Transactions of the Nigerian Association of Mathematical Physics
Volume 4, (July, 2017), **pp1 – 6**
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Existence of weak and classical solutions for linear elliptic partial differential equations with Dirichlet boundary condition in sobolev space

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Abstract

This work considers an Elliptic Partial Differential Equation (PDE) with Dirichlet boundary condition, with the aim of establishing a sufficient condition for the existence of weak and classical solution. Applying the Lax-Milgram theorem on linear elliptic PDE with Dirichlet Boundary Conditions, the results obtained indicate that the equation is continuous and coercive. This establishes the existence of a unique weak solution for the Dirichlet boundary conditions of the equation. The establishment of existence of classical solution shows that there exist a C^2 function u on Ω which satisfy the linear elliptic PDE.

Keyword*:* Existence; weak solutions; classical solutions; linear elliptic partial differential equations; Sobolev space

1.0 Introduction

Non linear elliptic system on a bounded and unbounded domains of R^N was investigated [1] and they were able to ascertain that the generalized formation of many stationary boundary value problem for partial differential equations lead to operator equation on a Banach space. The weak formulation consists in looking for an unknown function u from a Banach space V such that an integral identity containing u holds for each test function v from the space V. Denoting the terms containing the unknown u as the value of an operator A, they obtain an equation which is equivalent to a functional on V. Using the theory of monotone operator the existence of weak solution for these systems was established. The existence of weak solution of the non linear elliptic system by using the method of sub and super solutions was also established [2,3]. The variational method in a weighted Sobolev space was used to prove the existence of solution for certain class of singular nonlinear ordinary differential equations [4]. This type of equation arises in the context of standing wave solutions of nonlinear Klein-Gordon and Schrodinger equations as well as in self-focusing problem for intense optical beams. Solution was sought for singular nonlinear boundary value problem in a weighted Sobolev space designed to include the boundary condition at infinity as well as handle the singularity at $t = 0$. Some of the results was extended by allowing a more general weight function $p(t)$. The existence of the solution of the second-order impulsive differential equations with non constant coefficients was considered and the second-order impulsive partial differential equation was changed into the equivalent equation by transformation. Using the critical point theory of variational method and Lax-Milgram theorem, new results for the existence of the solution for the impulsive partial differential equations was obtained. Existence of global solutions for a class of second order impulsive abstract functional differential equations was also studied and result obtained using Leray-Schauder's Alternative fixed point theorem [5].

Some over determined elliptic system in a domain of R^3 which contains an axis were examined, assuming that the functions belonged to Sobolev spaces with weights proportional to a power of the distance from the axis, existence of solutions in the corresponding weighted Sobolev spaces was established [6]. The resolution of some elliptical problems in the half-space R_+^N, with $N \geq 2$ was investigated using the Dirichlet and Neumann conditions for the Laplace operator and existence and uniqueness of solution was established in L^P Spaces[7]. Existence of weak solutions in the weighted Sobolev spaces $W_0^{1,p}$ (Ω,w) for the Dirichlet problem to some degenerate qausilinear elliptic equations in a bounded open set $\Omega \subset R^n$ and $\partial\Omega$ is the boundary of Ω in R^n was established [8].

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Weight here means a locally integral function w on R^n such that $0 < w(x) < \infty$ for $x \in R^n$. They made use of two major theorems, the weighed Sobolev inequality and the Lax-Milgram theorem from [9] to establish the existence of weak solution in the weighted Sobolev spaces $W_0^{1,p}$ (Ω,w) for the Dirichlet problem above. Two elliptic partial differential equations: the homogeneous Dirichlet problem and the homogeneous Neumann problem using the Lax-Milgram representation theorem and some of properties of Sobolev spaces was investigated and it was established that there exist a weak solution for both problems [10]. The approach used consisted of weak formulation, existence and of weak solution and recovery of classical solution.

