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Power Analysis of the Likelihood Ratio Tests for Exponential Populations

[1]Yahya W.B., [1]Kolade E.I., [1]Garba M.K. and [2]Usman A.

[1]Department of Statistics, University of Ilorin, P.M.B. 1515, Ilorin, Nigeria.
[2]Department of Statistics, Federal University of Technology, Minna, Nigeria.

Abstract

The statistical power of the likelihood ratio (LR) test for testing the parameter (λ) of the exponential distribution under different parameter considerations and sample sizes was investigated. Up till now, considerations had only been on the effect and sample sizeswhile determining the power of statistical hypothesis tests, especially those that involve the parameters of exponential distributions of the form H_0: $\lambda = \lambda_0$ vs. H_1: $\lambda = \lambda_1$. Thus, literature is apparently silent on the impact of the sizes of the parameter pair (λ_0, λ_1) being tested on the power of the test. This was investigated in this study in addition to some other situations considered for determining power of LR test for exponential distributions through detail Monte Carlo studies. Part of the novel resultsobtained from this study showed that the power of the test is highly sensitive to the sizes of parameter pair (λ_0, λ_1) being tested irrespective the effect size $\Delta = |\lambda_0 - \lambda_1|$.In other words, at any given sample size, small values of the parameter pair (λ_0, λ_1) yielded appreciable power than the large values of the parameter pair (λ_0, λ_1) of the exponential distributions being tested even under equal effect sizes. Therefore, increasing the sample size at any point may only be desirable as a corrective measure to increase the power of the LR test whenever the power provided by the test is considered small, the situation that can possibly occur when the parameter pair (λ_0, λ_1) of the exponential distributions being tested is relatively large. The implication of these results is that fewer samples would be required to attain an appreciable power with small values of the parameter pair (λ_0, λ_1) while large samples would be needed to attain a similar feat of power size under large values of the parameter pair (λ_0, λ_1) even if the effect size is the same under the two test problems. Further results from this study indicated that fewer samples would be required by the LR test to achieve appreciable power as the chosen size α level of the test increases. Empirical illustrations are provided to validate the results from Monte Carlo experiments. It is therefore recommended that more attention should be given to the size of the parameters being tested in any statistical significant test if the prime interest is to achieve appreciable power.

Keywords: Likelihood ratio test; Exponential distribution; Effect sizes; Statistical power.
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1.0 Introduction

The concept of power in statistical hypothesis testing has received prominent discussion in the literature. Statistical hypothesis testing is a scientific process to examine if a proposition is plausible or not. The power of a statistical test therefore is the probability that the test will correctly lead to the rejection of a false proposition. A statistical power is the ability of a test to detect an effect, if the effect actually exists [1,2]. A statistical power analysis may be retrospective (post

Corresponding author: Yahya W. B., E-mail: wbyahya@unilorin.edu.ng, Tel.: +234703-057-8626

hoc) or prospective (a priori). A prospective analysis is often used to determine the required sample size to achieve target statistical power prior to analysis, while a retrospective analysis computes the statistical power of a test given a sample size and effect size [3].
Power can be determined for a number of statistical test procedureswhich include the significance tests[4]and the likelihood ratio tests[5,6]among others. This has been very useful in many practical life applications[7-10].
The likelihood ratio (LR) is the ratio of two likelihood functions by varying the parameters over two different sets in the numerator and denominator. A likelihood ratio testtherefore is a statistical test that compares the maximum value of the likelihood of the null hypothesis against the maximum value of the likelihood of the alternative hypothesisin order to make a decision between the two hypotheses. The procedure of this test has been widely discussed in the literature [5,11,12]and its use has been demonstratedon many statistical hypothesis testing problems [13-15].
Despite manybenefits of the LR test over some of its counterparts[13,16], the determinationof the sampling distribution of the LR statistic in any given test hypothesis problem and its associated rigorous computational tasks constitute major constraints on the flexibility and usage of the LR tests. As a result of these challenges, few discussions on power analysis of the LR tests have been reported in the literature especially forstatistical test problems on data that emanated from population with underlining exponential distribution(s)[17].
Not only this, up till the present moment to the best of our knowledge, researchers had only focused on the effect sizes and sample sizes while discussing issues relating to the power of statistical hypothesis tests, especially those that involve the parameters of exponential distributions. Thus, literature is apparently silent on the impact of the sizes of the parameter pair, for instance(λ_0, λ_1) being tested on the power of the test for the hypothesis H$_0$: $\lambda = \lambda_0$ vs. H$_1$: $\lambda = \lambda_1$.
The work here is therefore intended to examine the behaviour of power of the likelihood ratio test on data sets with underlining exponential distributions. Various conditions under which appreciable power of the likelihood ratio test can be achieved given this distribution function shall be investigated. More importantly, the impact of the size of the parameters being tested under the null and alternative hypotheses forms on the power of the LR test shall be investigated. However, our choice of exponential distribution in this study is premised on the importance of this distribution as the distribution of survival time of some events with constant rate of occurrence[18-21].

2.0 Theoretical Formulations

Suppose a random variable Xfollows an exponential process with parameter λ. Then, the density function of $X$is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0, \lambda > 0. \\ 0, & if\ otherwise \end{cases} \qquad (2.1)$$

withλ being the rate parameter that indexes the average rate of occurrence of an event over Xduration of time[14]. In this context, random variable X may represent the duration of time that a given biological or mechanical system manages to survive before it fails. Hence, $E(X) = \frac{1}{\lambda}$would represent the expected duration of survival per unit with a corresponding variance of $\frac{1}{\lambda^2}$.
Conversely, the exponential probability density function (pdf) in (2.1) can be re-parameterized as

$$f(x) = \begin{cases} \frac{1}{m} e^{-\frac{1}{m}x}, & x > 0,\ m > 0. \\ 0, & if\ otherwise \end{cases} \qquad (2.2)$$

where m is the scale parameter with $\lambda = \frac{1}{m}$ establishing the link between the two density functions(2.1) and (2.2).Here also, $E(X) = m$is the expected durationof time to failure with a variance, $Var(X) = m^2$.In density function (2.2), $\frac{1}{m}$ represents the constant failure rate per unit time.
For simplicity, the representationof exponential density in (2.1) shall be employed for our subsequent discussions in this study since the representations(2.1) and (2.2) would essentially lead to similar conclusions.
Consider a set of n random samples $x_1, \ldots, x_n$drawn from the population having an underline exponential distribution of the type in (2.1) with an unknown parameter λ. If it is desirable to test the hypothesis that these$n$ samples are drawn from an exponential population with parameter $\lambda_0$in the parameter space $\boldsymbol{\lambda}$, the hypothesis of interest would bea simple null hypothesis versus acomposite alternative of the form

H$_0$: $\lambda = \lambda_0$versus H$_1$: $\lambda \neq \lambda_0$ (2.3)

More specifically, if the alternative side of this hypothesis set is true, it simply shows that the true value of parameter λ of the exponential population where the data come actually from is another value, say λ_1 in the parameter space $\boldsymbol{\lambda}$with $\lambda_1 \neq \lambda_0$. Therefore, the working hypothesis from (2.3) then becomes

H$_0$: $\lambda = \lambda_0$ versus H$_1$: $\lambda = \lambda_1$,$\lambda_0 \neq \lambda_1$,$\lambda_0, \lambda_1 \in \boldsymbol{\lambda}$ (2.4)

which can be simply expressed in term of the density function in (2.1) as

H$_0$:$f(x|\lambda = \lambda_0)$ versus H$_1$: $f(x|\lambda = \lambda_1)$,$\lambda_0 \neq \lambda_1$ (2.5)

given that $x \sim exp\ (\lambda)$.
To test the hypothesis set (2.5) through the procedure of the likelihood ratio (LR) test, the test statistic is of the form[5,22],

$$\Lambda = \frac{max_{\lambda \epsilon H_0} L(\lambda|x)}{max_{\lambda \epsilon H_0 \cup H_1} L(\lambda|x)} = \frac{\prod_{i=1}^{n} \lambda_0\, e^{-\lambda_0 x_i}}{\prod_{i=1}^{n} \lambda_1\, e^{-\lambda_1 x_i}} \qquad (2.6)$$

Clearly, $\Lambda > 0$. In many instances, it is much convenient to work with the log of the likelihood ratio statistic in (2.6). This is given by

$$\Lambda^* = ln\ (\Lambda) = \ ln \left(\frac{\prod_{i=1}^{n} \lambda_0\, e^{-\lambda_0 x_i}}{\prod_{i=1}^{n} \lambda_1\, e^{-\lambda_1 x_i}} \right) \qquad (2.7)$$

which simply becomes

$$\Lambda^* = \sum_{i=1}^{n} ln\left(\lambda_0\, e^{-\lambda_0 x_i}\right) - \sum_{i=1}^{n} ln\left(\lambda_1\, e^{-\lambda_1 x_i}\right) \qquad (2.8)$$

The value of λ_0 is known but that of λ_1 is not and can be obtained from the sample data through its *maximum likelihood estimator* (MLE)$\hat{\lambda}_1 = \frac{1}{\bar{x}}$ given that the alternative hypothesis set H_1 is true.

For any value $\hat{\Lambda}^*$of the log-likelihood ratio (LLR) statistic Λ^*estimated from the sample data, the null hypothesis set in (2.5) would be rejected if the value of $\hat{\Lambda}^*$is very small at any given Type I error α.Explicitly, the LR test would reject the null that$\lambda = \lambda_0$in favour of the alternative that $\lambda = \lambda_1$if

$$\hat{\Lambda}^* < k_\alpha \qquad (2.9)$$

where the value of k_α depends on the sampling distribution of $\hat{\Lambda}^*$.
The same results and conclusions would be obtained if the ratio in the LR statistic (2.6) is interchanged to give another representation

$$\Lambda^c = \frac{max_{\lambda \epsilon H_0 \cup H_1} L(\lambda|x)}{max_{\lambda \epsilon H_0} L(\lambda|x)} = \frac{\prod_{i=1}^{n} \lambda_1\, e^{-\lambda_1 x_i}}{\prod_{i=1}^{n} \lambda_0\, e^{-\lambda_0 x_i}} \qquad (2.10)$$

of the LR statistic. Under this representation, the null hypothesis would be rejected in favour of the alternative set if

$$\hat{\Lambda}^c > k_\alpha \qquad (2.11)$$

where $\hat{\Lambda}^c$ is the point estimate of Λ^c fromthe data.

