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Dynamic Performance of Variable Bi-parametric Elastic Subgrade on Simply Supported Uniform Rayleigh Beam Travelling with Constant Velocity

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Abstract

This study presents the comprehensive investigation of the dynamic characteristics of bi-parametric elastic foundation on prismatic simply-supported thick beam carrying moving distributed masses at uniform speed. An approach involving Generalized Galerkin's method, the Struble's asymptotic technique and Laplace method is developed. Analytical solutions of the model equations describing the motion of the vibrating structures are obtained and the results show that the higher the values of the variable bi-parametric elastic foundation, the lower the dynamic response amplitude of the prismatic simply supported Rayleigh beam at constant speed.

Keywords: Prismatic, Prestressed, Resonance, Vibration, Galerkin's method

1.0 Introduction

The reliability of structural members under moving load has attracted the attention of researchers in the area of engineering and applied mathematics [1-3]. The safety and effects of these moving mass on the structural element are of great importance. In most of the existing literature in dynamics of structures under moving loads, moving loads have been idealized as moving concentrated loads which acts at a certain point on the structure and along a single line in space. That is, the moving load is modeled as a lumped load. In practice, it is known that loads are actually distributed over a small segment or over the entire length of the structural member as they traverse the structure. Such moving loads are termed uniform distributed loads. Concentrated forces are mere mathematical idealization but cannot be found in the real world, where all forces are body forces acting over an area.

Several authors have worked on the concentrated load problems described above. Among them is the work of Timoshenko [6] who investigated the case of concentrated load problem moving with a constant velocity along a beam neglecting the effect of damping. He obtained analytical solution to the governing initial boundary problems.

Oni and Awodola [7] studied the response of uniform Rayleigh beam carry moving masses resting on variable Winkler elastic foundation. They obtained an analytical solution to the fourth order partial differential equation. They observed that the deflections of the Rayleigh beams under the actions of moving masses are higher than the deflections when only the force effects of the moving loads are conserved.An efficient analytical method for vibration of Euler-Bernoulli beam on elastic foundation with elastically restrained ends using Fourier sine series with Stoke's Transformation is used by Mustafa Ozguret [8] to obtain the vibration response.

For the two-dimensional structures problem, Gbadeyan and Dada [9] investigated the dynamic analysis of rectangular plate on a Pasternak foundation and subjected to uniformly partially distributed masses neglecting the effects of shear deformation and rotatory inertia. The critical speeds of the moving masses and forces were calculated.

In all the above works, studies have been limited to the cases where foundation on which the structure is resting is constant. The more practical case which considers the one-dimensional structural problems resting on variable bi-parametric elastic foundation is scanty in literature.

In this paper, the problem of the vibrations under moving distributed masses of a uniform simply supported Rayleigh beams on a variable bi-parametric elastic subgrade moving with constant velocity is investigated. The method in [10] is employed to solve the governing equation. The analysis is carried out for various parameters in the model equation.

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2.0 Model Equation

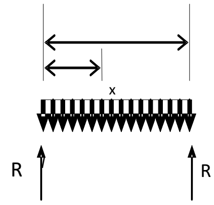

Fig.1: Simply supported Beam with a distributed load

Consider the vibration of a highly prestressed uniform Rayleigh beam of length L resting on variable bi-parametric elastic foundation and subjected to travelling distributed loads.

Let us assume that the distributed masses M move across the beam starting at time $t=0$ and advances on the beam from end $x=0$ to end $x=L$ of the beam with uniform velocity c. The governing fourth order partial differential equation of the undamped system is given by

$$EI\frac{\partial^4 Y(x,t)}{\partial x^4} - N\frac{\partial^2 Y(x,t)}{\partial x^2} + \bar{\mu}\frac{\partial^2 Y(x,t)}{\partial t^2} - \bar{\mu}R^o\frac{\partial^4 Y(x,t)}{\partial x^2 \partial t^2} + Q_k(x)Y(x,t) = P(x,t) \tag{1}$$

Where x is the spatial co-ordinate, t is the time, $Y(x,t)$ is the transverse displacement, EI is the flexural rigidity of the structure, $\bar{\mu}$ is the mass per unit length of the beam, N is the axial force, R^o is the rotatory inertia factor, $Q_k(x)Y(x,t)$ is the foundation reaction and $P(x,t)$ is the moving distributed load.

