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Analytical Solution of the Schrodinger Equation for the Ring-Shaped Multi-parameter Exponential Type Potential

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Abstract

We study the nonrelativistic Schrodinger equation for Ring- shaped Multiparameter potential using the generalized parametric form of Nikiforov- Uvarov method. The energy eigenvalues and corresponding normalized wavefunctions are obtained analytically. We have also discussed the special cases of this potential.

Keywords: Nonrelativistic Schrodinger equation, Ring shaped Multiparameter Potential, Nikiforov-Uvarov method

1.0 Introduction

In various physical applications including those in nuclear physics and high energy physics [1,2], one of the fundamental problems is to obtain the exact and approximate solutions of the Schrodinger, Klein-Gordon (KG) and Dirac wave equations with potentials of interest. However, exact solutions of these equations are very rare such that many quantum systems have to rely on approximate methods to obtain their solutions. These wave equations are used to describe particle dynamics in quantum mechanics. Over the past years, intensive efforts have been devoted by several authors to solve these wave equations for a number of potentials. Some of these potentials include the Coulomb and harmonicpotentials [3,4], Manning-Rosen potential [5] , Hulthen potential [6], Eckart potential [7], and others [8]. For some quantum mechanical systems the most common approximation schemes are Supersymmetric quantum mechanics (SUSYQM) and Shape- invariance method [9], shifted $\frac{1}{N}$ expansion [10], Nikiforov-Uvarov method [11],the variational [12], the standard method [13], path integral approach [14], the asymptotic interaction method AIM [15] and others.

The ring-shaped like potentials which belong to the classof noncentral potentials have been an area of special interest in physics and chemistry, in recent times. These potentials consist of radial and angular dependent potentials. Studies have shown that ring-shaped like potential have found applications in many areas of nuclear physicsand quantum chemistry such as the study of ring shaped molecules like benzene [16-18].Furthermore, the shape forms of this potentials play an important role when studying the structure of deformed nuclei or the nuclear interactions.Over the years, appreciable efforts have been made by many authors to obtain the solutions of different wave equations with ring-shaped potentials both in relativistic and non relativistic limits. For example, Zhang *et al*[19-21] obtained the complete solutions of the Schrödinger and Dirac equations with a spherically harmonic oscillatory RS potential. Ikhdair and Sever obtained the exact solutions of the D-dimensional Schrödinger equation with RS pseudo-harmonic potential [22], modified Kratzer potential [23] and the D-dimensional KG equation with ring-shaped pseudo-harmonic potential [24]. Hamzavi *et al* found the exact solutions of Dirac equation with Hartmann potential [25] and RS pseudo-harmonic oscillatory potential [26] by using NU method. Berkdemir and Sever [27] investigated the

diatomic molecules subject to central potential plus RS potential. Some years ago, Chen and Dong [28] proposed a new ring – shaped potential and obtained the exact solution of Schrodinger equation for the coulomb potential. This type of potential used by Cheng and Dong [28] appears to be very similar to the potential used by Yasuk et al [29]. Moreover, Cheng and Dai [30], proposed a new potential consisting from the modified Kratzer's potential [27] plus the new proposed ring-shaped

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potential in [26]. They have presented the energy eigenvalues for this proposed exactly-solvable non-central potential in three dimensional (i.e., D = 3) Schrodinger equation by means of the NU method. Recently, Ikot et al. [31] proposed an improved ring- shaped potential and obtained the exact solution of the Schrodinger equation for the non-spherical harmonic oscillator and coulomb potential plus this improved ring-shaped potential which has possible applications to ring-shaped organic molecules like cyclic polyenes and benzene.
Motivated by the study of the ring-shaped-like potential, we attempt to propose an improved ring –shaped multiparameter potential of the form,

$$V(r,\theta)=\frac{Ae^{bL}+Be^{-b(r-L)}}{q+pe^{-b(r-L)}}+\frac{Ce^{-b(r-L)}+De^{-2b(r-L)}}{(q+pe^{-b(r-L)})^2}+\frac{\hbar^2}{2\mu r^2}\left(\frac{\beta\sin^2\theta+\gamma\cos^2\theta+\lambda}{\sin\theta\cos\theta}\right), \quad (1)$$

