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The Mathematics of Fractal Geometry Applied to Cantor Set

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Abstract

Fractal geometry is the geometry of most objects occurring in nature. To describe complicated and unconventional geometrical objects like a tree, the coast of a river, the surface of the moon, cloud, mountain, and human brain we use fractal geometry. In this article we explore the properties of some fractal geometrical figures. In particular we discuss the methods of construction and peculiar mathematical features of the Cantor set especially the properties of self- similarity and fractal dimension.

Keywords and Phrases: Fractal Geometry, Cantor set, unit interval, middle third, fractal dimension, chaotic behavior

1.0 Introduction

Euclidean geometry is often described as cold and dry because of its in ability to describe unconventional geometric figures that occur in nature such as mountains, clouds, tree, costline of a river etc. Since most patterns in nature are so irregular and fragmented, B. Mandelbrot [1]a mathematician at IBM in 1975 introduced the concept of fractal. Fractal is a complicated geometric figure that, unlike a conventional complicated figure, does not simplify when it is magnified. Fractal geometry is also used to describe trajectories and structures produced by chaotic dynamical systems [2].It is generally acknowledged that fractals have some or all of the following properties: complicated structure at a wide range of length scales, repetition of structures at different length scales (self- similarity) [3], and a fractal dimension that is not an integer. The simplest geometrical object which is a fractal is the Cantor set. In this article we will discuss the construction and peculiar mathematical properties of the Cantor set.

A fractal is a mathematical set that exhibits a repeating pattern displayed at every scale. It is also known as expanding symmetry or evolving symmetry. If the replication is exactly the same at every scale, it is called self-similar pattern [4], [5] and [6].

By geometric intuition, one can describe the idea of stability of fixed points. Supposing that the discrete-time system exists to model real phenomena, not all fixed points are alike. A stable fixed point has the property that points near it are moved even closer to the fixed point under the dynamical system [7] and [8].

2.0 Preliminaries and Basic Definitions of Terms

Definition 2.1.Self- similarity: Objects which looks the same on magnification or have the property of repetition of structures at different length scales have the property of self – similarity. For example fractal objects have such property.

Definition 2.2.Fractal dimension: The dimension of a point is zero and of a line is one. An object with fractal dimension has dimension which is a fraction $D = p/q$, where $q \neq p \neq 0$, i.e D is a non-integer positive number. For example fractals like the Cantor set has fractal dimension.

Definition 2.3.Bounded set: We say a set is bounded if we can draw a circle around it. More precisely a set S is bounded means there is a number b such that all points in the set are within distance b of one another. For example the unit interval [0,1] of the real line is bounded.

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Definition 2.4.Closed set: Let S be a set and let p be a point not in S. We say that p is separated from S if there is some number d so that all points within distance d of p are also not in S. A set S is closed means that every point not in S is separated from S. For example in every closed interval the end points belong to the interval. Thus the unit interval [0,1] and the unit circle $\{(x,y) : x^2 + y^2 = 1\}$ are closed sets.
Definition 2.5.Compact set: A compact set is a set which is closed and bounded. For example a circle is a compact set.
Definition 2.6.Connected set:A set S is totally disconnected means that whenever p and q are points of S, then there is some point between p and q which is not in S. A set is connected if if is not totally disconnected.

Definition 2.7.Itinerary of a point: This is a book keeping device that allows much of the information concerning the location of a point to be coded in terms of discrete symbols. For example given points in [0,1] assign the symbol L to the left subinterval [0,1/2] and R to the right subinterval [1/2,1]. In the case of the points { 1/3, 8/9, 32/81} the itinerary is given by: LRL.
Definition 2.8.Set of measure zero: A set S is said to have measure zero if it can be covered with intervals whose total length is arbitrarily small. In other words, for each predetermined $\in > 0, one can find$ a countable collection of intervals containing S whose total length is at most $\in. For example$ the set {1, 2, 3,...,10} has measure zero, since for any predetermined $\in > 0, the set can be covered by$ 10 $intervals$ of length $\in/10$ $centered at the$ 10 $integers$. Therefore, it has a covering set of length $\in, for \in as small as you want.$