In this system, when the effect of the mass of the moving distributed load on the uniform Rayleigh beam is considered, $P(x,t)$ takes the form [4]

$$P = MH(x-ct)\left[g - \left(\frac{\partial^2}{\partial t^2} + 2c\frac{\partial^2}{\partial x \partial t} + c^2\frac{\partial^2}{\partial x^2}\right)Y(x,t)\right] \tag{2}$$

Where g is the acceleration due to gravity and $H(\cdot)$ is the well-known Heaviside function.

At this juncture, the boundary conditions for the dynamical system is arbitrary and the initial conditions without any loss of generality are taken as

$$Y(x,0) = 0 = \frac{\partial Y(x,0)}{\partial t} \tag{3}$$

The beam is assumed to rest on variable elastic foundation and the relationship between the foundation reaction and the lateral deflection $Y(x,t)$ takes the form

$$Q_k(x,t) = K(x)(x)Y(x,t) - \frac{\partial}{\partial x}\left(G(x)\frac{\partial Y(x,t)}{\partial x}\right) \tag{4}$$

where $K(x)$ and $G(x)$ are two variable parameters of the elastic foundation and specifically, $K(x)$ is the variable foundation stiffness and $G(x)$ is the variable shear modulus.

In this paper, an example of variable elastic foundation in [5] is adopted namely,

$$K(x) = K_0(4x - 3x^2 + x^3) \tag{5a}$$

$$G(x) = G_0(12 - 13x + 6x^2 - x^3) \tag{5b}$$

Using equations (2) to (5b), equation (1) yields

$$EI\frac{\partial^4 Y(x,t)}{\partial x^4} - N\frac{\partial^2 Y(x,t)}{\partial x^2} + \bar{\mu}\frac{\partial^2 Y(x,t)}{\partial t^2} - \bar{\mu}R^o\frac{\partial^4 Y(x,t)}{\partial x^2 \partial t^2} - K_0(4x - 3x^2 + x^3)Y(x,t)$$

$$+G_0(-13 + 12x - 3x^2)\frac{\partial Y(x,t)}{\partial x} + G_0(12 - 13x + 6x^2 - x^3)\frac{\partial^2 Y(x,t)}{\partial x^2}$$

$$+MH(x-ct)\left[\frac{\partial^2}{\partial t^2} + 2c\frac{\partial^2 Y(x,t)}{\partial x \partial t} + c^2\frac{\partial^2}{\partial x^2}\right]Y(x,t) = MgH(x-ct) \tag{6}$$

3.0 Operational Simplification

Equation (6) is a fourth order partial differential equation. Evidently, a closed form solution does not exist and therefore, an approximate solution is sought. In this section, use is made of the Generalized Galerkin's method described in [11], to reduce the equation to a sequence of ordinary differential equation. Thus a solution of the form

$$Y_n(x,t) = \sum_{m=1}^{\infty} Z_m(t) U_m(x) \tag{7}$$

is sought.

An appropriate selection of functions for beam problems are beam mode shapes. Thus, the mth normal mode of vibration of a uniform beam

$$U_m(x) = sin\frac{\lambda_m x}{L} + A_m cos\frac{\lambda_m x}{L} + B_m sinh\frac{\lambda_m x}{L} + C_m cosh\frac{\lambda_m x}{L} \tag{8}$$

is chosen such that the boundary conditions are satisfied. λ_m is the mode frequency, A_m, B_m, C_m are constants which are obtained by substituting (8) into the appropriate boundary conditions. In this paper, it is assumed that the beam has a simple supports at both end i.e $x = 0$ and $x = L$. In this case, both the bending moment and the deflections vanish. Thus, for this case it can be shown in equation (8) that

$$A_m = B_m = C_m = 0 \text{ and } \lambda_m = m\pi \tag{9}$$

Using (9) in (7), one obtains

$$Y_n(x,t) = \sum_{m=1}^{\infty} Z_m(t) \sin\frac{m\pi x}{L} \tag{10}$$

Substituting (10) into equation (6) and using an appropriate expansion of Fourier series for the Heaviside step function, after some simplification and arrangements, we obtain