Where $V(r)=\dfrac{Ae^{bL}+Be^{-b(r-L)}}{q+pe^{-b(r-L)}}+\dfrac{Ce^{-b(r-L)}+De^{-2b(r-L)}}{(q+pe^{-b(r-L)})^2}$ Radial part (1a)

$$V_{RS}\theta=\frac{\hbar^2}{2\mu r^2}\left(\frac{\beta\sin^2\theta+\gamma\cos^2\theta+\lambda}{\sin\theta\cos\theta}\right) \quad \text{Ring Shape Part} \quad (1b)$$

$V(r)$ is the multiparameter exponential potential (MPETP) in which A,B,C,D,P, and q are all variable parameters. We can demonstrate that with appropriate choices of the parameters the MPETP can reduce to specific exponential potentials that have applications in the relativistic and non-relativistic quantum mechanics, for example Hulthen [32], Manning and Rosen [33], Eckart [34], and Woods Saxon [35]. Further, $V_{RS}(\theta)$ is a new RS potential identical to the RS part of the non-spherical harmonic oscillator potential [31].
It is therefore the aim of this work to study the Schrodinger equation with the improved ring-shaped multiparameter potentials using the NU method which to the best of our knowledge, has never been reported before in any available literature.
This work is organized as follows: Section II describes the principles of NU formalism. The Schrodinger equation with the potential under consideration is examined in section III. Discussions on special cases are discussed in section IV.Finally, a brief conclusion is given in section V.

2.0 Nikiforov – Uvarov Method

The NU method can be used to solve a second- order differential equation of the form [36]

$$\psi''(s)+\frac{\tilde{\tau}(s)}{\sigma(s)}\psi'(s)+\frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi(s)=0 \quad (2)$$

Where $\sigma(s)$ and $\tilde{\sigma}(s)$ must be polynomials of at most second degree and $\tilde{\tau}(s)$ is a polynomial with at most first degree and $\psi(s)$ is a function of the hypergeometric type.
The parametric generalization of the NU method that is valid for both central and non- central exponential type potential is given by the generalized hypergeometric- type equation as [37],

$$\psi''(s)+\frac{(\alpha_1-\alpha_2)}{s(1-\alpha_3 s)}\psi'(s)+\frac{1}{S^2(1-\alpha_3 s)}\left[-\xi_1 s^2+\xi_2 s-\xi_3\right]\psi(s)=0 \quad (3)$$

Comparing Eq. (2) with Eq. (3), the following polynomials are obtained.

$$\tau(s)=(\alpha_1-\alpha_2);\quad \sigma(s)=s(1-\alpha_3 s);\quad \tilde{\sigma}(s)=-\xi_1 s^2+\xi_2 s-\xi_3$$

with the bound state condition , the following parameters are deduced

$$\alpha_4=\frac{1}{2}(1-\alpha_1);\quad \alpha_5=\frac{1}{2}(\alpha_2-2\alpha_3);\ \alpha_6=\alpha_5^2+\xi_1;\ \alpha_7=2\alpha_4\alpha_5-\xi_2;\ \alpha_8=\alpha_4^2+\xi_3, \quad (4)$$

$$\alpha_9=\alpha_3\alpha_7+\alpha_3^2\alpha_8+\alpha_6,\ \alpha_{10}=\alpha_1+2\alpha_4+2\sqrt{\alpha_8},$$

$$\alpha_{11}=\alpha_2-2\alpha_5+2(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8}),\ \alpha_{12}=\alpha_4+\sqrt{\alpha_8},\ \alpha_{13}=\alpha_5-(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8}) \quad (5)$$

According to the NU method, the energy eigenvalues equation is given by the equation:

$$\alpha_2 n-(2n+1)\alpha_5+(2n+1)(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8})+n(n-1)\alpha_3+\alpha_7+2\alpha_3\alpha_8+2\sqrt{\alpha_8\alpha_9}=0 \quad , (6)$$

The wave functions are:

$$\rho(s) = s^{\alpha_{10}}(1-\alpha_3 s)^{\alpha_{11}}, \tag{6a}$$

$$\phi(s) = s^{\alpha_{12}}(1-\alpha_3 s)^{\alpha_{13}}, \quad \alpha_{12} .> 0, \ \alpha_{13} > 0 \tag{6b}$$

$$y_{(s)} = P_n^{(\alpha_{10}, \alpha_{11})}(1-2\alpha_3 s), \ \alpha_{10} > -1, \ \alpha_{11} > -1 \tag{6c}$$

Thus, the total wave function becomes

$$\phi(s) = N_{n,l}\, s^{-\alpha_{12}}(1-\alpha_3 s)^{-\alpha_{12}-(\alpha_{13}/\alpha_3)} P_n^{((\alpha_{10}-1),\,(\alpha_{11}/\alpha_3)-\alpha_{10}-1}(1-2\alpha_3 s), \tag{7}$$

where $P_n^{(\mu,\nu)}(x)$, $\mu > -1, \nu > -1$, and $x \in [-1, 1]$ are Jacobi polynomials with

$$P_n^{(\alpha,\beta)}(1-2s) = \frac{(\alpha+1)n}{n\iota} 2F1(-n, 1+\alpha+\beta+n, \alpha+1, s),$$

Where $N_{n,l}$ is the normalization constant.

