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Formulation of the Klein-Gordon Equation from the Oyibo Grand Unification Theorem

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Abstract

Recently we formulated a generic second order partial differential equation (PDE) from the Oyibo grand unification theorem (GUT). The generic nature of our PDE means its solution continuously depend on the data of the problem hence for every application of our PDE, we are dealing with a well-posed problem which is key to all PDEs. We were able to recover the second order PDEs important to mathematical physics from our generic PDE. One of these PDEs is the wave equation. Since in its basic form, the Klein-Gordon Hamiltonian is the wave equation with a mass term, it can in principle be directly recovered from our generic PDF. This is the study undertaken here. The successful formulation of the Klein-Gordon equation from our PDE means it is has been obtained from the Oyibo GUT and therefore all the applications of the Klein-Gordon equation can also be obtained as the extension of the Oyibo God Almighty Grand Unification Theory (GAGUT).

Keywords: Grand unification theorem, Wave component, Partial differential Equation, Generic, Klein Gordon Equation

1.0 Introduction

The Oyibo grand unified theorem (GUT) is a theorem comprising of a set of equations from which one can possibly construct the mathematical formulations of all known and unknown forces in the universe. Therefore it is proposed to be a potential mathematical candidate for the formation of the theory of everything. Thus its application to a number of aspects of both classical and quantum mechanicsas well as proposed applications beyond physics was designated by Oyiboas the God Almighty Grand Unification Theory (GAGUT)[1-3]. There has been serious opposition to both the GUT and GAGUT: the former for its esoteric approach and the latter for its relatively few demonstrated applications but very many bogus claims by Oyibo. We have repeatedly argued that the GUT should be treated differently from the GAGUT. The reason is that the former is a theorem and the latter is a theory emerging from it. By definition, a theorem is a mathematical statement that is proved using rigorous mathematical reasoning while a theory is a set of ideas explaining physical behaviors of a given category of physical systems and is therefore capable of producing experimental predictions for them. The implication is that a theorem could be sound yet a wrong theory can be developed from it. The Oyibo GUT has been tested even by the American Mathematical Society and they found it very sound even with its esoteric approach [3]. Now the Oyibo GAGUT may have flaws as all new theories do. For example, the Bohr model of the atom which together with the Somerfield elliptical model of the atom formed what is today known as the old quantum theory was developed by Bohr merely by invoking the Max Planck idea of quantization into the classical Rutherford model of the atom. The limited success of the Bohr theory especially in obtaining the Rydberg constant from atomic constants was a major step in developing a quantum model of the atom and the need to beautify it as well as extend it strongly boosted the study of microscopic physics which today has become a very established aspect of physics with all its vast applications, even as the beautification and extension continue. Another example is the Klein-Gordon equation which historically was first formulated by Schrodinger but rejected it and settled for the non-relativistic equation now known as the Schrodinger equation. The basic reason why he rejected the former was because its solution could not account for the electronic energy levels of the hydrogen atom while the latter was successful with that description since the electrons are reasonably non- relativistic and the spin-dependent effects are fairly small [4]. It is a textbook knowledge that aside the reason for its initial rejection, the Klein-Gordon equation as man first step to a relativistically invariant quantum mechanical equation suffer other limitations. However, its adoption in field

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theory for spin zero particlesh as been successful [5,6] and encourages its beautification and extension [7,8]. This is why we have always insisted that we should strive to beautify and extend the GAGUT. Further, we have posited that one way to do so is to first recover already known solution from the GUT [9-12]. In one of our previous studies in line with that thinking, we developed a generic second order partial differential equation (PDE) from the Oyibo GUT from which we were able to recover the second order PDEs that have very important applications in physics [12, 13]. One of these PDE is the wave equation. It follows that if we adopt the approach of Ref. [14], then one can recover a number of solutions of the Klein-Gorden equation from this wave equation. Basically, the Klein-Gordon Hamiltonian is the wave equation with a mass term[15]. Therefore there is strong motivation that one can obtain the Klein-Gordon equation directly from our generic PDE. This is the study undertaken here using the following plan. In Section 2, a brief review of the Klein-Gordon equation is given to help guide the formalism of the generic Klein-Gordon equation in Section 3. The Klein-Gordon equations for photon which is a massless particle and then for a particle having mass are obtained from the generic Klein-Gordon in Sections 4 and 5 respectively. Thisis followed by a brief conclusion in Section 6.