$$\sum_{m=1}^{n} Q_A(m,k)\ddot{Z}_m(t) + Q_B(m,k)ZV_m(t)$$

$$+\varepsilon_0\left[\left(Q_{C1}(m,k) + \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{c\,\text{os}(2n+1)\pi ct}{2n+1}Q_{C2}(m,k) - \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{sin(2n+1)\pi ct}{2n+1}Q_{C3}(n,m,k)\right)\ddot{Z}_m(t)\right.$$

$$+2c\left(Q_{D1}(m,k) + \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{c\,\text{os}(2n+1)\pi ct}{2n+1}Q_{D2}(m,k) - \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{sin(2n+1)\pi ct}{2n+1}Q_{D3}(n,m,k)\right)\dot{Z}_m(t)$$

$$\left.+c^2\left(Q_{E1}(m,k) + \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{c\,\text{os}(2n+1)\pi ct}{2n+1}Q_{E2}(m,k) - \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{sin(2n+1)\pi ct}{2n+1}Q_{E3}(n,m,k)\right)Z_m(t)\right]\Bigg\}$$

$$= \frac{PL}{\bar{\mu}\lambda_m}\left[(-1)^m - \cos\frac{m\pi ct}{L}\right] \tag{11}$$

Where

$$\varepsilon_0 = \frac{M}{\bar{\mu}L} \tag{12}$$

Equation (11) represents the transformed equation governing the motion of a uniform prestressed simply supported uniform Rayleigh beam on variable bi-parametric elastic subgrade subjected to moving distributed loads. To this end, we consider two special cases of equation (11) namely: The **moving distributed force (MDF)**and the **moving distributed mass problem (MDM)**.

3.1 Simply Supported Uniform Rayleigh Beam Traversed by MDF

Considering only the force effect of the moving distributed loads in equation (11) above, i.e. setting $\varepsilon_0 = 0$ then the entire equation (11) now reduces to

$$\frac{d^2}{dt^2}Z_m(t) + \Omega_{ss}^{s} Z_m(t) = \frac{PL}{H_{mn}}\left[-(-1)^m + Cos\frac{m\pi ct}{L}\right] \tag{13}$$

Where

$$\Omega_{ss}^2 = \frac{EIm^4\pi^4 p^*}{2\bar{\mu}L^3} - \frac{Nm^2\pi^2 p^*}{2\bar{\mu}L} - \frac{K_0L^2p^*}{\bar{\mu}\pi^2}\left[\frac{1}{(m+k)^2} - \frac{1}{(m-k)^2}\right]$$
$$-\frac{6K_0L^3p^*}{\bar{\mu}\pi^2}\left[\frac{1}{(m-k)^2} - \frac{1}{(m+k)^2}\right] + \frac{12K_0L^3p^*}{(m-k)^3\bar{\mu}\pi^3} + \frac{26G_0m^2p^*}{(k^2-m^2)\bar{\mu}} - \frac{6m^2\pi^2LG_0p^*}{\bar{\mu}L}$$
$$-\frac{12G_0mLp^*}{\bar{\mu}}\left[\frac{1}{(m+k)^2} - \frac{1}{(m-k)^2}\right] + \frac{4m^2LG_0p^*}{(m-k)^3\bar{\mu}\pi} - \frac{6G_0L^3p^*m\pi}{\bar{\mu}}\left[\frac{(k+m)^4-(k-m)^4}{(k-m)^4(k+m)^4} - \frac{1}{(m-k)^2}\right]$$
$$+\frac{3m^2G_0p^*}{\bar{\mu}}\left[\frac{1}{(m+k)^2} - \frac{1}{(m-k)^2}\right] - \frac{12m^2LG_0p^*}{\bar{\mu}}\left[\frac{1}{(m-k)^2} - \frac{1}{(m+k)^2}\right] \quad (14)$$

$$p^* = \frac{2L}{L^2 - R^o m^2\pi^2}, \quad H_{mn} = \frac{\mu m\pi(L^2 - R^o m^2\pi^2)}{2L} \quad (15)$$