2.1 The Power of the Likelihood Ratio Test

If the probability that the LR test (2.6) or (2.10)accepts falsely, the null hypothesis set (2.5) is β (the Type II error), then,the power of the test, which is the probability that the test would reject a false null hypothesis[9,23]in (2.5)is defined by $P_w = p(\Lambda^* rejects H_0 | H_1 is true) = 1 - \beta$.This is theprobability that the LR test would reject the null hypothesis H_0:$f(x|\lambda = \lambda_0)$ given that the alternative set H_1: $f(x|\lambda = \lambda_1)$ is true. This statement can be expressed in term of the LLR test statistic Λ^*(using (2.8) and (2.9)) as

$$P_w = p\left(\hat{\Lambda}^* < k_\alpha | H_1 \colon f(x|\lambda = \lambda_1)\right) = 1 - \beta \qquad (2.12)$$

where the value of P_w depends on the sampling distribution of LLR statistic Λ^* in (2.8).

2.2 The Sampling Distribution of the LLR Test statistic Λ^*

As a review, the sampling distribution of the log-likelihood ratio statistic of the type in (2.7) is determined through the following derivations.

Suppose the parameter $\lambda_1$of the exponential distribution in the likelihood ratio statistic (2.6) is replaced by its MLE $\hat{\lambda}_1 = \frac{1}{\bar{X}}$, then, the LR statistic (2.6) becomes

$$\Lambda = (\lambda_0 \bar{x})^n e^{-n\lambda_0 \bar{x} + n} \qquad (2.13)$$

If the log of Λ in (2.13) is taken,an equivalent of the log-likelihood ratio statistic Λ^*given in (2.7) shall be obtained as

$$\Lambda^* = log\ \Lambda = n\ log(\lambda_0 \bar{x}) - n\lambda_0 \bar{x} + n$$

Hence, we have the log-likelihood ratio test statistic

$$\Lambda^* = n\{log(\lambda_0 \bar{x}) - (\lambda_0 \bar{x} - 1)\} \qquad (2.14)$$

Suppose we define a function

$$g(y) = log(\lambda_0 y) - (\lambda_0 y - 1) \qquad (2.15)$$

from (2.11) with $y = \bar{x}$, then (2.14) reduces to

$$\Lambda^* = ng(y). \qquad (2.16)$$

Obviously, the first and second derivatives of $g(y)$in (2.15) are $g^{'}(y) = \frac{1}{y} - \lambda_0$ and $g^{''}(y) = -\frac{1}{y^2}$ respectively.

Now, expansion of the function $g(y)$ about the mean $\bar{X} = E_0(X) = \frac{1}{\lambda_0}$ under the assumption that the null hypothesis holdsusing the Taylor's series expansion up to second degree yield

$$g(y) \approx g(y_0) + g^{'}(y_0)(y - y_0) + \frac{1}{2!} g^{''}(y_0)(y - y_0)^2$$

With $y_0 = \frac{1}{\lambda_0}$ this becomes,

$$g(y) \approx g\left(\frac{1}{\lambda_0}\right) + g'\left(\frac{1}{\lambda_0}\right)\left(y - \frac{1}{\lambda_0}\right) + \frac{1}{2!}g''\left(\frac{1}{\lambda_0}\right)\left(y - \frac{1}{\lambda_0}\right)^2. \quad (2.17)$$

Direct substitution of $\frac{1}{\lambda_0}$ for y in $g(y)$,$g'(y)$ and $g''(y)$ in (2.15)yield$g\left(\frac{1}{\lambda_0}\right) = 0$,$g'\left(\frac{1}{\lambda_0}\right) = 0$and $g''\left(\frac{1}{\lambda_0}\right) = -\lambda_0^2$respectively. Therefore, the series expansion$g(y)$in (2.17)reduces to

$$g(y) \approx -\frac{1}{2}\lambda_0^2(y - \frac{1}{\lambda_0})^2 \quad (2.18)$$

If (2.18) is substituted back into (2.16) we have

$$\Lambda^*(y) \approx -\frac{1}{2}n\lambda_0^2(y - \frac{1}{\lambda_0})^2$$

which can be expressed further as

$$-2\Lambda^*(y) \approx n\lambda_0^2(y - \frac{1}{\lambda_0})^2 \quad (2.19)$$

Since $y = \bar{x}$ from our representations in (2.14) and (2.15), equation (2.19) becomes

$$-2\Lambda^*(\bar{X}) \approx n\lambda_0^2(\bar{x} - \frac{1}{\lambda_0})^2$$

with a final form

$$-2\Lambda^*(\bar{X}) \approx \left(\sqrt{n}\frac{\left(\bar{x} - \frac{1}{\lambda_0}\right)}{1/\lambda_0}\right)^2 \sim\chi_1^2 \quad (2.20)$$

This implies that statistic $-2\Lambda^*(\bar{X})$or simply $-2\Lambda^*$has the chi-square distribution with 1 degree of freedom. Obviously, the statistic $\sqrt{n}\frac{\left(\bar{x} - \frac{1}{\lambda_0}\right)}{1/\lambda_0}$ in (2.20) is distributed N(0,1), hence the result.

2.3 Parameterization of the LR Test Statistic

It is important to note that the log-likelihood ratio statistic Λ^*in (2.7) can be re-parameterized in terms of the effect size $\lambda_1 - \lambda_0$ and the ratio $R = \frac{\lambda_0}{\lambda_1}$ of the parameters λ_0 and $\lambda_1$of the exponential population being tested under the null and alternative hypotheses as stated in (2.5).

Let the LR test statistic Λ in (2.6) be re-expressed as

$$\Lambda = \left(\frac{\lambda_0}{\lambda_1}\right)^n \left(\frac{e^{-\lambda_0 n\bar{X}}}{e^{-\lambda_1 n\bar{X}}}\right) \quad (2.21)$$

Taken the logarithm of (2.21) to have

$$log\ (\Lambda) = n\left(log\left(\frac{\lambda_0}{\lambda_1}\right) + \bar{X}(\lambda_1 - \lambda_0)\right) \quad (2.22)$$

Since the MLE of λ_1 is $1/\bar{x}$ under H$_1$, a direct substitution of $1/\lambda_1$for $\bar{X}$ in (2.22) yields an equivalent form of the log-likelihood ratio test statistic Λ^*in (2.7) given by

$$\Lambda^* = n\left(log\left(\frac{\lambda_0}{\lambda_1}\right) + \frac{\lambda_1 - \lambda_0}{\lambda_1}\right) \quad (2.23)$$

Similarly, the LR test statistic Λ^c in (2.10)can be re-expressed in terms of λ_0 and λ_1 as in (2.23)to yield

$$\Lambda^{*c} = n\left(log\left(\frac{\lambda_1}{\lambda_0}\right) + \frac{\lambda_0 - \lambda_1}{\lambda_1}\right) \quad (2.24)$$

Thus, under the power consideration of the LR test on the parameter of an exponential population, the form of the log-likelihood ratio test statistic in (2.23) or (2.24) shows that the power of the LR test is related to both the effect size and the ratio of the parameters of the exponential population being tested as stated under the null and alternative hypotheses set (2.5). Therefore, apart from the effect size, the effects of the parameter ratios,$R = \frac{\lambda_0}{\lambda_1}$ on the power of the LR tests areequally examined in this work.

3.0 Simulation Studies

To compute the power of the LR test for the hypothesis set (2.5), considerations were given to the choice of different possible values of the parameters of a classical exponential distributionon which statistical hypothesis test may be required, the various sample sizes and the desirable number of iterations for Monte-Carlo studies.

Random samples of sizes between 1 and 250 were drawn from exponential distributions of the type given by (2.1) at different values of parameter λ. For a given sample size$n$, the LR statistic (2.8)was computed for each hypothesis set (2.4) or (2.5)at specified $\lambda_0$values under H$_0$ and $\lambda_1$valuesunder H$_1$and at r = 50, 100, 250, 500, and 1000 iterations.At each sample size n considered, the power of theLR test was computed forsomeType I error rate$\alpha \in (0,1)$.

However, for each hypothesis set of the form (2.5) tested in this context, the power of the LR test is simply determined by estimating the proportion of cases, out of the rLR tests constructed (iterations),in which the LR tests rejected the nullhypothesis (H_0: $\lambda = \lambda_0$) given that the true value of the parameter of the distribution is λ_1(i.e. H_1: $\lambda = \lambda_1$is true) at some significancelevels$\alpha \in (0,1)$.We later examined the behaviours of the power of the LR test at some reasonable values of α within the interval $(0,1)$ in this study.

Basically, two different hypotheses schemes were employedto examine the behaviour of the power of the LR test under different parameter considerations of the exponential density. The two schemes areas follows:

a.) In the first scheme, wedetermine the power of the LR test at a fixed$\lambda_0 = 0.2$under H_0but at different values 0.3, 0.4, 0.5 and 0.6 ofλ_1 under H_1. Thus, the following four hypothesis sets

i. H_0: $\lambda = 0.2$ versus H_1: $\lambda = 0.3$, effect size $\Delta = |\lambda_0 - \lambda_1| = 0.1$, ratio $R = \frac{\lambda_0}{\lambda_1} = \frac{2}{3}$

ii. H_0: $\lambda = 0.2$ versus H_1: $\lambda = 0.4$, effect size $\Delta = |\lambda_0 - \lambda_1| = 0.2$,ratio $R = \frac{\lambda_0}{\lambda_1} = \frac{1}{2}$

iii. H_0: $\lambda = 0.2$ versus H_1: $\lambda = 0.5$, effect size $\Delta = |\lambda_0 - \lambda_1| = 0.3$,ratio $R = \frac{\lambda_0}{\lambda_1} = \frac{2}{5}$

iv. H_0: $\lambda = 0.2$ versus H_1: $\lambda = 0.6$, effect size $\Delta = |\lambda_0 - \lambda_1| = 0.4$, ratio $R = \frac{\lambda_0}{\lambda_1} = \frac{1}{3}$

Were considered under different effect sizes $\Delta = |\lambda_0 - \lambda_1|$ and ratios $R = \frac{\lambda_0}{\lambda_1}$ of the two parameters λ_0 and λ_1. Here, only the values of $\lambda_1$underH_1were varied and the parameter ratios R were monotonically decreasing.

b.) In the second scheme, we consider the power of the LR tests at different $\lambda_0$values 0.1, 0.2, 0.3, 0.4under H_0 and at different $\lambda_1$values 0.3, 0.4, 0.5, 0.6, under H_1 but with equal effect sizes$\Delta = |\lambda_0 - \lambda_1| = 0.2$. Thus, the following four hypothesis sets

i. H_0: $\lambda = 0.1$ versus H_1: $\lambda = 0.3$, effect size $\Delta = |\lambda_0 - \lambda_1| = 0.2$, ratio $R = \frac{\lambda_0}{\lambda_1} = \frac{1}{3}$

ii. H_0: $\lambda = 0.2$ versus H_1: $\lambda = 0.4$, effect size $\Delta = |\lambda_0 - \lambda_1| = 0.2$, ratio $R = \frac{\lambda_0}{\lambda_1} = \frac{1}{2}$

iii. H_0: $\lambda = 0.3$ versus H_1: $\lambda = 0.5$, effect size $\Delta = |\lambda_0 - \lambda_1| = 0.2$, ratio $R = \frac{\lambda_0}{\lambda_1} = \frac{3}{5}$

iv. H_0: $\lambda = 0.4$ versus H_1: $\lambda = 0.6$, effect size $\Delta = |\lambda_0 - \lambda_1| = 0.2$, ratio $R = \frac{\lambda_0}{\lambda_1} = \frac{2}{3}$

were considered. Here, the values of both$\lambda_0$andλ_1 were increased simultaneously in the four hypotheses casesbut with equal effect size of 0.2.