Also, the above wave functions can be expressed in terms of the hypergeometric function through

$$\psi_{nk}(s) = N_{nk} s^{c_{12}}(1-c_3 s)^{c_{13}} 2F1(-n, 1+c_{10}+c_{11}+n, c_{10}+1, c_3 s), \tag{7a}$$

where $c_{12} > 0$, $c_{13} > 0$ and $s \in [0, 1/c_3], c_3 \neq 0.$

3.0 Bound State Solution of the Schrodinger Equation: Calculation of the Energy Eigenvalues and Eigen Functions

The Schrodinger equation in spherical coordinate for a particle with energy E moving in an external potential is written in the following form (in natural units $\hbar = m = 1$):

$$\left\{-\frac{1}{2}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2}{\partial\phi^2}\right)\right] + V(r) - E\right\}\psi_{nlm}(r,\theta,\phi) = 0 \tag{8}$$

By choosing the wavefunction $\psi_{nlm}(r,\theta,\phi) = r^{-1} R_n, l(r) Y_{lm}(\theta, \phi)$ and performing the necessary calculations, we obtain the following equations:

$$\frac{d^2 R_{nl}}{dr^2} + \left\{2[E - V(r)] - \frac{\Lambda}{r^2}\right\} R_{nl} = 0 \tag{9}$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dH(\theta)}{d\theta}\right) + \left[\Lambda - \frac{1}{\sin^2\theta} - \left(\frac{\beta\sin^2\theta + \gamma\cos^2\theta + \lambda}{\sin\theta\cos\theta}\right)^2\right] H(\theta) = 0 \tag{10}$$

$$\frac{d^2}{d\varphi^2} + m^2\Phi(\varphi) = 0 \tag{11}$$

A. Solutions of the polar angular equation

To obtain the energy eigenvalues and wave functions of Eqn. (10), we make use of an appropriate variables, $x = \cos^2\theta$ and Eqn. (10) becomes

$$\frac{d^2H(x)}{dx^2} + \frac{1-3x}{2x(1-x)}\frac{dH(x)}{dx} + \frac{1}{4x^2(1-x)^2}\left\{-\left[(\beta-\gamma)^2+\lambda\right]x^2 + \left[\Lambda - m^2 - 2(\beta+\lambda)(\beta-\gamma)x\right] - (\beta+\lambda)^2\right\} H(x) = 0. \tag{12}$$

Comparing Eqn. (3) with Eqn. (12), we have the following,

$$\alpha_1 = \frac{1}{2}, \ \alpha_2 = \frac{3}{2}, \alpha_3 = 1, \ \xi_1 = \frac{(\beta-\gamma)^2+\Lambda}{4}, \xi_2 = \frac{\Lambda - m^2 - 2(\beta+\lambda)(\beta-\gamma)}{4}, \xi_3 = \frac{(\beta+\gamma)}{4}. \tag{13}$$

$$\alpha_4 = \frac{1}{4}, \ \alpha_5 = -\frac{1}{4}, \ \alpha_6 = \frac{1}{16} + \frac{(\beta-\gamma)^2+\lambda}{4}, \ \alpha_7 = -\frac{1}{8} - \frac{\Lambda - m^2 - 2(\beta+\lambda)(\beta-\gamma)}{4}, \ \alpha_8 = \frac{1}{16} + \frac{(\beta+\lambda)^2}{4}$$

$$\alpha_9 = \frac{(\beta-\gamma)^2+(\beta+\lambda)^2 \ +m^2+2(\beta+\lambda)(\beta-\lambda)}{4}, \ \alpha_{10} = \frac{1}{2}\left(2+\sqrt{1+4(\beta+\lambda)^2}\right),$$

$$\alpha_{11} = 2\left(1+\frac{1}{2}\sqrt{(\beta-\gamma)^2+(\beta+\lambda)^2+m^2+2(\beta+\lambda)(\beta-\lambda)} \ +\frac{1}{4}\sqrt{1+4(\beta+\lambda)^2}\right)$$

$$\alpha_{12} = \frac{1}{4}\left(\sqrt{1+4(\beta+\lambda)^2}\right)$$

$$\alpha_{13} = -\frac{1}{4}\left(1+2\sqrt{(\beta-\gamma)^2+(\beta+\lambda)^2+m^2+2(\beta+\lambda)(\beta-\gamma)} +\sqrt{1+4(\beta+\lambda)^2}\right). \qquad (14)$$