Further arrangements and simplification of equation (13) using Laplace method defined by

$$(\tilde{\cdot}) = \int_0^\infty (\cdot)e^{-st}dt \quad (16)$$

and when solved in conjunction with initial conditions, one obtains an expression for $Z_m(t)$. Thus,

$$Z_m(t) = \frac{2PL}{m\pi(L^2 - R^o m\pi)}\left[\frac{cosq_m t - cos\Omega_{ss}t}{\Omega_{ss}^2 - q_m^2} + \frac{(1-cos\Omega_{ss}t)}{\Omega_{ss}}\right] \quad (17)$$

which on inversion yields,

$$Y_n(x,t) = \sum_{m=1}^{n}\frac{2PL}{m\pi(L^2 - R^o m\pi)}\left[\frac{cosq_m t - cos\Omega_{ss}t}{\Omega_{ss}^2 - q_m^2} + \frac{(1-cos\Omega_{ss}t)}{\Omega_{ss}}\right]\times\frac{sinmx}{L} \quad (18)$$

Equation (18) represents the transverse displacement response of distributed force moving at a constant velocity of a simply supported beam resting on variable bi-parametric elastic foundation.

3.2 Simply Supported Uniform Rayleigh Beam Traversed by MDM

If the force effect and the inertia effects are considered, the solution to the entire equation (11) is sought and we term this moving distributed mass (MDM) problem. To this end, use is made of the asymptotic method of Struble extensively discussed in [4]. It requires that the asymptotic solution of the homogeneous part of equation (11) be written in the form

$$Z_m(t) = B(m,t)cos\left[\Omega_{ss}t - \phi(m,t)\right] + \varepsilon_1 ZV_1(t) + 0(\varepsilon_1^2) \quad (19)$$

where $B(m,t)$ and $\phi(m,t)$ are slowly varying functions.

To obtain the modified frequency, equation (19) and its derivatives are substituted into the homogeneous part of equation (11). The resulting variational equations describing the behavior of $B(m,t)$ and $\phi(m,t)$ during the motion of the mass determined by the modified frequency gives

$$-2\dot{B}(m,t)\Omega_{ss}sin\left[\Omega_{ss}t - \phi(m,t)\right] + 2B(m,t)\Omega_{ss}\dot{\phi}(m,t)cos\left[\Omega_{ss}t - \phi(m,t)\right]$$
$$+\frac{\varepsilon_1 m^2 c}{H_{mm}(m^2-k^2)}B(m,t)\Omega_{ss}sin\left[\Omega_{ss}t - \phi(m,t)\right] - \frac{\varepsilon_1 L}{8H_{mm}(m^2-k^2)}B(m,t)\Omega_{ss}\cos\left[\Omega_{ss}t - \phi(m,t)\right]$$
$$-\frac{\varepsilon_1 m^2c^2\pi^2}{8LH_{mm}(m^2-k^2)}B(m,t)\cos\left[\Omega_{ss}t - \phi(m,t)\right] = 0 \quad (20)$$

Where terms higher than $0(\varepsilon_1^2)$ have been neglected.

Therefore, where the effect of the mass of the particle is considered, the first approximation to the homogeneous system is

$$Z_m(t) = B(m,t)cos\left[\gamma_m t - \phi(m,t)\right] \quad (21)$$

Where

$$\gamma_m = \Omega_{ss}\left[1-\frac{\varepsilon_1}{16H_{mm}}\left(\frac{c^2m^2\pi^2}{\Omega_{ss}^2}-L\right)\right] \qquad (22)$$

is the modified frequency representing the frequency of the free system due to the presence of the moving distributed mass. Hence, the entire equation (11) reduces to

$$\frac{d^2}{dt^2}Z_m(t)+\gamma_m^2 Z_m(t)=\frac{PL}{H_{mm}(1+H_{m1}(n,m,t))}\left[-(-1)^m+cos\frac{m\pi ct}{L}\right] \qquad (23)$$

which when solve with the initial conditions leads to

$$Z_m(t)=\frac{\varepsilon_1 L^3 g}{m\pi(L^2-R^0m^2\pi^2)}\left[\frac{cosq_m t-cos\gamma_m t}{\gamma_m^2-q_m^2}+\frac{(1-cos\gamma_m t)}{\gamma_m}\right] \qquad (24)$$

which on inversion yields

$$Y_n(x,t)=\sum_{m=1}^{n}\frac{\varepsilon_1 L^3 g}{m\pi(L^2-R^0m^2\pi^2)}\left[\frac{cosq_m t-cos\gamma_m t}{\gamma_{ss}^2-q_m^2}+\frac{(1-cos\gamma_m t)}{\gamma_m}\right]\times\frac{sinmx}{L} \qquad (25)$$

Equation (25) represents the transverse displacement response of moving distributed mass moving at a constant velocity of a simply supported uniform Rayleigh beam resting on variable bi-parametric elastic foundation.