All the Monte-Carlo experimentswere performed using R statistical package (www.cran.org)[24].

4.0 Results

This section presents the results from our Monte Carlo experiments as well as the results ofsome empirical applications to validate the Monte Carlo results.

4.1 Monte Carlo Results

This section presents the results obtained from the Monte-Carlo studies to examine the behaviour of the power of LR test for exponential populations under different chosen levels of iterations, parameterizations and sample sizes.

Table 3.1 presents the estimated powers of the LR tests under the five chosen iteration levels for the first hypothesis set H_0: $\lambda = 0.2$ vs. H_1: $\lambda = 0.3$ under the first Monte Carloscheme a.) in section 3. The effect size Δ of the test is 0.1 with parameter ratio $R = \frac{2}{3}$. This is intended to evaluate the quality of the Monte Carlo experiment as engaged here.

Beginning from 50 to 1000 iterations, the estimated powers of the LR tests constructed at various sample sizes are very stable and consistent. The standard deviations of the estimated powers across the samples are almost zero while those across the levels of iteration are all below 1. This is an indication that the simulation experiment and its results are very stable.

Consequently, the results in Table 3.1 indicated that the various LR tests across the five chosen iteration levels attained about the same level of power at each sample size irrespective of the number of iterations employed. For instance, the LR tests attained about 95% power at around 80 sample size across all the levels of iterations as indicated in Table 3.1. Although, results of the LR tests at higher levels of iterations appeared more stable than those obtained at lower levels of iteration going by the estimated standard deviations of the powers of the tests across all the chosen sample sizes. However, these apparent differences in the estimated standard deviations are not actually significant (p = 0.215). These results are clearly shown in Fig 3.1 by the plots of the powers of the LR tests against the various sample sizes between 1 and 250 and at 50, 100, 250 and 1000 iterations.

Table 3.1: Table of the estimated powers of the likelihood ratio (LR) tests on the parameters of the exponential distribution for the hypothesis set H$_0$:$f(x|\lambda = 0.2)$ vs. H$_1$: $f(x|\lambda = 0.3)$, $x{\sim}exp\ (\lambda)$, (parameter ratio = 2/3) at various sample sizes over 50, 100, 250, 500 and 1000 fitted models (iterations). (*) indicates the sample size at which the LR test yielded about 95% power under each iteration.

Sample size (*n*)	Power of The LR Test H$_0$:$f(x|\lambda = 0.2)$ vs. H$_1$: $f(x|\lambda = 0.3)$					**Standard Deviation**
	Number of Iterations (r)					
	50	**100**	**250**	**500**	**1000**	
1	0.040	0.040	0.052	0.078	0.120	0.034
6	0.180	0.120	0.124	0.164	0.140	0.026
11	0.280	0.240	0.220	0.266	0.190	0.034
16	0.320	0.260	0.348	0.324	0.340	0.035
21	0.320	0.460	0.392	0.446	0.400	0.055
26	0.560	0.550	0.556	0.510	0.510	0.025
31	0.620	0.600	0.596	0.578	0.560	0.023
36	0.620	0.670	0.648	0.654	0.610	0.025
41	0.700	0.780	0.696	0.724	0.670	0.042
46	0.780	0.820	0.760	0.774	0.770	0.023
51	0.860	0.810	0.828	0.830	0.800	0.023
56	0.880	0.820	0.844	0.852	0.840	0.022
61	0.920	0.850	0.876	0.888	0.860	0.027
66	0.940	0.880	0.896	0.912	0.930	0.024
71	0.900	0.880	0.904	0.944	0.930	0.025
76	0.940	0.910	0.944	0.949	0.941	0.015
81	***0.960**	***0.945**	***0.948**	***0.956**	***0.955**	0.006
86	1.000	0.950	0.968	0.968	0.980	0.018
91	0.960	1.000	0.968	0.972	0.980	0.015
96	1.000	0.970	0.988	0.978	0.980	0.011
101	0.960	1.000	0.988	0.988	0.990	0.015
106	0.980	0.980	0.988	0.988	0.990	0.005
111	0.980	0.990	1.000	0.972	0.980	0.011
116	1.000	1.000	0.996	0.994	0.980	0.008
121	0.980	0.990	0.996	0.996	0.990	0.007
126	1.000	1.000	0.992	0.990	0.980	0.008
131	1.000	1.000	0.992	0.998	0.980	0.008
136	1.000	1.000	1.000	0.998	1.000	0.001
141	1.000	1.000	1.000	0.998	1.000	0.001
146	1.000	1.000	1.000	1.000	1.000	0.000
151	1.000	1.000	0.996	1.000	1.000	0.002
156	1.000	1.000	1.000	1.000	1.000	0.000
161	0.980	1.000	1.000	0.998	1.000	0.009
>160	1.000	1.000	1.000	1.000	1.000	0.000
StandardDev.	**0.244**	**0.246**	**0.246**	**0.239**	**0.241**	

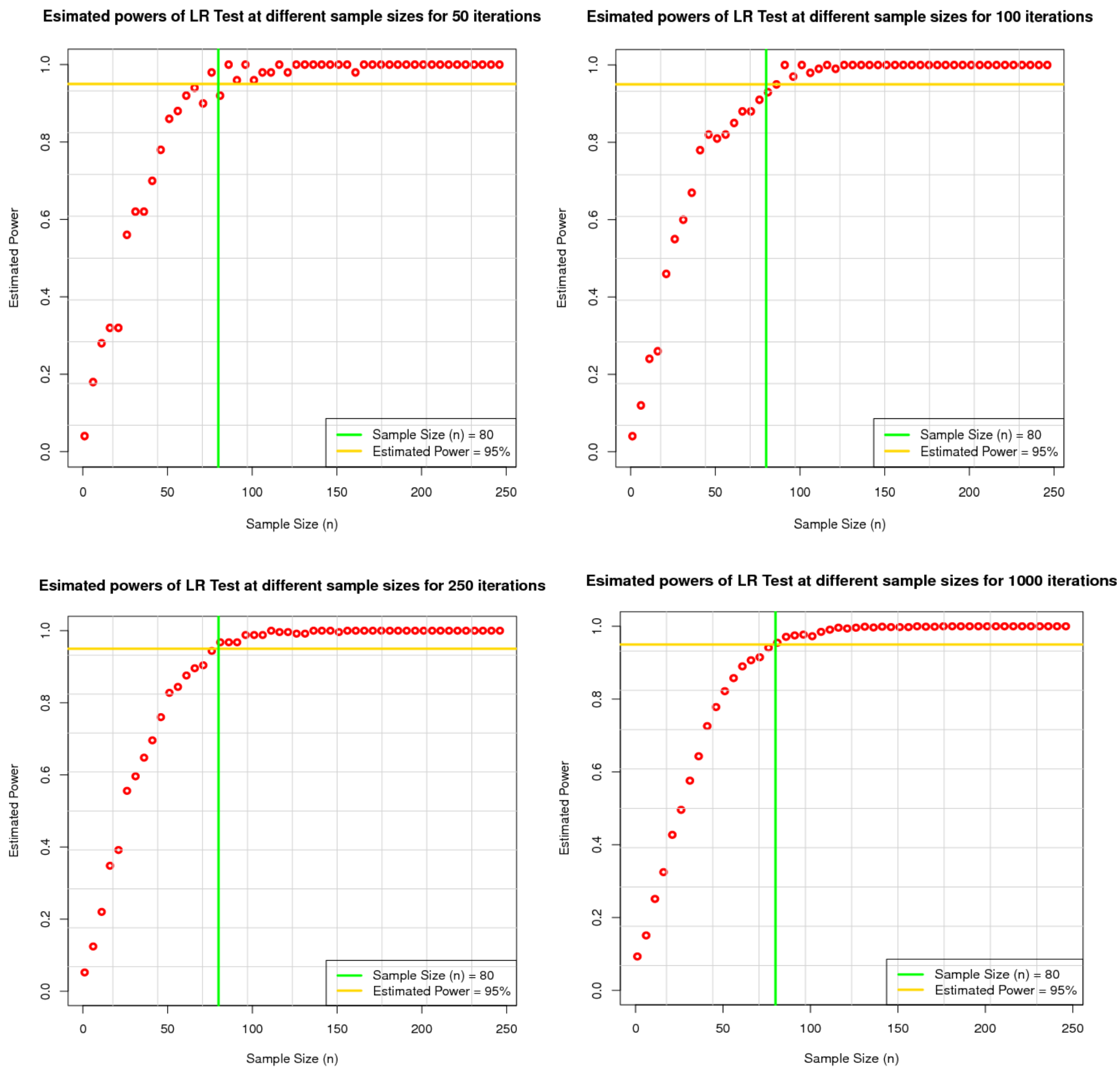


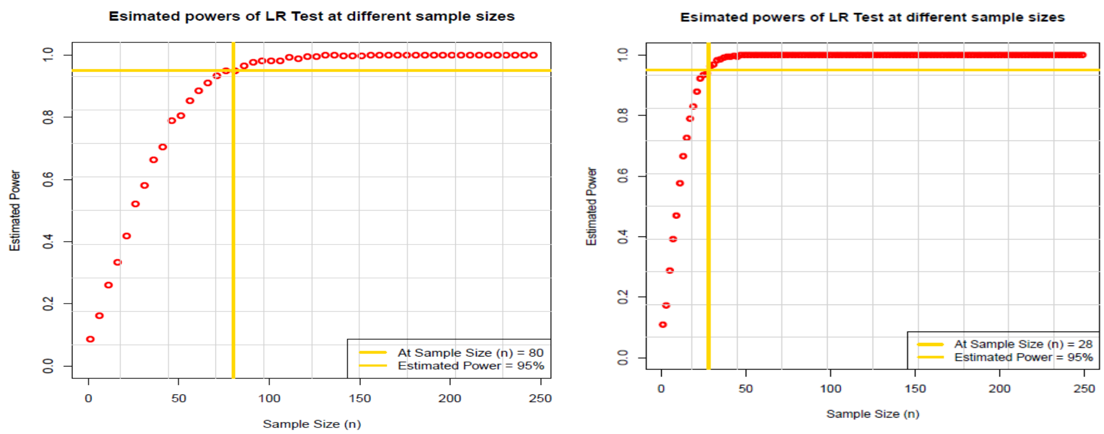
Fig 3.1: The graphs of the estimated powers of the likelihood ratio tests for the hypothesis set H_0:$f(x|\lambda = 0.2)$ vs. H_1: $f(x|\lambda = 0.3)$, $x\sim exp\ (\lambda)$,at various sample sizes and at 50, 100, 250and 1000 iterations. The vertical line in all the graphs indicates the sample size (n = 80) at which the LR test achieves the same 95% power across the four levels of iterations.