3.0 Constructions of the Cantor Set

3.1 Interval Construction of the Cantor Set

We will use the letter C to denote the Cantor's set. The Cantor's set can be constructed as follows:
Begin with the unit interval I = [1,0], we remove from I= [0,1] the open interval $(\frac{1}{3}, \frac{2}{3})$ which is the middle third of I. The set of points that remain after this step is given by:
$K_1 = [0, 1/3] \cup [2/3, 1]$…………………………………………………………………… (1)
In the second step remove the middle thirds of the two segments of K_1. That is, remove (1/9, 2/9) ∪ (7/9, 8/9) and what is remaining after this step is given by:
$K_2= [0, 1/9] \cup [2/9, 3/9] \cup [6/9, 7/9] \cup [8/9, 1]$………………………………………. (2)
Since given [2/3,1] = [6/9,9/9] = [6/9,7/9,8/9,9/9] = [6/9,7/9] ∪ (7/9,8/9) ∪[8/9,9/9]. Remove the middle third (7/9,8/9) and what remains is [2/3,7/9]∪[8/9,1]. Similarly we subdivide and delete all the intervals accordingly.
Deleting the middle thirds of the four remaining segments of K_2 we get K_3. This process can be continued, and at each stage we remove the open middle third of all the closed intervals in the previous stage. If this process is continued *adinfinitum* what we get as the limiting set C = $K_\infty is called$ the middle third Cantor set or the Cantor set. The set C is the set of points that belong to all of K_n. Thus the set C is contained in K_n for each n. The set K_1 consist of two intervals of length 1/3, the set K_2 consists of four intervals of length $1/9 = (1/3)^2$ and in general K_n consists of 2^n intervals, each of length $(1/3)^n$, so the total length of the 2^n closed intervals is $(2/3)^n$ which tends to zero as $n \to \infty$. Thus the total length of C is 0 or the set C has measure zero.

3.2 Probabilistic Construction of the Cantor Set

Consider the following game: start with any point in the unit interval [0,1] and flip a coin. If the coin comes up heads, move the point two - thirds of the way towards 1. If tails,then move the point two-thirdsofthe way to 0. Plot the point that results. Then repeat the process. Flip the coin again and move two - thirds of the way from the new point to 1 (if heads) or to 0 (if tails). Plot the result and continue. Aside from the first few points plotted, the points that are plotted appear to fill out amiddle - third Cantor set. More precisely, the Cantor set is a chaotic attractor for the probabilistic process described, and the points plotted in the game approach the chaotic attractor at an exponential rate.

3.3 Representation of the Cantor Set By Itineraries

When we delete (1/3, 2/3) from the unit interval, we are left with two pieces. We call the left piece L and the right piece R as follows:
L = [0, 1/3] and R = [2/3, 1]. Each of the pieces L and R is in turn broken into two. We can name the two pieces on the left as LL = [0, 1/9] and LR = [2/9, 1/3] and the two pieces on the right as RL = [2/3, 7/9] and RR = [8/9, 1]. We can name the eight closed intervals from the third stage as LLL through RRR, and so on. An infinite sequence of L's and R's such as LLRRLRL……, gives rise to a nested sequence of closed intervals:
$... LLRRL \subset LLRR \subset LLR \subset LL \subset L$……………………………………..……………. (3)
and the intersection of such a sequence is non empty. So we get that:
$L \cap LL \cap LLR \cap LLRR \cap LLRRL$ ………………………………………………...…… (4)
is nonempty and contains points of the Cantor set C.

4.0 Results and Discussions

4.1 Peculiar Properties of the Cantor Set

The Cantor set have some topological properties which include the following: the Cantor set is a fixed point of a set valued function, a set that isbounded, closed, compact, totally disconnected and has a fractal dimension which is fractional.