4.0 Comments on Closed Form Solutions

In this section, it is pertinent to establish the conditions under which the phenomenon of resonance occurs. This has a great interest for Structural engineers as it is the root cause of cracks and deformation of structures. Equation (18) clearly shows that the simply supported beam resting on a variable bi-parametric elastic foundation and traversed by a moving distributed force reaches a state of resonance whenever

$$\Omega_{ss}=\frac{m\pi c}{L} \qquad (26)$$

While equation (25) shows that the same beam under the action of moving mass will experience resonance effect whenever

$$\gamma_m=\frac{m\pi c}{L} \qquad (27)$$

From equation (22)

$$\gamma_{mm} = \Omega_{ss}\left[1-\frac{\varepsilon_1}{16H_{mm}}\left(\frac{m^2\pi^2c^2}{\Omega_{ss}^2}-L\right)\right] \qquad (28)$$

which implies

$$\gamma_{mm} = \Omega_{ss}\left[1-\frac{\varepsilon_1}{16\Delta^\bullet}\left(\frac{m^2\pi^2c^2}{\Omega_{ss}^2}-L\right)\right]=\frac{m\pi c}{L} \qquad (29)$$

It is therefore evident that, for the same natural frequency, the critical speed for the system consisting of a simply supported Rayleigh beam resting on a variable bi-parametric elastic foundation and traversed by moving distributed force with uniform speed is greater than that of distributed mass problem. Thus for the same natural frequency, resonance is reached earlier in the moving distributed mass system than in the moving distributed force system.

5.0 Numerical Results and Discussion

Numerical results obtained from the analyses in this present study are presented by considering a homogenous beam of modulus of elasticity $E=3.1\times10^{10}N/m^2$, the moment of inertia $I=2.87698\times10^{-3}m^4$, velocity 8.123m/s, the beam span L=12.192m and the mass per unit length of the beam $\bar{\mu}=2758.291Kg/m$. The dynamic behaviour of the simply supported uniform Rayleigh beam are calculated and graphs are plotted for beam response against time for values of rotatory inertia correction factor Ro ,axial force N, mass ratio Eo, shear modulus G_0 and foundation stiffness K_0. For the simply supported uniform Rayleigh beam resting on variable bi-parametric elastic foundation, the results are presented on the various graphs below.

Figures 2 and 3displays the effects of axial force N on the flexural vibrations of a simply supported uniform Rayleigh beam on variable bi-parametric elastic foundation at constant velocity in both cases of moving distributed force and moving distributed mass respectively. The graphs show that the response amplitude decreases as the value of the axial force increases.

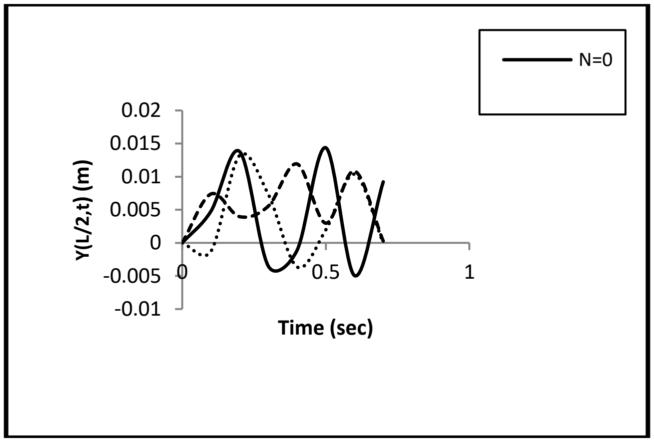

Fig. 2: Deflection profile of a Simply supported uniform Rayleigh beam on variable Pasternak foundation and traversed by moving distributed force for Go=30000, Ko=10000, Ro=0.2 and various values of No.