Using the energy equation (6) and the coefficients given in Eqns. (13) and (14), we obtain the relationship between the separation constant Λ and the non-negative integer $n = n_r$ as,

$$\Lambda = \left(2n_r+1\sqrt{(\beta-\gamma)^2 \ +(\beta+\lambda)^2+m^2+2(\beta+\lambda)(\beta-\gamma)} \ \right)\times$$
$$\left(2n_r+1+\sqrt{(\beta-\gamma)^2+(\beta+\lambda)^2+m^2+2(\beta+\lambda)(\beta-\gamma)} \ +\sqrt{1+4(\beta+\lambda)^2}\right)-4[(\beta-\gamma)^2+(\beta+\lambda)^2] \ (15)$$

Equation (15) is the contribution of the angle-dependent part of the IRNHO. However, by setting $\gamma = \beta = \lambda = 0$, the ring-shaped term potential in Eqn.(1) disappears and the separation constant turns into $\Lambda = l(l+1)$, where $l = 2n_r + 1+ |m|, \ m = 0, 1, 2....$ To find the polar angular part of the wave function, we first obtain the weight function from Eqn. (5) as,

$$\rho(\theta)=(\cos^2\theta)^{(1/2)(2+\sqrt{1+4(\beta+\lambda)^2}),} \ (\sin^2\theta)^{2[1+(1/2)\sqrt{(\beta-\gamma)^2+(\beta+\lambda)^2+m^2+2(\beta+\lambda)(\beta-\gamma)} \ + (1/4)\sqrt{1+4(\beta+\lambda)^2} \ ]} (16),$$

Which leads to the solutions of the first part of the angular wave function from Eqn. (6c) in terms of the Jacobi polynomial as,

$$y_n(\theta) = P_n^{[(1/2)(2+\sqrt{1+4(\beta+\lambda)^2}), \ 2(1+(1/2)\sqrt{(\beta-\gamma)^2+(\beta+\lambda)^2+2(\beta+\lambda)(\beta-\lambda)} \ +(1/4)\sqrt{1+4(\beta+\lambda)^2} \ ]} \ (1-2\cos^2\theta), \ (17)$$

From Eqn. (6b), we obtain the second part of the angular function as,

$$\phi(\theta) = (\cos^2\theta)^{(1/4)(2+\sqrt{1+4(\beta+\lambda)^2}),} \ (\sin^2\theta)^{-(1/4)(1+(1+2)\sqrt{(\beta-\gamma)^2+(\beta+\lambda)^2+m^2+2(\beta+\lambda)(\beta-\lambda)} \ +\sqrt{1+4(\beta+\lambda)^2} \ ]} \quad (18)$$

Thus, the total angular wave function can be obtained namely, $H(\theta) = \phi_n(\theta)$ or from Eqn. (7a) in terms of the hypergeometric function as,

$$H_{lm}(\theta) = A_n(\cos^2\theta)^{(1/4)(\sqrt{1+4(\beta+\lambda)^2})}(\sin^2\theta)^{c_{13} = -(1/4)(1+2\sqrt{(\beta-\gamma)^2+(\beta+\lambda)^2+2(\beta+\lambda)(\beta-\lambda)} \ + \ \sqrt{1+4(\beta+\lambda)^2})}$$
$$\times F1[-n,4+\frac{1}{2}(\sqrt{1+4(\beta+\lambda)^2}) +(\sqrt{(\beta-\gamma)^2+(\beta+\lambda)^2+m^2+2(\beta+\lambda)(\beta-\gamma)} \ \frac{1}{2}\sqrt{1+4(\beta+\lambda)^2})$$
$$+n;\frac{1}{2}(\sqrt{1+4(\beta+\lambda)^2}) \ +2;\cos^2\theta]. \qquad (19)$$