4.2 Cantor Set as a Fixed Point Of Set Valued Function

Let f(x) = $\frac{1}{3}x$ and g(x) = $\frac{1}{3}x + \frac{2}{3}$ …………………………………………….. (5)

The fixed point of f is 0 and the fixed point of g is 1. As setwise mappings, the fixed point of f is {0} and the fixed point of g is {1}. To get the fixed point of F = f ∪g we proceed as follows:

F([0,1]) = [0,1/3] and g(0,1) = [2/3,1], so F([0,1]) = [0,1/3] ∪[2/3,1] which is what we obtained in the first step in constructing the Cantor set.

What do we get if we apply F again? We work it out as follows:

f([0,1/3]) = [0,1/9], g([0,1/3]) =[2/3,7/9], f([2/3,1]) = [2/9,1/3] and g([2/3,1]) = [8/9,1]

So we get the following:

F^2 ([0,1]) = [0,1/9] ∪ [2/9,1/3]∪ [2/3,7/9] ∪ [8/9,1]………………………… (6)

which is the second step in the construction of the Cantor set. In a similar way we will see that F^3 ([0,1]) gives the third step in the construction of C and so on. Thus F^k ([0,1]) is converging to the Cantor set. To get the fixed point of F(C) note that f(C) shrinks C by a factor of 3, and therefore f(C) is the portion of C between 0 and 1/3. Similarly, g(C) shrinks C by factor of 3 and then slides the result to the right a distance of 2/3. Thus g(C) is the portion of C between2/3 and 1. Together,

f(C)∪g(C) = C which implies that F(C) = C………………………………….. (7)

That is to say the Cantor set C is a fixed point of C.

4.3 The Cantor Set is Bounded

The Cantor set is bounded since all its points are within distance 1 of one another and the entire set lies inside the unit interval [0,1].

4.4 The Cantor Set is Closed

A set S is closed means that every point not in S is separated from S. The Cantor set is a closed set. Consider a point p not in the Cantor set. If p > 1 $orp < 0$, we see that p is separated from C. However if p is between 0 and 1 we know that p is deleted at some stage in the construction of C. in all this cases p is separated from C, and thus every point not in C is separated from C, therefore C is closed.

4.5 The Cantor Set is Compact

The Cantor set is closed and bounded and therefore compact. Most fractals are compact sets.

4.6 The Cantor Set is Totally Disconnected

To show that Cantor set is totally disconnected, consider two numbers p and q belonging to the Cantor set. We have to find a number between them which is not in C. Now let p ∈RRLRLL while q ∈RRLRLR. This means that any number x in the middle third of the interval RRLRL is not in C and is between p and q. Therefore C is totally disconnected.

4.7 The Cantor Set has Fractional Dimension

A point is a zero dimensional object since it has no length. A real interval or straight or curved lines are one dimensional object; they have length but no area. The interior of a square or the surface of a sphere is example of two dimensional objects, which have area but no volume. Consider the Cantor set: it has no length, so it makes sense that its dimension is less than 1. However there are a lot of points in the Cantor set and a lot of structure. The dimension of the Cantor set cannot therefore be zero. The dimension is the fraction log 2/log 3 ≈0.630

4.8 The Cantor Set has Self-Similarity

The Cantor set look the same under a microscope as it does to the naked eye. For example we can look at the whole Cantor set C or just the tiny bit that lies between 2/81 and 3/81 in the interval LLLR. The tiny portion is simply a 1/81th scale of the original. No matter how much we magnify a portion of C, what we observe looks exactly like C. Self similarity is the repetition of structures at different length scales.

5.0 Conclusion

In the article we have discussed the Cantor set and its various methods of construction. We have seen that the Cantor set is the simplest fractal geometric figure. As a fractal figure, the Cantor set possesses peculiar features such as self-similarity, chaoticity and fractal or fractional dimension. As a chaotic structure the Cantor set is a stable fixed point of a function F whose orbits on the interval [0,1] converge to the Cantor set. The Cantor set exhibits the chaotic behavior of a dynamical system model. Chaotic behavior can be observed in many natural and artificial systems or models and shall always be a mystery, a paradox, a puzzle, an enigma and a riddle in nature.

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