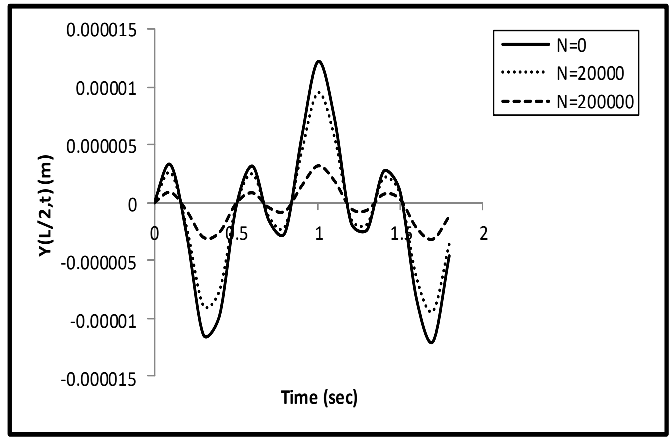

Fig. 3: Dispacement response of a Simply supported uniform Rayleigh beam on variable Pasternak foundation and traversed by moving distributed mass for Go=30000, Ko=10000, Eo=0.5, Ro=0.2 and various values of N.

Figures 4 and 5 shows the effect of rotatory inertia Ro on the transverse displacement of the simply supported Rayleigh beam in both cases of moving distributed force and moving distributed mass respectively. The curves show that the response amplitude decreases as the value of the rotatory inertia correction factor increases.

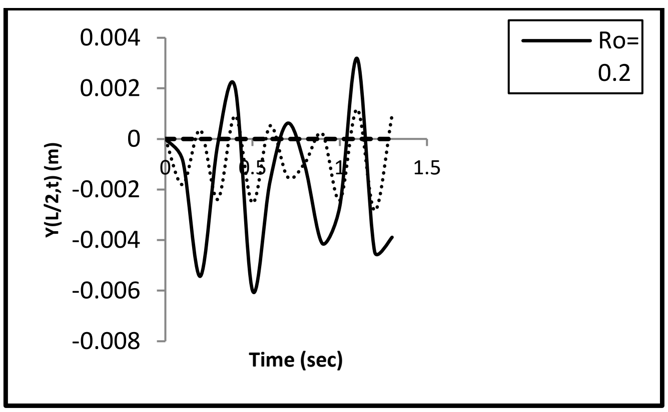

Fig. 4: Transverse displacement of a Simply supported uniform Rayleigh beam on variable Pasternak foundation and traversed by moving distributed force for N=2000, Ko=10000, Go=30000 and various values of Ro.

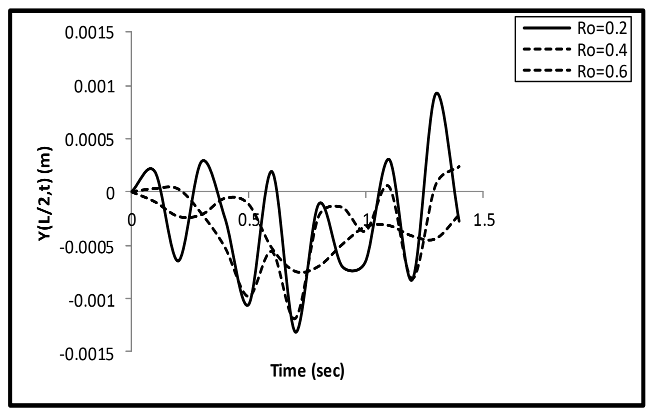

Fig. 5: Deflection profile of a Simply supported uniform Rayleigh beam on variable Pasternak foundation and traversed by moving distributed mass for N=2000, Eo=0.5, Ko=10000, Go=30000 and various values of Ro.

Figure 6 and 7 display the effects of shear modulus Go on deflection amplitude of the simply supported Rayleigh beam transverse by moving distributed force and moving distributed mass for Ko=10000N/m^2, N=2000 N/m^2 and Ro=0.2 respectively. It can be seen from the graphs that the response amplitude decreases as the value of the Go increases.