2.0 Brief Review of the Klein Gordon Equation

The relativistic energy equation is given by

$$E^2 = c^2P^2 + m^2c^4 \tag{1}$$

Where E is the total energy, p is the momentum, m is the mass and c is the speed of light. Then it is easy to see that for a mass less particle like photon, Eq.(1) reduces to

$$E^2 = c^2P^2 \tag{2}$$

The quantization of Eq.(1) by introducing the quantum analogues of p and E:

$$p = -i\hbar\nabla \quad \text{and}\; E = i\hbar\frac{d}{dt} \tag{3}$$

Will yield the Klein-Gordon Hamiltonian

$$\nabla^2 - \frac{d^2}{dt^2} - m^2 = 0 \tag{4a}$$

Which can also re-expressed as

$$[]^2 - m^2 = 0 \tag{4b}$$

Where $\hbar$ = c = 1 and the d'Alembertian operator is defined by

$$[]^2 = \nabla^2 - \frac{1}{c^2}\frac{d^2}{dt^2}. \tag{5}$$

It is easy to infer from Eq.(4) that the Klein_Gordon Hamiltonian is the wave equation or d'Alembertian equation with a mass term [15]. This is why it was possible to derive the solutions of the Klein-Gordon equation from the solutions of the wave equation [14]. Thereafter, the choice of the wave function $\phi_{(r,t)}$ will determine the applicability.

$$([]^2 - m^2)\psi_{(r,t)} = 0 \tag{6}$$

The general consensus is that it should be a spinless scalar field. For if we do not restrict it to a scalar field with zero spin, then the Klein-Gordon equation will become impracticable as such field will include particles with non-zero spin.

3.0 Formalism of the Generic Klein-Gordon Equation

The Oyibo generic conservation equation which is an arbitrary function of space and time coordinates (x,y,z,t), velocities $(\dot{x}, \dot{y}, \dot{z})$, density (ρ), fluid or gas viscosity (μ), temperature (T), pressure (P), etc is given by (See pedagogical review in Ref. [9]):

$$G_{mn}(x, y, z, t, \dot{x}, \dot{y}, \dot{z}, \rho, \mu, T, P, ...) = 0. \tag{7}$$

Eq.(7) can be generalized to a system of partial differential equations (PDEs) of order n given by

$$G_j\left[X^1, X^2, ...X^k, Y^1, Y^2, ...Y^q, ..., \frac{\partial^n Y^1}{(\partial X^1)^n} \cdots \frac{\partial^n Y^q}{(\partial X^p)^n}\right] = F\left[X^1, X^2, ...X^k, Y^1, Y^2, ...Y^q, ..., \frac{\partial^n Y^n}{(\partial X^p)^n} .G_j(X^1, X^2 \cdots \frac{\partial^n y^q}{(\partial x^p)^n}\right] \tag{8}$$

Which is conformally invariant under the transformations T_k^n,

$$T_K : \begin{cases} X^1 = g^i(x^i, ..., x^p, k) \\ Y^1 = h^i(y^i, ..., y^q, k) \end{cases}. \quad (9)$$

Where p and q are any integers, X^1 and g^1, Y^1 and h^1 are functions of x^i and y^i respectively and k is a single group parameter.

Theorem

Let us suppose that the form of G_j in Eq.(8) for the system of partial differential equations are conformally invariant under the nth enlargement of the group T_k ,then the invariant solution of G_j may be expressed in terms of a new system of partial differential equations

$$F\left[\eta_1, \eta_2, ..., \eta_{p-1} ... F_1, F_2, ..., F_q, ... \frac{\partial^n F_1}{(\partial \eta_1)^n} ... \frac{\partial^n F_q}{(\partial \eta_{p-1})^n} = 0\right] \quad (10)$$

Where η_1 are the absolute invariant of the subgroups of the transformation for just the independent variables and

$$F_i = F_i(\eta_i). \quad (11)$$

A more detailed explanation and proof of the above theorem can be found in Ref. [1] and the references therein.
For the space-time coordinate, Eq.(7) can be expressed as [1-3, 9-12]:

$$(G_{0n})_t + (G_{1n})_x + (G_{2n})_y + (G_{3n})_z = 0, \quad (12)$$

and its generic solution given by

$$\eta_n = g_{n0}(ct)^{n+1} + g_{n1}x^{n+1} + g_{n2}y^{n+1} + g_{n3}z^{n+1} \quad (13)$$