In the four graphs in Fig 3.1, the LR tests attained an appreciable power of 95% at 80 samples across all the chosen levels of iterations. Although, the power curves of the test at higher iteration levels (250 and 1000) are relatively smoother than those at the lower iteration levels (50 and 100) which simply justify the differences noticed in the estimated standard deviations of powers of the test in Table 3.1 as earlier remarked. The smoothness of these power curves notwithstanding, the same information is provided by the four curves as shown in Fig 3.1. Therefore, one can conclude that the number of iterations adopted for the construction of the LR test has no significant impact on the power of the test generally. Nonetheless, further results on the LR tests based on the hypothesis set (2.5) in this work shall be reported only for 1000 iterations.

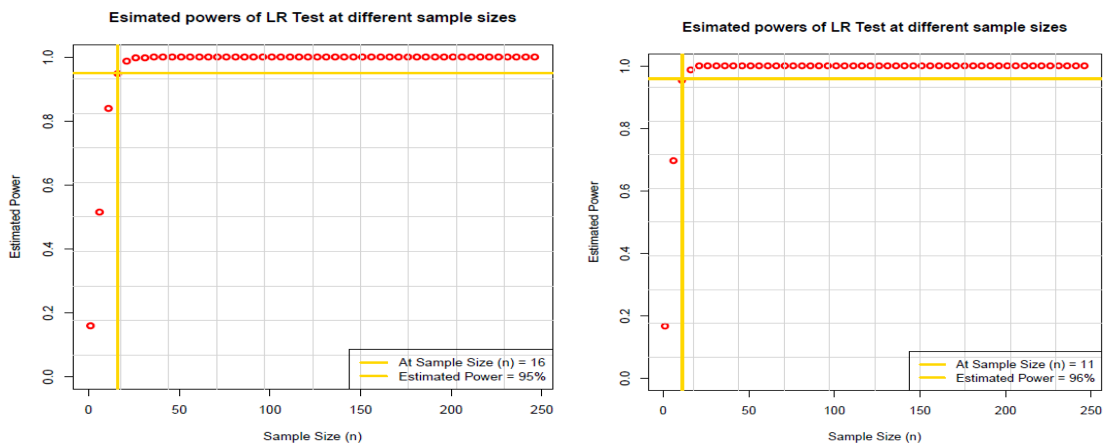
Table 3.2: Table of the estimated powers of the likelihood ratio (LR) tests on the parameters of an exponential distribution at different sample sizes for **LR Test 1** (H_0:$f(x|\lambda = 0.2)$ vs. H_1: $f(x|\lambda = 0.3)$, effect size$\Delta = 0.1$, parameter ratio R = 2/3), **LR Test 2** (H_0:$f(x|\lambda = 0.2)$ vs. H_1: $f(x|\lambda = 0.4)$, effect size$\Delta = 0.2$, parameter ratio R =1/2), **LR Test 3** (H_0:$f(x|\lambda = 0.2)$ vs. H_1: $f(x|\lambda = 0.5)$, effect size$\Delta = 0.3$, parameter ratio R = 2/5) and **LR Test 4** (H_0:$f(x|\lambda = 0.2)$ vs. H_1: $f(x|\lambda = 0.6)$, effect size$\Delta = 0.4$, parameter ratio R = 1/3) over 1000 fitted models (iterations), $x \sim exp\ (\lambda)$. (*) indicates the sample size at which the LR test yielded about 95% power under each hypothesis set.

Sample size (n)	**LR Test 1**	**LR Test 2**	**LR Test 3**	**LR Test 4**
	$\Delta = 0.1$, R = 2/3	$\Delta = 0.2$, R = 1/2	$\Delta = 0.3$, R = 2/5	$\Delta = 0.4$, R = 1/3
1	0.086	0.122	0.160	0.168
6	0.162	0.338	0.516	0.698
11	0.260	0.600	0.840	***0.955**
16	0.333	0.774	***0.950**	0.989
21	0.418	0.881	0.987	1.000
26	0.521	0.941	0.999	1.000
28	0.536	***0.950**	0.999	1.000
31	0.580	0.981	0.999	1.000
36	0.662	0.990	1.000	1.000
41	0.704	0.996	1.000	1.000
46	0.789	0.996	1.000	1.000
51	0.804	0.999	1.000	1.000
56	0.852	1.000	1.000	1.000
61	0.886	1.000	1.000	1.000
66	0.910	1.000	1.000	1.000
71	0.934	1.000	1.000	1.000
76	0.948	1.000	1.000	1.000
81	***0.949**	1.000	1.000	1.000
86	0.965	1.000	1.000	1.000
91	0.976	1.000	1.000	1.000
96	0.981	1.000	1.000	1.000
101	0.981	1.000	1.000	1.000
106	0.980	1.000	1.000	1.000
111	0.993	1.000	1.000	1.000
116	0.988	1.000	1.000	1.000
121	0.995	1.000	1.000	1.000
126	0.995	1.000	1.000	1.000
131	1.000	1.000	1.000	1.000
136	0.999	1.000	1.000	1.000
141	0.996	1.000	1.000	1.000
146	0.998	1.000	1.000	1.000
151	0.998	1.000	1.000	1.000
156	0.999	1.000	1.000	1.000
160	1.000	1.000	1.000	1.000
>160	1.000	1.000	1.000	1.000

Table 3.2 presents the results of the likelihood ratio tests for testing the hypotheses i. to iv. under the first simulation scheme as highlighted in Section 3. The table presented thepowersof the LR tests for each of the hypotheses at various sample sizes over 1000iterations. The number of cases in which the tests rejected the false null hypothesis H_0 out of 1000 LR tests constructed are equally determined (results not shown). Thus, at each sample size, the power of the LR test for each hypothesis set was determined by the proportion of cases, out of 1000 LR testsconstructed, in which the testscorrectly rejected the null hypothesis H_0.



H0: $\lambda = 0.2$ vs. H1: $\lambda = 0.3$, $\Delta = 0.1$, R=2/3H0: $\lambda = 0.2$ vs. H1: $\lambda = 0.4$, $\Delta = 0.2$, R=1/2



H0: $\lambda = 0.2$ vs. H1: $\lambda = 0.5$, $\Delta = 0.3$, R =2/5H0: $\lambda = 0.2$ vs. H1: $\lambda = 0.6$, $\Delta = 0.4$, R=1/3

Fig 3.2: The graphs of the estimated power of the likelihood ratio tests at various sample sizes underdifferent effect sizes $\Delta = |\lambda_0 - \lambda_1|$ and parameter ratios $R = \frac{\lambda_0}{\lambda_1}$over 1000 iterations (tests). The sample size required (indicated by vertical line in each graph) by the LR test to achieve 95% power decreases i.) as the value of the parameter ratio Rdecreases and ii.) as the effect size Δincreases.

It can be generally observed from the results in Table 3.2 that the power of the LR test increases as the sample size increases. However, the effect size $\Delta = |\lambda_0 - \lambda_1|$ also plays prominent roles on the behaviour of the power of the LR tests. It can be observed from Table 3.2 that very few samples are required to attain a reasonable power at large effect sizes while relatively large samples are needed to attain a similar level of power at smaller effect sizes. For instance, from Table 3.2, while about 80 samples are required by the LR test to attain 95% power when the effect size Δ of the test is 0.1 (LR Test 1), only 28, 16 and 11 samples are required to attain the same fit of power when the effect size of the test increases to 0.2, 0.3 and 0.4 respectively.

In terms of the ratio of the two parameters λ_0 under H0 and $\lambda_1$under H1 being tested, that is $R = \frac{\lambda_0}{\lambda_1}$ with $\lambda_0 < \lambda_1$, the above results equally showed that more samples would be needed to attain a reasonable power at higher value of parameter ratio R (i.e. as $R \to 1$) while relatively fewer samples would be required to attain a similar fit as the value of R gets smaller (i.e. as $R \to 0$) as evident from the results in Table 3.2.

The results above are clearly presented in Fig 3.2 by the plots of the estimated power of the LR test at different sample sizes for the four test hypotheses i. to iv. under the first Monte Carlo scheme a.). The behaviours of the powers of the LR tests at different sample sizes, effect sizes and parameter ratios are clearly shown on the four graphs over 1000 iterations.

It is evident from the graphs in Fig 3.2 that, with $\lambda_0 < \lambda_1$, the sample size required by the LR test to attain an appreciable power of about 95% reduces from 80 to 11 as the parameter ratio reduces from 2/3 to 1/3 or as the effect size of the test increases from 0.1 to 0.4. Another important consideration that is crucial to this work is on the effects of the magnitude of both λ_0(under H0) and λ_1(under H1) on the behaviour of power of the LR tests as captured by the second simulation scheme b.).