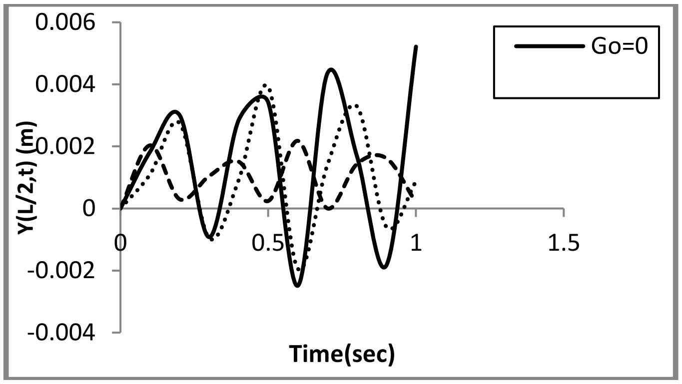

Fig. 6: Displacement response of a Simply supported uniform Rayleigh beam on variable Pasternak foundation and traversed by moving distributed force for N=2000, Ko=10000, Ro=0.2 and various values of Go.

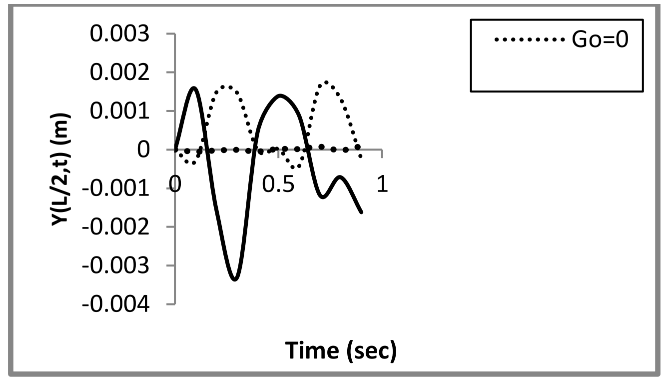

Fig. 7: Transverse displacement of a Simply supported uniform Rayleigh beam on variable Pasternak foundation and traversed by moving distributed mass for N=2000, Eo=0.5, Ko=10000, Ro=0.2 and various values of Go.

Figure 8 and 9 displays the deflection of the foundation stiffness Ko for the uniform Rayleigh beam traversed by moving distributed force and moving distributed mass. As the value of Ko increases, the response amplitude of the beam for moving distributed force and moving distributed mass decreases.

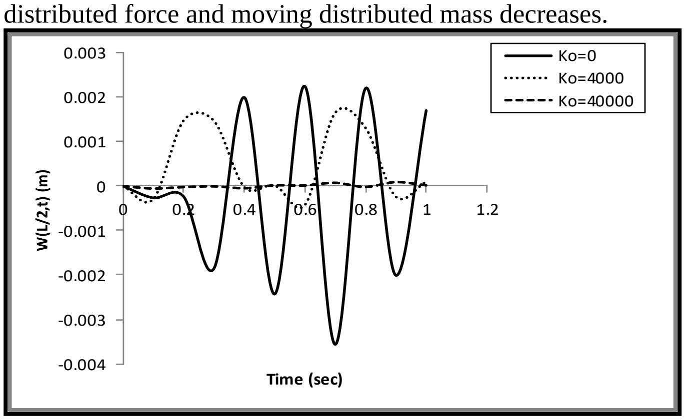

Fig. 8: Deflection profile of a Simply supported uniform Rayleigh beam on variable Pasternak foundation and traversed by moving distributed force for N=2000, Go=30000, Ro=0.2 and various values of Ko.

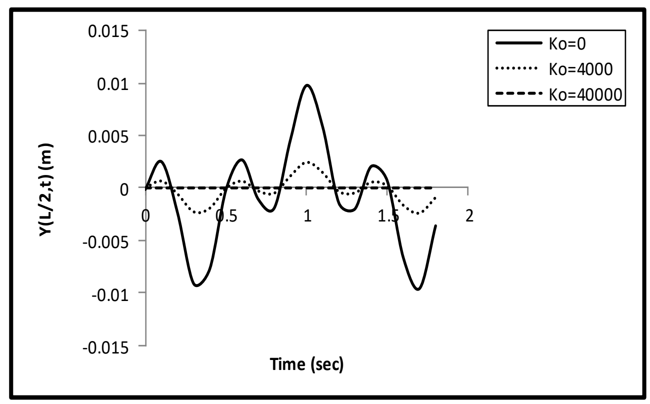

Fig. 9: Displacement response of a Simply supported uniform Rayleigh beam on variable Pasternak foundation and traversed by moving distributed mass for N=2000, Go=30000, Eo=0.5, Ro=0.2 and various values of Ko.