Where η_n which is now purely a function of space-time coordinates (t, x, y, z) and the metric parameters ($g_{00}, g_{11}, g_{22}, g_{33}$) as well as n = 0, 1, 2, 3, 4 have applications in all aspect of physics. We have focused on the case n = 0 which is the generic wave solution [9,10,12]:

$$\eta_0 = g_{00}ct + g_{10}x + g_{20}y + g_{30}z. \quad (14)$$

It has been shown that the unified field wave component, $F(\eta_0)$ representing wave is

$$F(\eta_0) = F_G(\eta_0) + F_{EM}(\eta_0) + F_{SF}(\eta_0) + F_{WF}(\eta_0) + F_{OF}(\eta_0), \quad (15)$$

where F_G is for gravitation, F_{EM} is for electromagnetism, F_{SF} is for strong force, F_{WF} is for weak force and F_{OF} is for other forces [1,2,9,11].
In our construction of the generic partial differential equation [12], we have proposed a corollary which we then proved:
Corollary: The partial differential equations important in physics form a subset of the generic universal conservation equations.
By applying the technique of partial differentiation [17,18] we have shown that the generic PDE of the wave component of the Oyibo GUT is [12]

$$\frac{\partial^2 F(\eta_0)}{\partial \eta_0^{\ 2}} = \frac{1}{g_{00}c}\frac{\partial^2 F(\eta_0)}{dt^2} + \frac{1}{g_{m0}}\nabla^2 F(\eta_0) \quad (16)$$

Where the Laplacian operator, $\nabla^2 = \frac{\partial^2}{dr^2}$ with *r = x, y, z* and *m = 1, 2, 3.*

It is pertinent to emphasize that the generic nature of Eq.(11) means it is formulated to inherently solve well-posed problem and therefore meets an important goal of the study of PDE which is to determine the conditions under which a problem is well-posed [19]. In general, one of the important conditions for a problem involving a PDE to be a well-posed problem is that its solution depends continuously on the data of the problem. Thus as already stated, the generic nature of our PDE means its solution will continuously depend on the data of the problem hence for every application of our PDE, we will be dealing with a well-posed problem.
Now by differentiating Eq.(14), it is straightforward to show that

$$\frac{g_{m0}}{g_{00}} = \frac{d\eta_0}{dr}\frac{dt}{d\eta_0} = \frac{dt}{dr} = \frac{1}{c} \quad (17)$$

where$c$ is the corresponding wave speed.
Taking Eq. (17) into account in Eq.(16), the d'Alembertian operator emerges naturally in the RHS of Eq.(16) so that it can be re-expressed as

$$\frac{\partial^2 F(\eta_0)}{\partial \eta_0{}^2} = \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{dt^2}\right)F(\eta_0) \quad (18a)$$

or $$\frac{\partial^2 F(\eta_0)}{\partial \eta_0{}^2} = []^2 F(\eta_0) \quad (18b)$$

Where the change of the sign in Eq. (18a) can be subsumed in our choice of the metric parameters.
By comparing Eq.(6) and Eq.(18), one can claim that the latter is the generic Klein–Gordon equation obtained from Oyibo GUT. Again let it be emphasized that being 'generic' means that the final form of Eq.(18) depends on the system initial/boundary conditions and other system specifics. In this current case, the LHS of Eq.(18) has to be transformed to a mass term and the wavefield to a scalar field for it to be the form of Eq.(6) which is the standard Klein-Gordon equation that is applicable in physics.

4.0 The Klein-Gordon Equation for Photon from the Generic Klein–Gordon Equation

The quantum electromagnetic field is characterized by photons which have vanishing rest mass and no electric charge. Coherent states of the quantum electromagneticfields which contains many photons are well approximated by classical electromagnetic fields that satisfy the Maxwell equations.
In the consideration of the electromagnetic field within the purview of the Oyibo GUT, the starting corollary was [1]
Corollary: Maxwell's electromagnetic field equations are a subset of the generic universal conservation equations.
One of the interesting outcomes from the aforementioned consideration is that Oyibo was able to show that the Maxwell's electromagnetic field of the electromagnetic wave component given by Eq.(15) is reproduced from the generic universal equations as:

$$G(\eta_0) = \overline{F}_{EM}(\eta_0) \quad (19)$$

The η_0 is a function of space and time, that is, $\eta_0 = r,t$. [12]. If the particle is a photon so that it is massless, then the left hand side of Eq.(18) will be zero and we can argue that the wavefunction can be constructed as an electromagnetic wavefield with $\overline{F}_{EM}(\eta_0) = F(\eta_0) = F(r,t)$ from the Oyibo approach to Maxwell equations:

$$G_{00} = g_{00}t = g_{00}[E_t^2 + H_t^2 - W] \quad (20a)$$

$$G_{01} = g_{01}x = g_{01}[-E_y H_z + E_z H_y] \quad (20b)$$

$$G_{02} = g_{02}y = g_{02}[-E_z H_x + E_x H_z] \quad (20c)$$

$$G_{03} = g_{03}z = g_{03}[-E_y H_x + E_x H_y] \quad (20d)$$

where the electric field E is a scalar field described by

$$E(r,t) = E_0 e^{i(kr - wt)}. \quad (21)$$

Introducing this scalar field as a wavefunction of free waves $F(r,t) \Rightarrow E(r,t) \Rightarrow \phi(r,t)$ into Eq. (18), we obtain the Klein-Gordon equationfor photon as:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{dt^2}\right)\phi(r,t) = 0. \quad (22)$$

It is easy to show that if we use the dispersion relationship in free space, the deBroglie relation, $E = \hbar\omega$ and Einstein-Planck relation, $P = \hbar k$, then we can work back to recover the classical relativistic total energy for photon as done in Ref. [16]:

$$E^2 = p^2 c^2. \quad (23)$$

5.0 The Klein-Gordon Equation for Particle with Mass from the Generic Klein–Gordon Equation

For a particle having mass *m*, the LHS of Eq.(18) will yield the mass and scalar field for it to correspond to Eq.(6). As already stated, $\eta_0 = r,t$ so that from Eq.(15), $F(\eta_0) = F(r,t)$: therefore we adopt the transformation,

$$\frac{\partial^2}{\partial\eta_0^2} = \frac{\partial}{\partial r}\frac{\partial}{\partial t} \quad \text{and} \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial r}\frac{\partial r}{\partial t} \qquad (24)$$

In the LHS of Eq.(18) to obtain

$$\frac{\partial^2}{\partial\eta_0^2}F(\eta_0) = \frac{\partial}{\partial r}\frac{\partial}{\partial r}\frac{\partial r}{\partial t}F(\eta_0)\ . \qquad (25)$$

Taking into account in Eq.(25) the common knowledge that the momentum for a particle moving with the speed of light is $p = mc$ and that it is related to the quantum analogue for p given by Eq.(3) which is introduced here in natural unit, then Eq.(18) can be expressed as

$$m^2c^2\partial rF(\eta_0) = \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{dt^2}\right)F(\eta_0)\partial t \qquad (26)$$

Finally, since $F(\eta_0) = F(r,t)$, we can introduce the scalar field, $\varphi(r,t)$ into Eq.(26) using the transformations

$$\partial rF(\eta_0) = rF(\eta_0) \cong F(\eta_0) \Rightarrow \varphi(r,t) \qquad (27a)$$

and $$F(\eta_0)\partial t = F(\eta_0)t \cong F(\eta_0) \Rightarrow \varphi(r,t) \qquad (27b)$$

to obtain the Klein-Gordon equation for a particle having mass as

$$m^2c^2\varphi(r,t) = \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{dt^2}\right)\varphi(r,t) \qquad (28)$$

6.0 Conclusion

We have been able to obtain the Klein-Gordon equation from the Oyibo GUT generic wave component. It follows then that all applications of the Klein-Gorden equations can now be recovered from the Oyibo GUT. This includes its solution for relativistic spin-zero particle in D-dimension yielding exact bound state spectra important in atomic and molecular physics and chemistry[20]. An interesting observation is that since the Klein-Gordon equation has been obtained from the GUT, then in principle one can recover the relativistic energy equation from it and then in turn using any of the standard methods to obtain the Dirac equation [5,6]. This will imply that all the solutions and applications of the Dirac equation can also be obtained from the Oyibo GUT [6]. Thus the study here is a major boost to the Oyibo GUT as it has laid the foundation for bringing quantum field theory within the purview of the GAGUT.

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