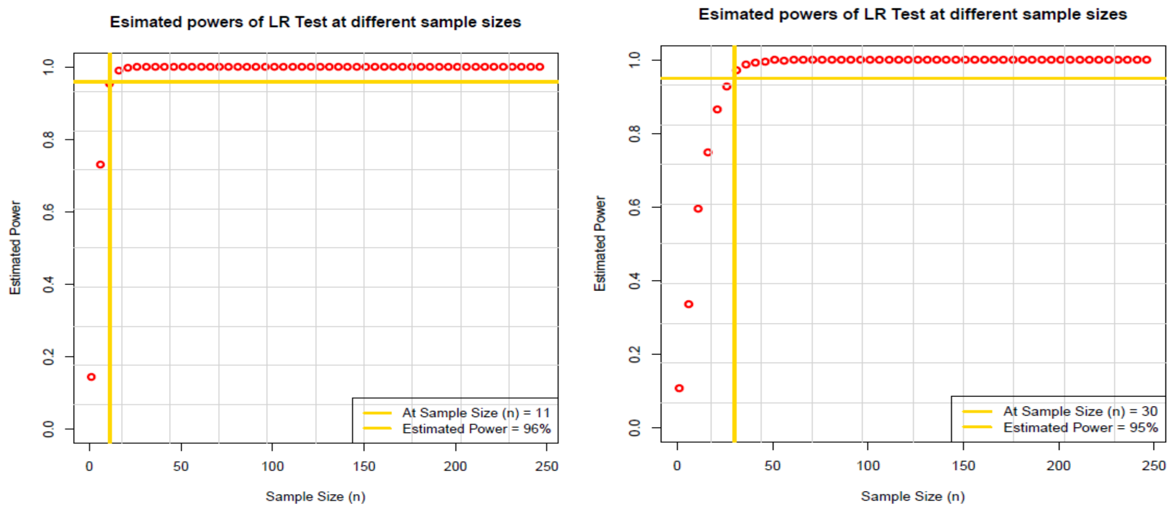
Interestingly, the results in Table 3.3 showed that the size of the parameters λ_0 and λ_1 of the exponential distributions being tested play significant role on the power of the test. For instance, the LR test with small values of exponential parameters λ_0

and λ_1 attains reasonable power faster (with fewer samples) than the oneswith relatively large values of both λ_0 and $\lambda_1$irrespective of the effect sizesas shown in Table 3.3.In other words, fewer samples are needed by the LR test to achieve reasonable power in anhypothesis test that involves small values of λ_0 and $\lambda_1$relative to the ones that involve relatively large values of both λ_0 and $\lambda_1$even under equal effect sizes. These results are clearly presented in Fig 3.3 by plotting the estimated powers of the four LR tests all with equal effect size of 0.2 against the various sample sizes. The performance of the LR tests at different sizes of λ_0 and $\lambda_1$showed clearly that more samples would be required by the test to attain a reasonable power as the sizes of both λ_0 and λ_1 become larger, even with equal effect size.

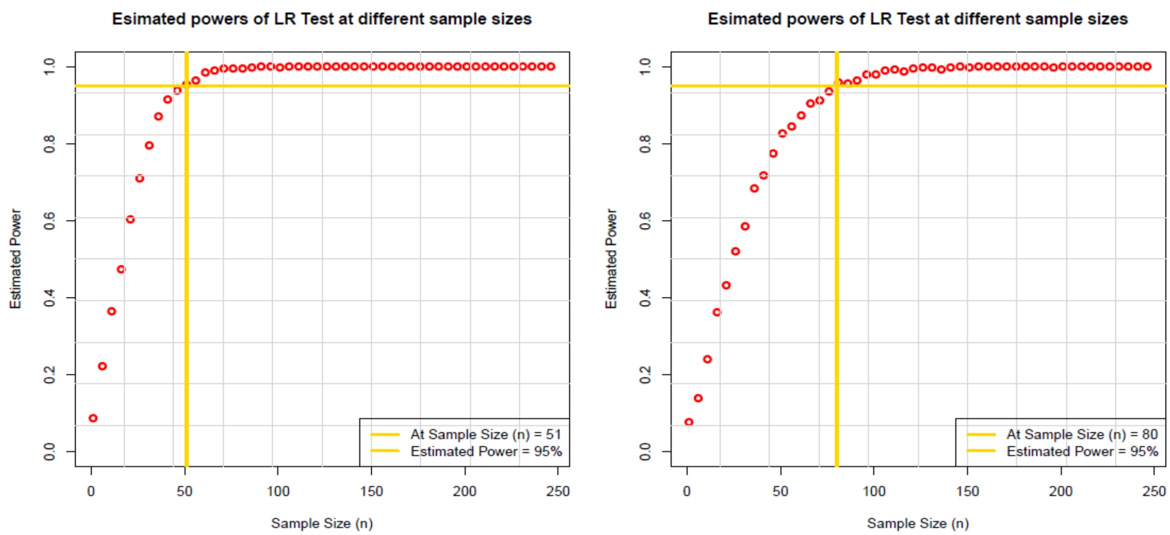
More specifically, the results from Table 3.3 and Fig 3.3showed that only about 10 samples are needed by the LR test to attain 95% power for the hypothesis test when $\lambda_0 = 0.1$ and $\lambda_1 = 0.3$(LR Test 1) with the effect size 0.2whereas, up to 30, 50 and 80 samples arerequired by the LR tests toattainthe samefit of 95% powerwhen $\lambda_0 = 0.2$ and $\lambda_1 = 0.4$(LR Test 2), $\lambda_0 = 0.3$ and $\lambda_1 = 0.5$(LR Test 3) and $\lambda_0 = 0.4$ and $\lambda_1 = 0.6$(LR Test 4) respectively, with all of them having equal effect size $\Delta = 0.2$.

Table 3.3: Table of the estimated powers of the likelihood ratio (LR) tests on the parameters of an exponential distribution at different sample sizes for **LR Test 1**(H$_0$:$f(x|\lambda = 0.1)$ vs. H$_1$: $f(x|\lambda = 0.3)$, effect size$\Delta = 0.2$, parameter ratio R = 1/3), **LR Test 2** (H$_0$:$f(x|\lambda = 0.2)$ vs. H$_1$: $f(x|\lambda = 0.4)$, effect size$\Delta = 0.2$, parameter ratio R = 1/2), **LR Test 3**(H$_0$:$f(x|\lambda = 0.3)$ vs. H$_1$: $f(x|\lambda = 0.5)$, effect size$\Delta = 0.2$, parameter ratio R = 3/5) and **LR Test 4**(H$_0$:$f(x|\lambda = 0.4)$ vs. H$_1$: $f(x|\lambda = 0.6)$, effect size$\Delta = 0.2$, parameter ratio R = 2/3)over 1000 fitted models (iterations). (*) indicates the sample size at which the LR test yielded 95% power under each hypothesis set.

Sample size (n)	**LR Test 1** $\lambda_0 = 0.1, \lambda_1 = 0.3$	**LR Test2** $\lambda_0 = 0.2, \lambda_1 = 0.4$	**LR Test3** $\lambda_0 = 0.3, \lambda_1 = 0.5$	**LR Test4** $\lambda_0 = 0.4, \lambda_1 = 0.6$
	$\Delta = 0.2$, R = 1/3	$\Delta = 0.2$, R = 1/2	$\Delta = 0.2$, R = 3/5	$\Delta = 0.2$, R = 2/3
1	0.145	0.108	0.087	0.076
6	0.731	0.336	0.222	0.138
11	***0.955**	0.597	0.364	0.241
16	0.991	0.750	0.474	0.362
21	0.999	0.866	0.603	0.431
26	1.000	0.929	0.709	0.521
30	1.000	***0.950**	0.768	0.557
31	1.000	0.972	0.795	0.586
36	1.000	0.989	0.872	0.685
41	1.000	0.993	0.914	0.718
46	1.000	0.997	0.939	0.776
51	1.000	1.000	***0.954**	0.828
56	1.000	0.999	0.964	0.844
61	1.000	1.000	0.985	0.873
66	1.000	1.000	0.990	0.904
71	1.000	1.000	0.997	0.912
76	1.000	1.000	0.997	0.936
80	1.000	1.000	0.997	***0.950**
81	1.000	1.000	0.997	0.959
86	1.000	1.000	0.999	0.958
91	1.000	1.000	1.000	0.964
96	1.000	1.000	1.000	0.98
101	1.000	1.000	0.998	0.98
106	1.000	1.000	1.000	0.991
111	1.000	1.000	1.000	0.993
116	1.000	1.000	1.000	0.989
121	1.000	1.000	1.000	0.996
126	1.000	1.000	1.000	0.998
131	1.000	1.000	1.000	0.999
136	1.000	1.000	1.000	0.994
141	1.000	1.000	1.000	0.999
146	1.000	1.000	1.000	1.000
150	1.000	1.000	1.000	1.000
>150	1.000	1.000	1.000	1.000



H0: $\lambda = 0.1$ vs. H1: $\lambda = 0.3$, $\Delta = 0.2$,R = 1/3H0: $\lambda = 0.2$ vs. H1: $\lambda = 0.4$, $\Delta = 0.2$, R = 1/2



H0: $\lambda = 0.3$ vs. H1: $\lambda = 0.5$, $\Delta = 0.2$, R = 3/5H0: $\lambda = 0.4$ vs. H1: $\lambda = 0.6$, $\Delta = 0.2$, R = 2/3

Fig 3.3: The graphs of estimated power of the likelihood ratio tests at different sample sizes for four hypothesesset with equal effect size $\Delta = |\lambda_0 - \lambda_1| = 0.2$but at varying parameter ratios $R = \frac{\lambda_0}{\lambda_1}$ over 1000 iterations (LR tests). The vertical line in each graph indicates the sample size at which the LR test achieves about 95% power.The sample size required (indicated by vertical golden line in each graph) by the LR test to achieve 95% power increases as the value of the parameter ratio Rdecreases even with equal effect size of 0.2 across the various tests.

In term of the parameter ratios $R = \frac{\lambda_0}{\lambda_1}$ with $\lambda_0 < \lambda_1$, the results above showed that more samples would be needed by the LR tests to attain a reasonable power as the parameter ratio R increases ($R \to 1$) while relatively fewer samples would be needed to attain a similar fit as R gets smaller ($R \to 0$), thus, confirming the earlier results in Table 3.2.

Table 3.4: The powers of the likelihood ratio (LR) tests on the parameters of the exponential distribution for the hypothesis set H$_0$:$f(x|\lambda = 0.2)$ vs. H$_1$: $f(x|\lambda = 0.3)$, $x \sim exp\ (\lambda)$,at selected Type I error rates in the interval $\alpha \in (0, 0.2]$ and at various sample sizes over1000 iterations. (*) indicates the sample size at which the LR test yielded about 95% power.