Figure 10 displays the deflection profile of the mass ratio for the uniform Rayleigh beam for N=2000N/m^2, Ro=0.2, Go=3000 N/m^2, Ko=1000 N/m^2. As the value of E_0 increases, response amplitude of the beam for the moving distributed mass decreases.

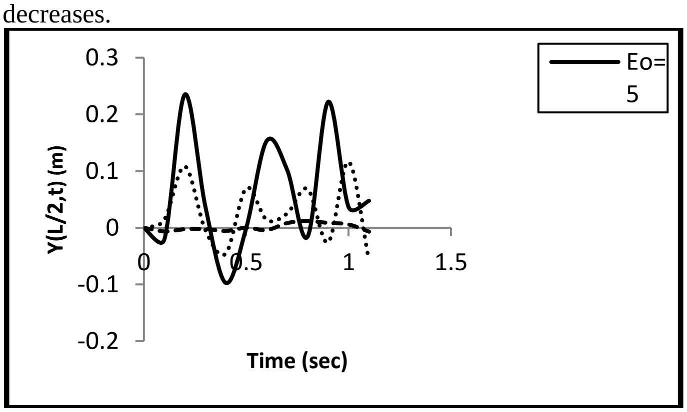

Fig. 10: Transverse displacement of a Simply supported uniform Rayleigh beam on variable Pasternak foundation and traversed by moving distributed mass for N=10000, Go=30000, Ko=2000, Ro=0.2 and various values of Eo.

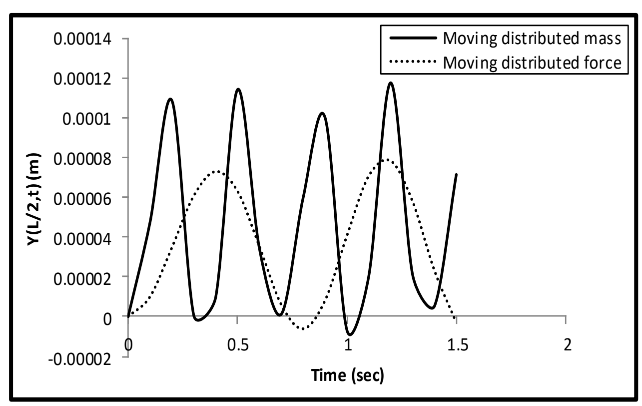

Fig. 11: Comparison of the displacement response of moving distributed force and moving distributed mass for Simply supported uniform Rayleigh beam on variable Pasternak foundation for fixed values of Ko=10000, Go=30000, N=2000, Ro=0.2 and of Eo=0.5.

Figure 11 compares the displacement curves of the moving distributed force and moving distributed mass for a simply supported Rayleigh beam with Ko=10000N/m^2. N=2000N/m^2, Ro=0.2 and Go=30000N/m^2. Clearly, the response amplitude of a moving distributed mass is greater than that of a moving distributed force pattern.

6.0 Concluding Remarks

In this study, a simply supported uniform Rayleigh beam on a variable Pasternak elastic subgrade subjected to mobile distributed mass is investigated by generalized Galerkin's method, an expansion of Heaviside function in series form and Struble's asymptotic method. The dynamic responses of the thick beam are obtained in closed forms and the conditions under which the finite system may experience resonance phenomenon are established. The effects of rotator inertia, foundation stiffness, axial force and shear modulli on the beam deflections are presented. Also, for the same natural frequency, the critical speed for moving distributed mass (mdm) problem is smaller than that of the moving distributed force (mdf) problem of the simply supported thick beam considered which show that the moving distributed force solution is not always an upper bound for an accurate solution of the moving distributed mass problem.

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