B. Solutions of the Radial Part

In this section we will consider the radial part of the Ring-Shaped Multiparameter potential and obtain the eigenvalues and corresponding wave functions. Substituting

$$V(r) = \frac{Ae^{bL} + Be^{-b(r-L)}}{q+pe^{-b(r-L)}} + \frac{Ce^{-b(r-L)} + De^{-2b(r+L)}}{(q+pe^{-b(r-L)})^2} , \qquad (20)$$

into Eqn. (9) and making use of the approximation to the centrifugal given by [38]

$$\frac{1}{(x+L)^2} = Co + \frac{C_1}{p+qe^{bx}} + \frac{C_2}{(p+qe^{bx})^2} \qquad (21)$$

With $r = (x+L)$, we have

$$\frac{d^2R_{nl}}{dx^2}+\left\{2\left[E-\frac{Ae^{bL}+B}{qe^{bx}+p}-\frac{Ce^{bx}+D}{(qe^{bx}+p)^2}\right]-\Lambda\left(Co+\frac{C_1}{p+qe^{bx}} \ \frac{C_2}{(p+qe^{bx})^2}\right)\right\}R_{nl} = 0, \qquad (22)$$

by applying the transformation $s = -\frac{q}{p}e^{bx}$, we obtain

$$\frac{du}{ds^2} + \frac{(1-s)}{(1-s)}\frac{du}{ds} + \frac{1}{s^2(1-s)^2}\left[-As^2 + Bs - C\right]R_{nl}, \tag{23}$$

Where,

$$A = \varepsilon^2 + \Lambda Co, \quad B = 2\varepsilon^2 + 2\Lambda Co + \frac{2}{p}\left(Ae^{bL} + B\right) + \frac{\Lambda C_1}{p},$$

$$C = \varepsilon^2 + \Lambda Co + \frac{\Lambda C_1}{p} + \frac{\Lambda C_2}{p^2} + \frac{2}{p}\left(Ae^{bL} + B\right) + \frac{2}{p^2}\left(Ce^{bx} + D\right),$$

Comparing Eqn. (23) with Eqn. (3), we derive the physical constants,

$$\alpha_1 = 1 \quad \xi_1 = \varepsilon^2 + \Lambda Co$$

$$\alpha_2 = 1 \quad \xi_2 = 2\varepsilon^2 + 2\Lambda Co + \frac{2}{p}\left(Ae^{bL} + B\right) + \frac{\Lambda C_1}{p} \tag{24}$$

$$\alpha_3 = 1$$

$$\alpha_4 = 0 \quad \xi_3 = \varepsilon^2 \Lambda Co + \frac{\Lambda C_1}{p} + \frac{\Lambda C_2}{p^2} + \frac{2}{p}\left(Ae^{bL} + B\right) + \frac{2}{p^2}\left(Ce^{bx} + D\right)$$

and

$$\alpha_5 = -\frac{1}{2}$$

$$\alpha_6 = \frac{1}{4} + \varepsilon^2 + \Lambda Co$$

$$\alpha_7 = -\left(2\varepsilon^2 + 2\Lambda Co + \frac{2}{p}\left(Ae^{bL} + B\right) + \frac{\Lambda C_1}{p}\right)$$

$$\alpha_8 = \varepsilon^2 + \Lambda Co + \frac{\Lambda C_1}{p} + \frac{\Lambda C_2}{p^2} + \frac{2}{p}\left(Ae^{bL} + B\right) + \frac{2}{p^2}\left(Ce^{bx} + D\right)$$

$$\alpha_9 = \frac{2}{p^2}\left(Ce^{bx} + D\right) + \frac{1}{4} + \frac{\Lambda C_2}{p^2}$$

$$\alpha_{10} = 1 + 2\sqrt{\varepsilon^2 + \Lambda C_O + \frac{\Lambda C_1}{p} + \frac{\Lambda C_2}{p^2} + \frac{2}{p}\left(Ae^{bL} + B\right) + \frac{2}{p^2}\left(Ce^{bx} + D\right)}$$

$$\alpha_{11} = 2 + 2\left(\sqrt{\frac{2}{p^2}\left(Ce^{bx} + D\right) + \frac{1}{4} + \frac{\Lambda C_2}{p^2}} + \sqrt{\varepsilon^2 + \Lambda Co + \frac{\Lambda C_1}{p} + \frac{\Lambda C_2}{p^2} + \frac{2}{p}\left(Ae^{bL} + B\right) + \frac{2}{p^2}\left(Ce^{bx} + D\right)}\right)$$