Sample size (*n*)	**Type I Error rate (Significance Level) α**						
	0.01	**0.03**	**0.05**	**0.07**	**0.10**	**0.15**	**0.2**
1	0.020	0.054	0.081	0.105	0.137	0.191	0.245
6	0.043	0.106	0.148	0.199	0.249	0.335	0.403
11	0.093	0.184	0.251	0.302	0.367	0.456	0.532
16	0.136	0.266	0.348	0.432	0.507	0.605	0.67
21	0.198	0.326	0.404	0.469	0.550	0.638	0.699
26	0.280	0.428	0.510	0.576	0.649	0.737	0.792
31	0.322	0.498	0.589	0.644	0.713	0.801	0.855
36	0.385	0.594	0.675	0.735	0.798	0.852	0.888
41	0.450	0.617	0.713	0.761	0.821	0.880	0.912
46	0.521	0.71	0.778	0.821	0.870	0.924	0.949
51	0.585	0.768	0.825	0.867	0.905	0.934	***0.952**
56	0.643	0.798	0.863	0.895	0.923	***0.955**	0.969
61	0.667	0.817	0.899	0.908	0.936	0.954	0.965
66	0.725	0.853	0.904	0.927	***0.951**	0.972	0.980
71	0.769	0.889	0.932	***0.950**	0.972	0.982	0.989
76	0.799	0.903	0.938	0.953	0.972	0.983	0.989
81	0.826	0.923	***0.954**	0.962	0.974	0.992	0.995
86	0.864	0.934	0.956	0.967	0.979	0.989	0.990
91	0.898	***0.964**	0.984	0.990	0.992	0.995	0.997
96	0.923	0.969	0.980	0.986	0.992	0.997	0.998
101	0.923	0.972	0.986	0.990	0.994	0.995	0.996
106	***0.950**	0.986	0.991	0.996	0.997	0.999	1.000
111	0.943	0.981	0.989	0.993	0.993	0.993	0.998
116	0.973	0.989	0.993	0.995	0.996	0.998	0.998
121	0.968	0.988	0.996	0.996	0.997	0.999	1.000
126	0.980	0.993	0.996	0.996	0.998	0.999	0.999
131	0.980	0.998	0.999	1.000	1.000	1.000	1.000
136	0.977	0.992	0.994	0.996	0.999	1.000	1.000
141	0.985	0.998	1.000	1.000	1.000	1.000	1.000
146	0.990	0.996	0.998	0.999	1.000	1.000	1.000
151	0.994	0.999	0.999	1.000	1.000	1.000	1.000
156	0.990	0.996	1.000	1.000	1.000	1.000	1.000
161	0.992	0.998	0.999	0.999	1.000	1.000	1.000
166	0.995	0.999	0.999	1.000	1.000	1.000	1.000
171	0.998	0.999	0.999	1.000	1.000	1.000	1.000
176	0.996	0.999	0.999	0.999	0.999	1.000	1.000
181	0.995	0.998	0.999	1.000	1.000	1.000	1.000
>181	1.000	1.000	1.000	1.000	1.000	1.000	1.000

The power analyses of the LR tests presented so far are based on 5% level of significance. In order to have a broader view of the behaviour of the power of the LR test under various values of αwithin the interval (0,1), we present in Table 3.4 the estimated powers of the LR tests for the hypothesis set H$_0$:$f(x|\lambda = 0.2)$ vs. H$_1$: $f(x|\lambda = 0.3)$at various values of $\alpha \in (0,1)$ over 1000 iterations. However, only the results for selected values of α within the interval (0, 0.2] are presented in Table 3.4 due to space.

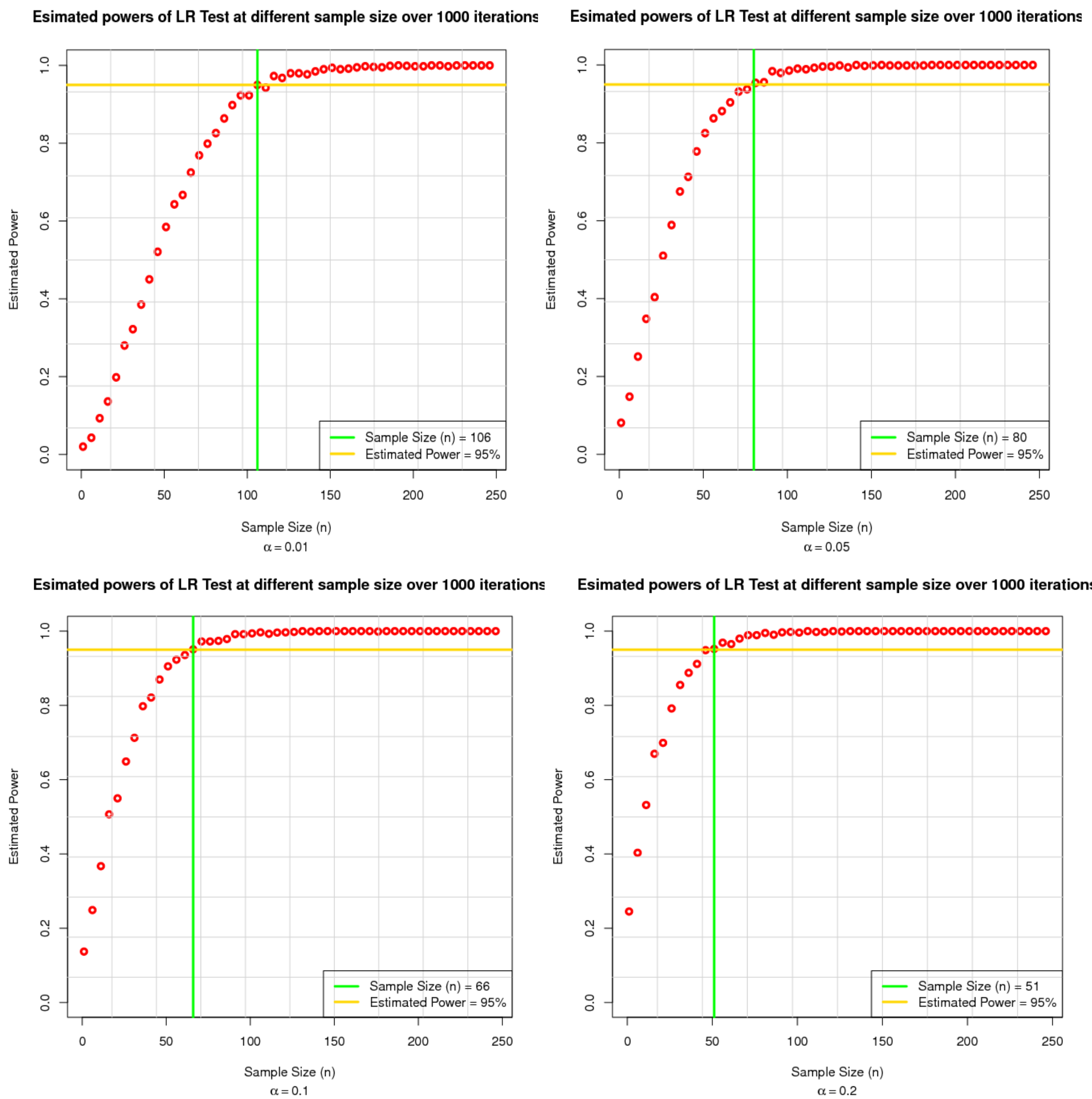


Fig 3.4: The plots of the estimated powers of the likelihood ratio tests at various sample sizes and at some selected Type I error rates α within the interval (0,1) for testing the hypothesis set H$_0$:$f(x|\lambda = 0.2)$ vs. H$_1$: $f(x|\lambda = 0.3)$over 1000 iterations where $x \sim exp\ (\lambda)$. The vertical line in each graph indicates the sample size at which the LR test achieves 95% power. The four graphs showed that fewer samples are required by the LR test to achieve 95% power as the chosen significance level αincreases (from 0.01 to 0.2).

It can be easily observed from Table 3.4 that the sample size required by the LR test to achieve a particular level of power differs depending on the size α of the test. For instance, to attain about 95% power by the LR test for the hypothesis set H$_0$:$f(x|\lambda = 0.2)$ vs. H$_1$: $f(x|\lambda = 0.3)$, more samples are employed by the test at small values of α (i.e. as $\alpha \rightarrow 0$) than at large values of α (i.e. as $\alpha \rightarrow 1$). This is so because, the strength of evidence required for the rejection of H$_0$ given that the H$_1$ is correct is relatively less at small values of α than at its higher values. Therefore, before a small shift in the hypothesized parameter values could be detected by the LR test, more samples would be required at lower values of α than at higher values of α. The various sample sizes at which the LR tests attained 95% power at some selected values of α within the interval (0, 0.2] are clearly presented in Fig 3.4 in which it can be observed that the sample size required to attain 95% power reduces as the value of size α of the LR test increases.

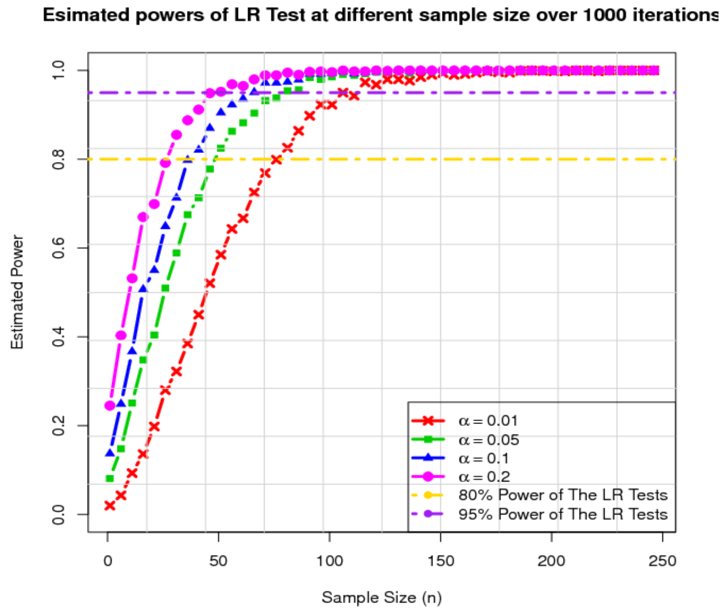


Fig 3.5: The plots of the estimated powers of the likelihood ratio tests at various sample sizes for some selected Type I error rates α= 0.01, 0.05, 0.1, 0.2 for testing the hypothesis set H_0:$f(x|\lambda = 0.2)$ vs. H_1: $f(x|\lambda = 0.3)$over 1000 iterations where $x \sim exp\ (\lambda)$. The various graphs showed that the morethe size $\alpha$of the LR test increases, the fewer the samples required by the test to achieve appreciable power.

Also, in order to have a quick overview of the influence of different values of α on the power and sample size of the LR test, we present in Fig 3.5 the plots of the power of the LR tests at four selected values of α (0.01, 0.05, 0.1 and 0.2) against the sample sizes between 1 and 250. The yellow and purple dashed horizontal lines passed through 80% and 95% power of each of the LR tests at the four chosen α values. It is again clear from the power graphs in Fig 3.5 that at higher values of α ($\alpha = 0.1, 0.2$), the LR test requires fewer samples to attain reasonable power (80% or 95%) as indicated by dashed horizontal lines while at lower values of α ($\alpha = 0.01, 0.05$), more samples are needed by the test to attain the same fit of power. All these results revealed the impact of Type I error rates on the power and sample size requirements by the LR test on the parameter of an exponential distribution.