Most of the work reviewed concentrated on existence of weak solution in one form or the other but failed to talk about the necessary condition that made weak solution in a Sobolev space a classical solution. That is what motivated us to try to establish the existences of weak solution using a different approach and establish sufficient conditions for a weak solution of linear elliptic partial differential equation with Dirichlet boundary condition to be a classical solution.

2.0 Methodology

2.1 Weak Solutions

The existence and uniqueness of weak solution of elliptic partial differential equations shall be established using the Lax-Milgram theorem in bilinear form.

Lemma 2.1 **(**Lax-Milgram):

Let H be a Hilbert space and let $a(u,v)$ be a continuous and coercive bilinear form on a Hilbert space. Given any $L \in H^*$, there exists a unique element $u \in H$ such that

$$a(u,v) = L(v) \quad \forall\, v \in H \tag{2.1}$$

Proof:

Case I: Existence: Let $u \in H$ fixed. Define on H the linear form l as follows:

$$l(v) = a(u,v) \quad \forall\, v \in H \tag{2.2}$$

From the assumptions above, it follows that

$$|l(v)| = |a(u,v)| \le M\|u\|_H\|v\|_H, \qquad \forall\, v \in H.$$

So L is continuous and moreover we have

$$\|l\|_{H'} \le M\|u\|_H \tag{2.3}$$

By the Riesz-Frechet representation theorem, there exists a unique vector $Au \in H$ such that

$$a(u,v) = l(v) = (Au,v)_H, \quad \forall\, v \in H,$$

From equation (2.3), we have

$$\|Au\|_H \le M\|u\|_H \tag{2.4}$$

Now let $A: H \to H$ defined as follows

$$(Au,v) = a(u.v)_H \quad \forall \quad u,v \quad \in H.$$

A is linear, continuous (because of inequality (2.4)) and from the H-Ellipticity of the bilinear form we have

$$\delta\|u\|_H \le \|Au\|_H \le M\|u\|_H, \ \forall\, u \in H$$

From the proceeding lemma, it follows that A is injective and it range $R(A)$ is closed in H.

Next is to show that A is onto which can be achieve by showing that $R(A)$ is dense in H. we will show that the orthogonal of $R(A)$ is reduced to zero. Let $Au \in R(A)$, then $(Au,u)_H = 0.$ But

$$0 = (Au,u) = a(u,u) \ge \delta\|u\|_H^2$$

So $u = 0$ and therefore $R(A)^\perp = \{0\}$, thus $R(A)$ is dense in H. since it is close in H, it follows that $R(A) = \overline{R(A)} = H.$

Now since $L \in H'$, by Riesz-Frechet representation theorem there is a unique $w \in H$ such that

$$L(v) = (w,v), \quad \forall\, v \in H.$$

A is bijective, then there exist a unique $u \in H$ such that $Au = w$ and so for all $v \in H$ we have

$$a(u,v) = (Au,v)_H = (w,v)_H = L(v)$$

Case II: Uniqueness. Let u_1, u_2 such that

$$a(u_1,v) = L(v), \ \forall\, v \in H.$$

$$a(u_2,v) = L(v), \ \forall\, v \in H.$$

It follows that

$$a(u_1 - u_2, v) = 0 \ \forall\, v \in H.$$

If we take $v = u_1 - u_2$ and we use the H-ellipticity of a then we get $u_1 = u_2$.

Lemma 2.2 (Poincare Inequality):

Suppose that $1 \le p \le \infty$ and Ω is a bounded open set. Then there exist a constant C (depending on Ω and p) such that

$$\|u\|_{L^p(\Omega)} \le C\|\nabla u\|_{L^p(\Omega)} \quad \forall\, u \in W_0^{1,p}(\Omega).$$

In particular the expression $\|u\|_{L^p(\Omega)}$ is a norm on $W_0^{1,p}(\Omega)$, and it is equivalent to the norm $\|u\|_{W^{1,p}}$; on $H_0^1(\Omega)$ the expression $\sum_{i=1}^N \int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}$ is a scalar product that induces the norm $\|\nabla u\|_{L^2}$ and is equivalent to the norm $\|u\|_{H^1}$.