$$\alpha_{12} = \sqrt{\varepsilon^2 + \Lambda Co + \frac{\Lambda C_1}{p} + \frac{\Lambda C_2}{p^2} + \frac{2}{p}\left(Ae^{bL} + B\right) + \frac{2}{p^2}\left(Ce^{bx} + D\right)}$$

$$\alpha_{13} = -\frac{1}{2} - \left(\sqrt{\frac{2}{p^2}\left(Ce^{bx} + D\right) + \frac{1}{4} + \frac{\Lambda C_2}{p^2}} + \sqrt{\varepsilon^2 + \Lambda Co + \frac{\Lambda C_1}{p} + \frac{\Lambda C_2}{p^2} + \frac{2}{p}\left(Ae^{bL} + B\right) + \frac{2}{p^2}\left(Ce^{bx} + D\right)}\right) \tag{25}$$

Using Equations (25), (24) and (6), and after carrying out some mathematical operations, we obtain the energy eigenvalues of the radial part as

$$E = \frac{1}{2}\left[\Lambda Co + \frac{\Lambda C_2}{p^2} + \chi\right] + \frac{1}{2}\left[\frac{(n+\sigma)}{2} + \frac{\chi}{2(n+\sigma)}\right]^2, \tag{26}$$

Where,

$$\sigma = \frac{1}{2} + \sqrt{\frac{2}{p^2}\left(Ce^{bx} + D\right) + \frac{1}{4} + \frac{\Lambda C_2}{p^2}} ,$$

$$\chi = \frac{\Lambda C_1}{p} + \frac{2}{p}\left(Ae^{bL} + B\right) + \frac{2}{p^2}\left(Ce^{bx} + D\right),$$

$$Co = \frac{(3+bL)p^2 - 2(-3+bL)pq + (3-3bL+b^2L^2)q^2}{b^2q^2L^4} ,$$

$$C_1 = \frac{2(p+q)^2\left[(3+bL)p + (3-2bL)q\right]}{b^2q^2L^4} ,$$

$$C_2 = \frac{(p+q)^3\left[(3+bL)p + (3-bL)q\right]}{b^2q^2L^4} .$$

Using Eqns. (7), (24), and (25), corresponding wave function of the radial part is obtained as

$$N_{nl}s^V(1-s)^{\mu+\frac{1}{2}} P_n^{2V,2+2\mu}(1-2s) \tag{27}$$

where,

$$\mu = \sqrt{\frac{2}{p^2}(Ce^{bx} + D) + \frac{1}{4} + \frac{\Lambda C_2}{p^2}} ,$$

$$V = \sqrt{\varepsilon^2 + \Lambda Co + \frac{\Lambda C_1}{p} + \frac{\Lambda C_2}{p^2} + \frac{2}{p}\left(Ae^{bL} + B\right) + \frac{2}{p^2}\left(Ce^{bx} + D\right)} ,$$

and N_{nl} is the normalization constant.

C. Effect of Angular dependent Part on the Radial Solutions

The total energy of the Ring-Shaped multiparameter potential is obtained by considering the effect of the angle dependent part on the radial part. Substituting Eqn. (15) into Eqn. (26) yields the energy spectra for this system as

$$E_{nl} = \frac{1}{2}\left[(\eta + \chi) + \left(\frac{(n+\sigma)}{2} + \frac{\chi}{2(n+\sigma)}\right)^2\right], \tag{28}$$

Where,

$$\eta = \left(\begin{array}{l}\left(2n_r + 1 + \sqrt{(\beta-\gamma)^2 + (\beta+\lambda)^2 + m^2 + 2(\beta+\lambda)(\beta-\gamma)}\right) \times \\ \left(2n_r + 1 + \sqrt{(\beta-\gamma)^2 + (\beta+\lambda)^2 + m^2 + 2(\beta+\lambda)(\beta-\gamma)} + \sqrt{1+4(\beta+\lambda)^2}\right) - 4[(\beta-\gamma)^2 + (\beta+\lambda)^2]\end{array}\right) C_0 +$$

$$\frac{1}{p^2}\left(\begin{array}{l}\left(2n_r + 1\sqrt{(\beta-\gamma)^2 + (\beta+\lambda)^2 + m^2 + 2(\beta+\lambda)(\beta-\gamma)}\right) \times \\ \left(2n_r + 1 + \sqrt{(\beta-\gamma)^2 + (\beta+\lambda)^2 + m^2 + 2(\beta+\lambda)(\beta-\gamma)} + \sqrt{1+4(\beta+\lambda)^2}\right) - 4[(\beta-\gamma)^2 + (\beta+\lambda)^2]\end{array}\right) C_2,$$

$$n, n_r = 0,1,2......,$$

4.0 Discussions on Special Cases

To test the validity of our results, several other potentials with different applications in physics can be derived from the Multiparameter potential by assigning values to the adjustable parameters.