Consequently, it can be established based on the results in Table 3.4 that atany given sample size *n*, the power of the LR test increases as the value of size α of the test increases. For instance, at $n \approx$ 100, the estimated power of the LR test is about 92% at smaller value of α ($\alpha = 0.01$) while at this same sample size, the power of the test increases to about 99% at higher values of α ($\alpha \geq 0.05$) as shown in Table 3.4.

Finally, we present in Table 3.5, the estimated powers of the LR tests for different (increasing) values of parameter$\lambda_1$for some selected sample sizes. This is intended to show how sensitive the power of the LR test is to changes in the values ofparameter$\lambda_1$under the alternative hypothesis H_1 for a fixed value of parameterλ (λ_0)under the null hypothesis H_0. It can be observed from the table thatat each selected sample size, the power of the LR test increases as the value of parameter $\lambda_1$increases (from 0.3 to 0.6) with λ_0 fixed at 0.2.This` increment approaches 1 faster with further little increment in the value of λ_1.

Table 3.5: Table of the powers of the LR tests at increasing values of parameter λ_1 under the alternative hypothesis H_1 for a fixed value of parameter λ_0 under the null hypothesis H_0 for different sample sizes (*n*) ranging from 10 to 100.

LR Test	$\lambda = \lambda_0$ (under H_0)	$\lambda = \lambda_1$ (under H_1)	$R = \frac{\lambda_0}{\lambda_1}$	**Sample Sizes**					
				10	**20**	**30**	**50**	**80**	**100**
				Power					
1	0.2	0.3	0.6667	0.2590	0.4170	0.5789	0.8038	0.9490	0.9810
2	0.2	0.4	0.5000	0.5980	0.8804	0.9810	0.9989	1.0000	1.0000
3	0.2	0.5	0.4000	0.8390	0.9868	0.9989	1.0000	1.0000	1.0000
4	0.2	0.6	0.3333	0.9540	0.9990	1.0000	1.0000	1.0000	1.0000

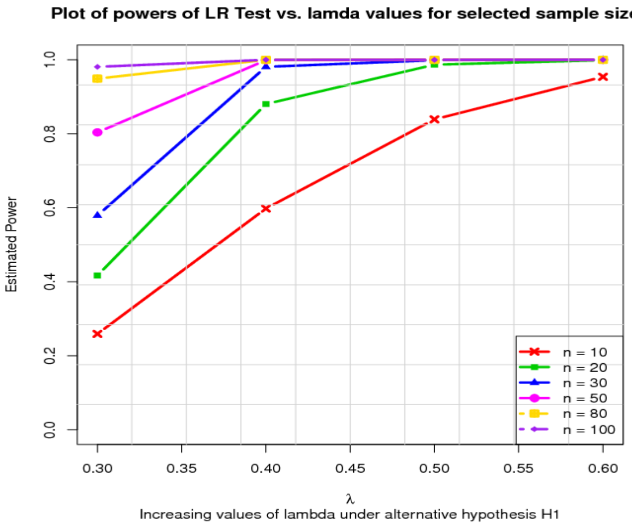


Fig 3.6: Plots of the powers of different LR testsat increasing values of parameter λ_1 under the alternative hypothesis H_1 for a fixed value of parameter λ_0 under the null hypothesis H_0 for different sample sizes (n) ranging from 10 to 100.

The sensitivity of the power of the LR test to changes in the size of parameter λ under H_1 for exponential population at different sample sizes is clearly presented in Fig 3.6 based on the results in Table 3.5. From the various graphs, it can be observed that the power of the LR test increases with increase in the size of parameter λ under H_1 at a fixed value of λ under H_0 for all the selected sample sizes. Nonetheless, the power of each LR test approaches 1 as the sample sizes keep increasing.

4.2 Empirical Results

The section presents a few empirical results on the estimation of the asymptotic power of the LR test to validate the results from the Monte Carlo studies.

Two cases are considered here. In the first case, we want to determine the power of the LR test for testing the hypothesis set

H_0: $f(x|\lambda = 0.2)$ vs. H_1: $f(x|\lambda = 0.3)$ (4.1)

with a sample size $n = 61$ and Type I error rate $\alpha = 0.01$, where $x \sim exp\ (\lambda)$ as earlier defined.

In the second case, the power of the LR test for the same hypothesis set (4.1) is desired using the same sample size $n =$ 61but at an increased Type I error rate $\alpha = 0.05$.

The test statistic of the LR test as given in (2.20) is

$$\psi = \left(\sqrt{n}\frac{\left(\bar{x} - \frac{1}{\lambda_0}\right)}{1/_{\lambda_0}}\right)^2 \sim \chi_1^2$$

This implies that statistic $Z = \sqrt{\psi} \sim N(0,1)$. Therefore, the decision rule for the hypothesis set (4.1), according to (2.9), is to reject the null hypothesis H_0 if

$$Z = \sqrt{n}\frac{\left(\bar{x} - \frac{1}{\lambda_0}\right)}{1/_{\lambda_0}} < -Z_{1-\alpha} \qquad (4.2)$$

where $\lambda_0 = 0.2$ where $Z_{1-\alpha}$ is the quantile of the standard normal distribution at Type I error rate α.

Power computations:

Case 1: $n = 61, \ \alpha = 0.01$.

The decision rule (4.2) simply implies that the null hypothesis H_0 is rejected if $\bar{x} < \frac{1}{0.2} - \frac{Z_{0.99}}{0.2\sqrt{61}}$. That is, the LR test rejects H_0 when $\bar{x} < 3.5107$. The power of this test is therefore computed, using (2.12),as

$P(\bar{x} < 3.5107|\lambda = 0.3)$ (4.3)

i.e. $$P\left(\sqrt{61}\frac{\left(\bar{x} - \frac{1}{0.3}\right)}{1/_{0.3}} < \sqrt{61}\frac{\left(3.5107 - \frac{1}{0.3}\right)}{1/_{0.3}}\right)$$

$$\rightarrow P(Z < 0.4156) = 0.6611 \qquad (4.4)$$

where $Z = \sqrt{61}\frac{\left(\bar{x} - \frac{1}{0.3}\right)}{1/_{0.3}}$ in (4.4). Hence, the power of the LR test for the hypothesis set (4.1) using 61 samples at Type I error rate of 1% is about 66%. This is a good agreement with66.7%power of the LR testfor this same hypothesis set which was obtained from our Monte Carlo experiment as presented in Table 3.4 with$n = 61$ and $\alpha = 0.01$.

Case2:$n = 61, \ \alpha = 0.05$.

Under this second case, the value of $\alpha$was increased from 0.01 to 0.05. Therefore, the value of$Z_{1-\alpha}$at $\alpha = 0.05$is 1.6448 and by decision rule in (4.2), it implies that H$_0$ is rejected only if $\bar{x} < 3.9470$.
Here again, the power of the LR test for the hypothesis set (4.1) is computed as

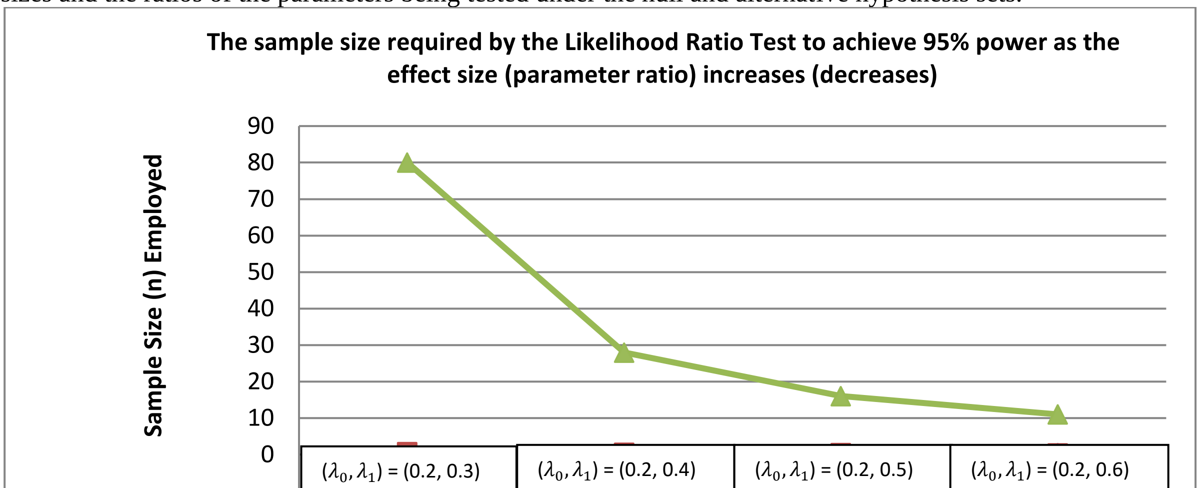
$$P(\bar{x} < 3.9470|\lambda = 0.3) \quad (4.5)$$

$$\rightarrow P(Z < 1.4379) = 0.9248 \quad (4.6)$$

This equally shows that the empirical power of the LR test for hypothesis (4.1) using 61 samples at 5% Type I error rate is about 90%. This value also agreed reasonably with Monte Carlo estimate of 89.9% power of the LR test for this same hypothesis at 5% significant level using 61 samples.
The above empirical results simply show that various Monte Carlo experiments performed in this study are quite efficient and reliable.

5.0 Discussions and Conclusion

This paper presents power analysis of the likelihood ratio test to compare the parameters of exponential distributions. As reported in some earlier works, it is equally established in this study that the number of samples employed in statistical hypothesis testing has a crucial role at influencing the power of such a test. More importantly, the results of this work indicated that the power of the likelihood ratio test for testing the parameters of exponential distributions is sensitive to effect sizes and the ratios of the parameters being tested under the null and alternative hypothesis sets.