Remark: the Poincare's inequality remains true if Ω has finite measure and also if Ω has a bounded projection on some axis.
Lemma 2.3 [12]:
Let $u \in L^p(\Omega)$ $with$ $1 \le p \le \infty. p'$ is the derivative of p.
The following properties are equivalent:

i. $u \in W^{1,p}(\Omega)$
ii. $there\ exist\ a\ consant\ C\ such\ that$

$$\left|\int_\Omega u\frac{\partial \varphi}{\partial x_i}\right| \le C\|\varphi\|_{Lp'(\Omega)} \ \forall \varphi \in C_c^\infty(\Omega). \forall i = 1,2, \ldots N$$

iii. $there\ exist\ a\ constant\ C\ ssuch\ that\ for\ all\ w \subset\subset \Omega, and\ all\ h \in R^N\ with\ |h| < dist\ (\omega, \partial\Omega)\ we\ have$
$\|\tau_h - u\|_{L^p(\omega)} \le C|h|$.

(Note that $\tau_h u(x) = u(x+h)$ make sense for$for\ x \in \omega\ and\ |h|\ < dist\ (\omega, \partial\Omega)$) furthermore, we can take $C = |\nabla u|_{L^p(\Omega)}\ in\ (ii)$ and $and\ (iii)$.
$If\ \Omega = R^N$ we have $\|\tau_h u - u\|_{L^p(R^N)} \le |h||\nabla u|_{L^p(R^N)}$.

2.2 Regularity of Weak Solutions

The theory for deriving the smoothness of the weak solution is called regularity condition. We shall formulate and establish the regularity condition for the Dirichlet problem.

3.1 Main Results

Let $\Omega \subset R^N$ be an open bounded set. We are looking for a function $u: \overline{\Omega} \to \mathbb{R}\ satisfing$

$$\begin{cases} -\nabla u + q(x)u = f\ in\ \Omega \\ \qquad u = 0\ on\ \partial\Omega \end{cases} \tag{3.1}$$

where $\nabla u = \sum_{i=1}^N \frac{\partial^2 u}{\partial x_i^2} =$ Laplacian of u and f is a given function on Ω. The boundary condition $u = 0$ on Γ is called the (homogeneous) Dirichlet condition.

Definition 3.1:
A classical or strong solution of equation (3.1) is a C^2on Ω satisfying equation (3.1) in a usual sense. Multiplying equation (3.1) by a test function $v \in C^1(\Omega)$ and integrate by part we obtain

$$\int_\Omega \nabla u. \nabla v + \int_\Omega uv = \int_\Omega fv, \ \forall\ v \in C^1(\Omega) \tag{3.2}$$

where

$$\nabla u. \nabla v = \sum_{i=1}^N \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}$$

A C^1 function u that satisfy equation (3.2) is called a weak solution of equation (3.1).

3.1.1 Existence and Uniqueness of Weak Solutions of the Homogeneous Dirichlet Problem

In this subsection we are going to establish the existence of a unique weak solution of equation (3.1) where $u \in H^1$ and as a mollifiers $v \in (H^1)^* = H^1$ consequently equation (3.1) mollified by $v \in H^1(\Omega)$ where H^1 is the Hilbert space $W^{1,2}$ becomes
$-\nabla u. v + q(x)uv = fv$
and integrating over **Ω**,

$$\int_\Omega -\nabla u. vdx + \int_\Omega q(x)uvdx = \int_\Omega fvdx.$$

Recall Green formula

$$\int_\Omega (\nabla u)v = \int_\Gamma \frac{\partial u}{\partial n} vd\sigma - \int_\Omega \nabla u. \nabla v \quad \forall\ u \in C^2(\overline{\Omega}),\ \forall v \in C^1(\overline{\Omega}),$$