A. The Hulthen Potential

If the parameters are arranged as $A = C = D = 0,\ B = -Vo,\ q = e^{bL},\ p = -1$, $\beta = \gamma = \lambda = 0$ and $b = 2a$, the Multiparameter potential reduces to the Hulthen Potential of the form [32]:

$$V_H(r) = -Vo\frac{e^{-2\alpha r}}{1-e^{-2\alpha r}} . \tag{29}$$

With the corresponding eigenvalue obtained from Eqn. (28) as

$$E_{nl}=\frac{1}{2}\left[(\eta+\chi)+\left(\frac{(n+\frac{1}{2}+\sqrt{\frac{1}{4}-\Lambda C_2})}{2}+\frac{\chi}{2(n+\frac{1}{2}+\sqrt{\frac{1}{4}-\Lambda C_2})}\right)^2\right] \tag{30}$$

Where,

$$\chi=-\Lambda C_1-V_0,$$

$$Co=\frac{(3+2\alpha L)+2(-3+2\alpha L)e^{2\alpha L}+(3-6\alpha L+4\alpha^2L^2)e^{4\alpha L}}{4\alpha^2e^{4\alpha L}L^4},$$

$$C_1=-\frac{2(e^{2\alpha L}-1)^2[(3+2\alpha L)]+e^{2\alpha L}(3-4\alpha L)}{4\alpha^2e^{4\alpha L}L^4},$$

$$C_2=-\frac{(e^{2\alpha L}-1)^3[(3+2\alpha L)+(3-2\alpha L)e^{2\alpha L}]}{4\alpha^2e^{4\alpha L}L^4},$$

B . The Manning- Rosen Potential

Manning – Rosen potential is one of the short range potential and it has been used to describe the diatomic molecular vibration [39]If we set $A=C=0,\quad B=-K/2m\beta^2,\ D=\alpha(\alpha-1)/2m\beta^2,\ q=e^{bL},\ p=-1$ $\beta=\gamma=\lambda=0$ and $b=\frac{1}{\beta}$, the Multiparameter potential changes into the Manning – Rosen potential of the form [33]:

$$V_{M-R}(r)=\frac{1}{2m\beta^2}\left[\frac{\alpha(\alpha-1)e^{-2r/\beta}}{(1-e^{-2r/\beta})^2}-\frac{Ke^{-r/\beta}}{1-e^{-2r/\beta}}\right], \tag{31}$$

with the corresponding eigenvalue obtained from Eqn. (28) as

$$E_{nl}=\frac{1}{2}\left[(\eta+\chi)+\left(\frac{(n+\frac{1}{2}+\sqrt{\frac{1}{4}-\alpha\frac{(\alpha-1)}{m\beta^2}-\Lambda C_2})}{2}+\frac{\chi}{2(n+n+\frac{1}{2}+\sqrt{\frac{1}{4}-\alpha\frac{(\alpha-1)}{m\beta^2}-\Lambda C_2}}\right)^2\right] \tag{32a}$$

Where,

$$\chi=2\left(\frac{K}{2m\beta^2}\right)-\Lambda C_1-\frac{\alpha(\alpha-1)}{m\beta^2},$$

$$Co=\frac{(3+\frac{L}{\beta})+2(-3+\frac{L}{\beta})e^{\frac{L}{\beta}}+(3-3\frac{L}{\beta}+\frac{L^2}{\beta^2})e^{\frac{L}{\beta}}}{\frac{L^6}{\beta^2}e^{\frac{L}{\beta}}},$$

$$C_1=-\frac{2(e^{\frac{L}{\beta}}-1)^2[3+\frac{L}{\beta}]+e^{\frac{L}{\beta}}(3-\frac{2L}{\beta})}{\frac{e^{2L/\beta}L^4}{\beta^2}},$$

$$C_2=-\frac{(e^{L/\beta}-1)^3[(3+\frac{L}{\beta})+(3-\frac{L}{\beta})e^{L/\beta}]}{\frac{e^{2L/\beta}L^4}{\beta^2}},$$

