}(\mu p\tau_\alpha+\lambda_\alpha)-\frac{B_2}{B_1}\le -1,$$

Which implies

$$\mu\le\frac{\eta_\alpha^2}{bp^2B_1\tau_\alpha}\left(\frac{B_2}{B_1}-1-\frac{bpB_1\lambda_\alpha}{\eta_\alpha^2}\right)=\sigma_1$$

By application of Lemma 1, we have

$$|b_{p+2}-\mu b_{p+1}^2|\le\frac{4bp^2B_1}{\tau_\alpha}\left(\frac{B_2}{B_1}-\frac{bpB_1\lambda_\alpha}{\eta_\alpha^2}(\mu p\tau_\alpha+\lambda_\alpha)\right)$$

Next if $v\ge 1$ then we have,

$$\frac{bpB_1}{\eta_\alpha^2}(\mu p\tau_\alpha+\lambda_\alpha)-\frac{B_2}{B_1}\ge 1$$

Which implies
$$\mu \geq \frac{\eta_\alpha^2}{bp^2B_1\tau_\alpha}\left(\frac{B_2}{B_1}+1-\frac{bpB_1\lambda_\alpha}{\eta_\alpha^2}\right)=\sigma_2$$
By lemma 1 we have
$$|b_{p+2}-\ \mu b_{p+1}^2| \leq \frac{4bp^2B_1}{\tau_\alpha}\left(\frac{bpB_1}{\eta_\alpha^2}(\mu p\tau_\alpha+\lambda_\alpha)-\frac{B_2}{B_1}\right)$$
If $-1 \leq v \leq 1$ then
$$-1 < \frac{bpB_1}{\eta_\alpha^2}(\mu p\tau_\alpha+\lambda_\alpha)-\frac{B_2}{B_1} \leq 1,$$
Which show by the application of Lemma 1
$$|b_{p+2}-\ \mu b_{p+1}^2| \leq \frac{4bp^2B_1}{\tau_\alpha}$$
which is the second part of (2.1).

The sharpness of the results is a direct consequence of Lemma 1. Furthermore when $\sigma_1 < \mu < \sigma_2$ the result can be improved as follows:

if $-1 < v \leq 0$, then
$$-1 < \frac{bpB_1}{\eta_\alpha^2}(\mu p\tau_\alpha+\lambda_\alpha)-\frac{B_2}{B_1} \leq 0$$
Which implies that $\sigma_1 < \mu \leq \sigma_3$ where
$$\sigma_3 = \frac{\eta_\alpha^2}{bp^2B_1\tau_\alpha}\left(\frac{B_2}{B_1}-\frac{bpB_1\lambda_\alpha}{\eta_\alpha^2}\right)$$
By lemma 1, (2.13) and (2.14), we have
$$\frac{\tau_\alpha}{4bp^2B_1}\left|b_{p+2}-\ \mu b_{p+1}^2\right| + \left(1-\frac{B_2}{B_1}+\frac{bpB_1}{\eta_\alpha^2}(\mu p\tau_\alpha+\lambda_\alpha)\right)|w_1^2| \leq 1, (2.15)$$
From (2.10) and (2.15), we have
$$\frac{\tau_\alpha}{4bp^2B_1}\left|b_{p+2}-\ \mu b_{p+1}^2\right| + \frac{\eta^2\left(\left(1-\frac{B_2}{B_1}+\frac{bpB_1}{\eta_\alpha^2}(\mu p\tau_\alpha+\lambda_\alpha)\right)\right)}{4b^2p^4B_1^2}|b_{p+1}|^2 \leq 1,$$
$$\left|b_{p+2}-\ \mu b_{p+1}^2\right| + \frac{4bp^2B_1}{\tau_\alpha}\left(1-\frac{B_2}{B_1}+\frac{bpB_1}{\eta_\alpha^2}(\mu p\tau_\alpha+\lambda_\alpha)\right)|b_{p+1}|^2 \leq \frac{4bp^2B_1}{\tau_\alpha}$$
Further, if $0 \leq v < 1$, then $\sigma_3 \leq \mu \leq \sigma_2$ by Lemma 1
$$\frac{\tau_\alpha}{4bp^2B_1}\left|b_{p+2}-\ \mu b_{p+1}^2\right| + \left(1+\frac{B_2}{B_1}+\frac{bpB_1}{\eta_\alpha^2}(\mu p\tau_\alpha+\lambda_\alpha)\right)|w_1^2| \leq 1$$
Which becomes
$$\left|b_{p+2}-\ \mu b_{p+1}^2\right| + \frac{4bp^2B_1}{\tau_\alpha}\left(1+\frac{B_2}{B_1}+\frac{bpB_1}{\eta_\alpha^2}(\mu p\tau_\alpha+\lambda_\alpha)\right)|b_{p+1}|^2 \leq \frac{4bp^2B_1}{\tau_\alpha}$$
By lemma 2, we can write
$$\left|b_{p+2}-\ \mu b_{p+1}^2\right| \leq \frac{4bp^2B_1}{\tau_\alpha}Max\left\{1, \left|\frac{bpB_1}{\eta_\alpha^2}(\mu p\tau_\alpha+\lambda_\alpha)-\frac{B_2}{B_1}\right|\right\}$$
For any complex number μ

By Lemma 3, (2.12) becomes
$$b_{p+3} = \frac{8bB_1p^2}{[(1-\alpha)Y+3\alpha p]}\{w_3+q_{\cdot 1}w_1w_2+q_2w_1^3\}$$
And further written as
$$|b_{p+3}| \leq \frac{8bB_1p^2}{[(1-\alpha)Y+3\alpha p]}H(q_1q_2),$$
Where
$$q_1 = \frac{(2B_2\tau_\alpha\eta_\alpha - bpB_1^2\gamma_k^\alpha)\eta_\alpha^2}{\tau_\alpha\eta_\alpha^3B_1}$$

And

$$q_2 = \frac{B_3\tau_\alpha\eta_\alpha^3 + [\lambda_\alpha\gamma_k^\alpha - (1-\alpha]k + \alpha p^3)\tau_\alpha]b^2p^2B_1^3 - bpB_1B_2\eta_\alpha^2\gamma_k^\alpha}{\tau_\alpha\eta_\alpha^3B_1}$$

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