Then using Green formula and the fact that $\frac{\partial u}{\partial n} = 0\ (since\ u = 0\ on\ \partial\Omega)$ we have

$$\int_\Omega \nabla u. \nabla vdx + \int_\Omega q(x)uvdx = \int_\Omega fvdx.$$

Next we use the Lax-Milgram lemma to formulate a theorem to establish the existence of a unique weak solution of equation (3.1)
Theorem 3.1:
Assume that $B: H \times H \to \mathbb{R}$ is a bilinear form on a Hilbert space such that for some constant $\alpha, \beta > 0$ we have

$$|B(u,v)| \le \alpha\|u\|_2\|v\|_2 \quad \forall\ u, v \in H \tag{3.3}$$

$$and\ B(u,u) \ge \beta\|u\|_2^2 \ \ \forall\ u \in\ H. \tag{3.4}$$

Then for any bounded linear functional $f \in H^*$ there exist a unique element $u \in H$ in equation (3.1) such that
$B(u,v) = \langle f, v\rangle\ \forall v \in H$.

Applying Theorem 3.1 on equation (3.1)
Let $H = H^1(\Omega)$. H is a Hilbert space as a close sub space of a Hilberrt space $H^1(\Omega)$.
Defining

$$B(u,v) = \int_\Omega \nabla u.\nabla v dx + \int_\Omega q(x)uvdx \; and \; L(v) = \int_\Omega fvdx \quad (3.5)$$

Next is to show that equation (3.5) satisfies inequality (3.3) and (3.4) above.
First is to show that is satisfy inequality (3.3)

$$B(u,v) = \int_\Omega \nabla u.\nabla v dx + \int_\Omega q(x)uvdx$$

Applying the Cauchy-Schwarz inequality

$$|B(u,v)| \leq \int_\Omega |\nabla u||\nabla v|dx + \int_\Omega |q(x)uv|dx$$
$$\leq ||\nabla u||_{L^2(\Omega)}||\nabla v||_{L^2(\Omega)} + ||q||_\infty ||u||_{L^2(\Omega)}||v||_{L^2(\Omega)}$$
$$\leq (1 + \|q\|_\infty)(||\nabla u||_{sL^2(\Omega)}||\nabla v||_{L^2(\Omega)} + ||u||_{L^2(\Omega)}||v||_{L^2(\Omega)})$$
$$\leq (1 + \|q\|_\infty)(||u||_{L^2}||v||_{L^2} + ||u||_{L^2}||v||_{L^2})$$
$$\leq \alpha||u||_{L^2}||v||_{L^2}$$

which shows that the equation (3.5) satisfies inequality (3.3) hence equation (3.5) can be said to be continuous.
Next is to show that equation (3.5) satisfies inequality (3.4)

$$B(u,u) = \int_\Omega |\nabla u|^2 dx + \int_\Omega q(x)|u|^2 \, dx$$
$$= \|\nabla u\|_{L^2}^2 + q\|\nabla u\|_{L^2}^2$$
$$\geq \beta \||\nabla u|\|_{L^2(\Omega)}^2$$

$\geq \beta||u||_{L^2}^2$ (by Poincare inequality)
which also show clearly that equation (3.5) satisfy inequality (3.4) hence its coercive.
Corollary 3.1:
From lemma (3.1) Since H is a Hilbert space and $B(u,v)$ in equation (3.5) is a continuous and coercive bilinear form on a Hilbert space then given any linear functional $f \in H^*$, there exists a unique element $u \in H$ such that $B(u,v) = \langle f, v\rangle \; \forall \; v \in H$. Which concludes the prove for the existence and uniqueness of solution of equation (3.1).