C. Eckart Potential

By substituting $A= D=0,\ B=-\alpha,\ C=\beta e^{bL},\ q=e^{bL},\ p=1 \quad \beta=\gamma=\lambda=0$ and $b=\frac{1}{\alpha}$, we obtain the Eckart Potential as follows [34]

$$V_E(r) = -\alpha\frac{e^{-r/\alpha}}{1-e^{-r/\alpha}} + \beta\frac{e^{-r/\alpha}}{(1-e^{-r/\alpha})^2} \tag{32b}$$

The energy eigenvalues for Eckart potential under this conditioncan be obtained from Eqn. (28) as:

$$E_{nl}=\frac{1}{2}\left[(\eta+\chi)+\left(\frac{(n+\frac{1}{2}+\sqrt{\frac{1}{4}+2\beta e^{xL/\alpha^2}+\Lambda C_2})}{2}+\frac{\chi}{2(n+\frac{1}{2}+\sqrt{\frac{1}{4}+2\beta e^{xL/\alpha^2}+\Lambda C_2})}\right)^2\right] \tag{33}$$

Where,

$$\chi=\Lambda C_1-2\alpha+2\beta e^{\frac{Lx}{\alpha^2}},$$

$$Co=\frac{(3+\frac{L}{\alpha})-2(-3+\frac{L}{\alpha})e^{\frac{L}{\alpha}}+(3-\frac{3L}{\alpha}+\frac{L^2}{\alpha^2})e^{\frac{2L}{\alpha}}}{\frac{1}{\alpha^2}e^{\frac{2L}{\alpha}}L^4},$$

$$C_1=\frac{2(1+e^{\frac{L}{\alpha}})^2\left[(3+\frac{L}{\alpha})+(3-2\frac{L}{\alpha})e^{\frac{L}{\alpha}}\right]}{\frac{1}{\alpha^2}e^{\frac{L}{\alpha}}L^4},$$

$$C_2=\frac{(1+e^{\frac{L}{\alpha}})^3\left[(3+\frac{L}{\alpha})+(3-\frac{L}{\alpha})e^{\frac{L}{\alpha}}\right]}{\frac{1}{\alpha^2}e^{\frac{2L}{\alpha}}L^4},$$

D. Woods Saxon Potential

If the parameters are chosen as $A=C=D=0,\ B=-Vo,\ q=1,\ p=1,\ b=\frac{1}{\alpha},\ \ \beta=\gamma=\lambda=0$ and $L=R$, the Multiparameter potential is transformed to the Woods – Saxon potential [35]

as: $$V_{W-S}(r) = -\frac{Vo}{1+e^{\frac{r-R}{\alpha}}}, \tag{34}$$

with the energy eigenvalues obtained from eqn. (28) as

$$E_{nl}=\frac{1}{2}\left[(\eta+\chi)+\left(\frac{(n+\sqrt{\frac{1}{4}+\Lambda C_2})}{2}+\frac{\chi}{2(n+\sqrt{\frac{1}{4}+\Lambda C_2})}\right)^2\right], \tag{35}$$

where,

$$\chi=\Lambda C_1-2V_0,$$

$$Co=\frac{(3+\frac{R}{\alpha})-2(-3+\frac{R}{\alpha})+(3-\frac{3R}{\alpha}+\frac{R^2}{\alpha^2})}{\frac{R^4}{\alpha^2}},$$

$$C_1 = \frac{8\left[(3+\frac{R}{\alpha}) + (3-2\frac{R}{\alpha})\right]}{\frac{R^4}{\alpha^2}},$$

$$C_2 = \frac{8\left[(3+\frac{R}{\alpha}) + (3-\frac{R}{\alpha})\right]}{\frac{R^4}{\alpha^2}}.$$

5.0 Conclusion

In this paper, we have obtain analytical solutions of the non relativistic Schrodinger equation with the Ring-shaped Multiparameter potential under the framework of Nikiforov-Uvarov method with the help of approximation scheme in Ref [33]to evaluate the centrifugal term. The energy eigenvalues and corresponding wavefunctions are obtained. By appropriate choice of parameters our potential in Eqn.(1) reduces to well known potentials: Hulthen, Woods-Saxon, Eckart, Manning Rosen, and with their respective eigenvaluesalso evaluated.
Finally, it is worth noting that the approximate solution obtained in the newly proposed potential may have some significant applications in the study of quantum mechanical systems in both chemical and molecular physics.

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