3.1.2 Existence of Classical Solutions of the Homogeneous Dirichlet Problem

In this subsection we considered the regularity condition that is, the conditions that will make a weak solution to be a classical solution for the homogeneous Dirichlet problem. We now proceed to formulate and prove a regularity theorem to establish it.
Theorem 3.2:
Let Ω be an open set of class C^2 with Γ bounded. Let $f \in L^2(\Omega)$ and $u \in H_0^1(\Omega)$ satisfy

$$\int_\Omega \nabla u.\nabla\varphi + \int_\Omega u\varphi = \int_\Omega f\varphi, \qquad \forall \; \varphi \in H_0^1(\Omega).$$

Then $u \in H^2(\Omega) \; and \; \left||u|\right|_{H^2} \leq C\left||f|\right|_{L^2}$ where C is a constant depending only on Ω.
Proof**:**
For the proof we shall consider two cases the case where $\Omega = \mathbb{R}^N$, then the case where $\Omega = \mathbb{R}_+^N$.
Case I: $\Omega = \mathbb{R}^{\boldsymbol{N}}$**.**
Given $h \in R^N, h \neq 0, set$

$$D_h u = \frac{1}{|h|}(\tau_h u - u), \quad i.e., \quad D_h u(x) = \frac{u(x+h) - u(x)}{|h|}$$

In (3.5) take $\varphi = D_{-h}(D_h u)$. this is possible, since $\varphi \in H^1(\mathbb{R}^N)$ and $\left(u \in H^1(\mathbb{R}^N)\right)$; we obtain

$$\int |\nabla \mathrm{D_h u}|^2 + \int |\mathrm{D_h u}|^2 = \int fD_{-h}(D_h u).$$

and thus

$$||D_h u||_{H^1}^{\mathbf{2}} \leq ||f||_{\mathbf{2}}||D_{-h}(D_h u)||_{\mathbf{2}} \quad (3.6)$$

On the other hand, recall from lemma (3.5) that

$$||D_{-h}v||_2 \leq ||\nabla v||_2 \; \forall v \in H^1. \quad (3.7)$$

Using inequality (3.6), (3.7) with $v = D_h u$, we obtain

$$||D_h u||_{H^1}^{\mathbf{2}} \leq ||f||_{\mathbf{2}}||\nabla(D_h u)||_{\mathbf{2}}.$$

and consequently

$$\left\|D_h \frac{\partial u}{\partial x_i}\right\|_2 \leq ||f||_2 \ \forall i = 1,2, \dots, N. \tag{3.8}$$

Then we see that $\frac{\partial u}{\partial x_i} \in H^1$ and thus $u \in H^2$

Case II: $\Omega = \mathbb{R}_+^N$.

We use again translations, but only in the tangential directions, i.e., in a direction $h \in \mathbb{R}^{N-1} \times \{0\}$: we say that h is parallel to the boundary, and denote this by $h||\Gamma$.

It is essential to note that

$u \in H_0^1(\Omega) \Longrightarrow \tau_h u \in H_0^1(\Omega)$ $if\ h||\Gamma$

Choosing $h||\Gamma$ and inserting $\varphi = D_{-h}(D_h u)$ in equation (3.5); we obtain

$$\int |\nabla \mathrm{D_h u}|^2 + \int |\mathrm{D_h u}|^2 = \int f D_{-h}(D_h u).$$

i.e.,

$$||D_h u||_{H^1}^2 \leq ||f||_2 ||D_{-h}(D_h u)||_2 \tag{3.9}$$

From equation (3.6) we have

$$\|D_h v\|_{L^2(\Omega)} \leq ||f||_2 \quad \forall \boldsymbol{v} \in H^1(\Omega), \quad \forall h||\Gamma. \tag{3.10}$$

Combining equation (3.8) and equation (3.9), we obtain

$$\|D_h u\|_{H^1} \leq ||f||_2 \qquad \forall h||\Gamma. \tag{3.11}$$

Let $1 \leq j \leq N, 1 \leq k \leq N-1, h = |h| e_k, \varphi \in C_c^\infty(\Omega)$. We have

$$\int D_h \left(\frac{\partial u}{\partial x_j}\right) \varphi = -\int u D_{-h} \left(\frac{\partial \varphi}{\partial x_j}\right)$$

From equation (3.10),

$$\left|\int u D_{-h} \left(\frac{\partial \varphi}{\partial x_j}\right)\right| \leq ||f||_2 ||\varphi||_2$$

Passing to the limit as $h \to 0$, this becomes

$$\left|\int u \frac{\partial^2 \varphi}{\partial x_j \partial x_k}\right| \leq ||f||_2 ||\varphi||_2 \quad \forall 1 \leq j \leq N \ \ \forall 1 \leq k \leq N-1 \tag{3.12}$$

Finally, we claim that

$$\left|\int u \frac{\partial^2 \varphi}{\partial x_N^2}\right| \leq ||f||_2 ||\varphi||_2 \ \forall \varphi \in C_c^\infty(\Omega) \tag{3.13}$$

From equation 3.5 we deduce that

$$\left|\int u \frac{\partial^2 \varphi}{\partial x_N^2}\right| \leq \sum_{i=1}^{N-1} \left|\int u \frac{\partial^2 \varphi}{\partial x_i^2}\right| + \left|\int (f-u)\varphi\right| \leq C\|f\|_2 \|\varphi\|_2$$

from equation (3.12). And combining equation (3.12) and equation (3.13), we end up with

$$\left|\int u \frac{\partial^2 \varphi}{\partial x_j \partial x_k}\right| \leq C\left||f|\right|_2 \left||\varphi|\right|_2 \ \forall \varphi \in C_c^\infty(\Omega), \ \ \forall\ 1 \leq j,k \leq N.$$

As a consequence, $u \in H^2$, since there exist functions $f_{jk} \in L^2(\Omega) such\ that$

$$\left|\int u \frac{\partial^2 \varphi}{\partial x_j \partial x_k}\right| = \int f_{jk} \varphi \qquad \forall \varphi \in C_c^\infty(\Omega)$$

Therefore we can have the following corollary as a result of theorem 3.2.

Corollary 3.2:

Given Ω an open set of class C^2 with Γ bounded (or else $\Omega = R_+^N$) and we were able to show that $f \in L^2(\Omega) and\ u \in H_0^1(\Omega)$ satisfy $\int_\Omega \nabla u . \nabla \varphi + \int_\Omega u\varphi = \int_\Omega f\varphi, \forall \varphi \in H_0^1(\Omega)$ then $u \in H^2(\Omega)\ and\ \left||u|\right|_{H^2} \leq C\left||f|\right|_{L^2}$ where C is a constant depending only on Ω. Therefore u is a classical solution of equation (3.1) from the definition of classical solutions.

4. Discussion

Cavalheiro [9] established the existence of weak solutions for the Dirichlet problem in a weighted Sobolev space using the weighted sobolev inequality and the lax-milgram theorem. In this research we apply the Lax-Milgram theorem of the bilinear form in section to establish the existence of a unique weak solution for linear elliptic partial differential equation with the Dirichlet boundary condition and We were able to obtain that there exists a weak solution to equation our equation and also the solution is unique in the sense that it cannot be different for the same equation no matter the method adopted to solve the equation. Therefore, we obtain almost a similar result apart from the fact that we also establish that the weak solution obtain is also unique.

Iyiola [11] and Cavalheiro [9] establish that they exist a weak solution for the Dirichlet problem but we improve on their result by formulating a regularity theorem to prove whether the unique weak solution obtained is a classical solution. We were able to prove that the weak solution meets the necessary condition to be a classical solution which is an improvement of their results.

5. Conclusion

From the research we were able to use the Lax-Milgram theorem to show that our linear elliptic PDE with Direchlet Boundary condition is continuous and coercive which lead us to conclude that they exist a unique weak solution to our equation. We were also able establish that there exist a C^2 function u on Ω which satisfy equation hence the conclusion that there exist a classical solution to equation.

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