	(λ_0, λ_1) = (0.2, 0.3)	(λ_0, λ_1) = (0.2, 0.4)	(λ_0, λ_1) = (0.2, 0.5)	(λ_0, λ_1) = (0.2, 0.6)
Effect Size	0.1	0.2	0.3	0.4
Parameter Ratio	2/3	1/2	2/5	1/3
Sample Size	80	28	16	11

Fig 5.1: The graph showing the sample size requirements by the likelihood ratio test to achieve 95% power as the effect size of the test ($\Delta = |\lambda_0 - \lambda_1|$) increases for the hypothesis set H$_0$:$\lambda = \lambda_0$ vs. H$_1$: $\lambda = \lambda_1$. In the four hypotheses considered, the value of parameter λ_0 of the exponential density under H$_0$ was set at 0.2 while that of $\lambda_1$under H$_1$were varied.Thus, the parameter pairs (λ_0, λ_1) = (0.2, 0.3), (λ_0, λ_1) = (0.2, 0.4), (λ_0, λ_1) = (0.2, 0.5) and (λ_0, λ_1) = (0.2, 0.6) with the respective effect sizes 0.1, 0.2, 0.3 and 0.4 were tested.

For a given value of the parameter of the exponential distribution λ_0 under the null hypothesis, any shift in the value of λ_0 as indexed by the effect size $\Delta = |\lambda_0 - \lambda_1|$ can be detected faster with a small sample size when the effect size Δ is large while relatively large samples would be required to detect such a shift when Δ is small (i.e. when λ_1 under H$_1$ is closer toλ_0 under H$_0$). For instance, at a value of $\lambda_0 = 0.2$ under H$_0$, the number of samples required to attain 95% power by the LR test as shown in Fig 3.2 reduces from 80 to 11 as the value of $\lambda_1$under H$_1$ progressively increases from 0.3 to 0.6 with corresponding increase in their effect sizes from 0.1 to 0.4 respectively. The influence of the effect sizes on the power behaviour of the LR tests at different sample sizes is clearly apparent in Fig 5.1. It can be observed from the graph that the number of samples required by the LR test to achieve 95% power decreases monotonically as the effect size of the tests increases.
Also, in term of the ratio $\lambda_0/\lambda_1$of the two parameters being tested, $\lambda_0 < \lambda_1$, the Monte Carlo results in Tables 3.2 and 3.3 generally showed that the LR test would require more samples to attain a reasonable power as the value of the ratio λ_0/λ_1 becomes large.

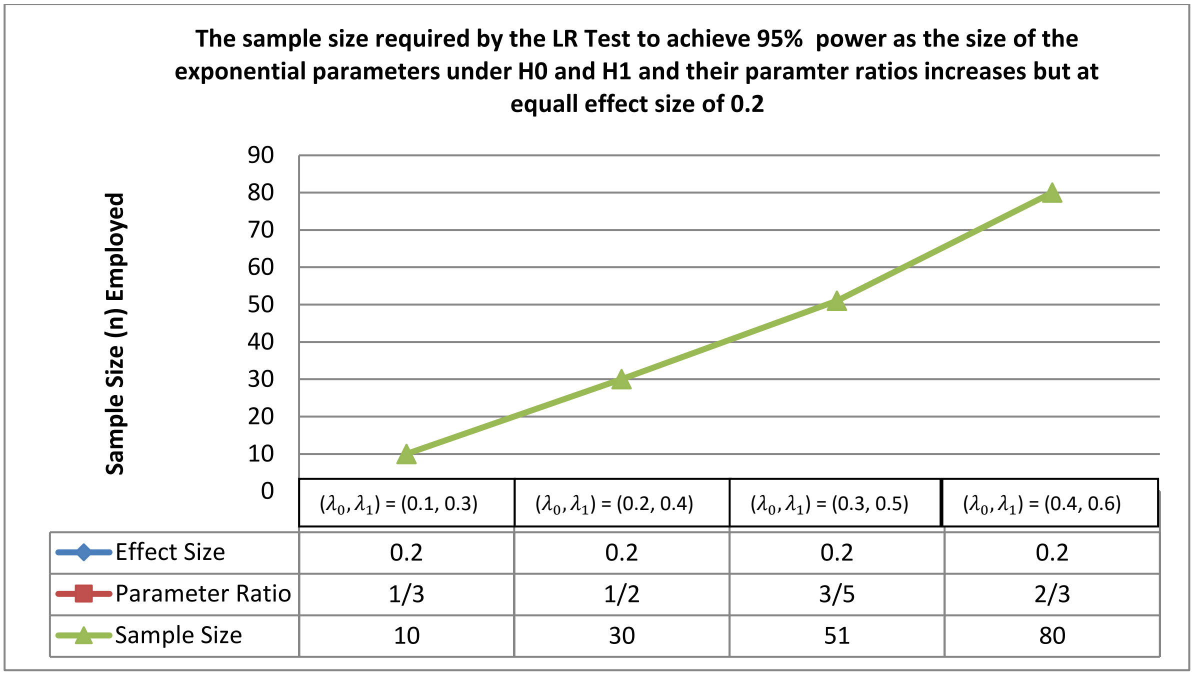


Fig 5.2: The graph of sample size requirements by the likelihood ratio test to achieve 95% power as the values of the parameter pair (λ_0, λ_1) being tested increasesbut with equal effect size of 0.2. In the four hypothesesof the form H$_0$:$\lambda = \lambda_0$ vs. H$_1$: $\lambda = \lambda_1$ considered, the parameter pairs (λ_0, λ_1) = (0.1, 0.3), (λ_0, λ_1) = (0.2, 0.4), (λ_0, λ_1) = (0.3, 0.5) and (λ_0, λ_1) = (0.4, 0.6) with equal effect size of 0.2in the four cases were tested. It is observed that the sample size required by the test to achieve 95% power increases as the values of parameter pair (λ_0, λ_1) being tested increase even with equal effect size of 0.2 in all cases.

A novel result obtained from this study isthat the power of a statistical test is not only determined by the effect and sample sizes, but also by the sizes of the parameters of the distribution being tested under the null and the alternative hypotheses. This is evident from the results of the Monte-Carlo study provided in Table 3.3. The results in Table 3.3showed the power of the LR tests1, 2, 3, and 4 under different values of parameter pairs (λ_0, λ_1) = (0.1, 0.3), (0.2, 0.4), (0.3, 0.5) and (0.4, 0.6) respectively all of equal effect size of 0.2. The graphical representation of these results is provided by Fig 5.2.

It can be observed from the results in Table 3.3 that at sample size of 21, the LR Test 1 with the smallest values of the parameters (λ_0, λ_1) = (0.1, 0.3) yielded 99.9% power while LR Tests 2 and 3 with relatively large parameter values (λ_0, λ_1) = (0.2, 0.4) and (λ_0, λ_1) = (0.3, 0.5) with the same effect size of 0.2 provided about 87% and 60% powers respectively at this same sample size.Surprisingly, the LR Test 4 with the largest parameter values (λ_0, λ_1) = (0.4, 0.6) but with the same effect size of 0.2 like others only yielded about 40% power at 21 sample size. This clearly showed the significant effect of the sizes of the parameters of the distributions being tested on the power of the LR test. Therefore, it can be concluded from these results that, the smaller the values of the parameters of the (exponential) distributions being tested under the null and alternative hypothesis, the higher the power of the LR tests irrespective the effect sizes.

Hence, forLR tests with relatively large values of the parameters of the (exponential) distributions to achieve reasonable power, more samples would be required. This is clearly evident from the results in Table 3.3. For instance, while only 11 samples are required by the LR Test 1 to achieve 95% power for testing the parameter pair (λ_0, λ_1) = (0.1, 0.3) with the least parameter ratio of 1/3, sample sizes 30, 50 and 80 were needed by LR Tests2, 3 and 4 to achieve the same fit of 95% power for testing the parameter pairs (λ_0, λ_1) = (0.2, 0.4), (λ_0, λ_1) = (0.3, 0.5) and (λ_0, λ_1) = (0.4, 0.6) respectively all at equal effect size of 0.2 but with a monotonic increase in the sizes of both λ_0 and λ_1 as well as their parameter ratios λ_0/λ_1.

As general conclusions, this present work re-affirms the general position in the literature that appreciable power of a statistical test can be achieved much more faster (with fewer samples) if the effective size of the test is fairly large. More importantly, results from this study established that, for test hypothesis regarding the parameter of an exponential distribution, the power of the test is majorly affected by the size of the parameter pair (λ_0, λ_1) being tested.Therefore, small values of parameter pair (λ_0, λ_1) would yield appreciable power than the large values of the parameter pair (λ_0, λ_1) of the exponential distributions. However, sample size increase may only be desirable as a corrective measure to increase the power of the LR test whenever small power of the test is obtained at possibly large values of the parameters of the exponential distributions being tested as can be observed from the results in Table 3.3.

The small and large values of parameter λ of the exponential distribution are of practical importance in real life situations. As a distribution of time to the occurrence of event[14], an exponential distribution of relatively small value of parameter

λrepresents the distribution of a fairly long period of time before the occurrence of an event of interest could be recorded such as time to death of persons. The results of this work therefore shows that, among the populations of individuals or objects with a known longer life span before death, fewernumber of samples would be needed before a shift in the death rate from λ_0 to λ_1 could be detected as shown,for instance, by the results of the LR test for the hypothesis set H_0: $\lambda = 0.1$ vs. H_1: $\lambda = 0.3$in Table 3.3. For this hypothesis, the LR test employs only 10 samples to achieve 95% power as shown by the graph in Fig 5.2. Here, the value of the parameter ratio λ_0/λ_1 is 1/3 which is relatively small compared to other parameter ratios in Table 3.3.

On the other hand,an exponential distribution with moderately large value of parameter λdescribes the distribution of short time period before an event of interest could occur such as the life span of an electric component.Therefore, among the populations of individuals or objects with a known short life span, it will require a fairly large number of samples from this group before a small shift in the death rate from λ_0 to λ_1 could be reasonably detected (with appreciable power). This is the scenario captured by the results of the LR test for the hypothesis set H_0: $\lambda = 0.4$ vs. H_1: $\lambda = 0.6$ in Table 3.3.For this hypothesis set (with a relatively large parameter ratio λ_0/λ_1 of 2/3) and in contrast to the earlier results, the LR test requiresup to 80 samples to achieve the same 95% power as determined under hypothesis test (H_0: $\lambda = 0.1$ vs. H_1: $\lambda = 0.3$) shown in Fig 4.3.

The simple conclusion from the above two resultsis that, whenever the LR test is to be constructed around a large values of parameter λ of the exponential distribution, a large sample size might be desirable to achieve appreciable power. However, if a relatively small values of λ are involved, a small samplesize might be sufficient.

However, it is quite instructive to add that whenever the values of the parameters under the null and the alternative hypotheses to be tested are specified, it is desirable to determine the sample size requirement of the test that would yield a reasonable power as desired by the investigator prior to the commencement of the experiment and the hypothesis testing proper.

This study focused discussions on power considerations of the LR test for exponential distributions. In future works, the behaviours of powers of the LR tests under some other forms of probability distributions shall be investigated within the framework of the current study.

6.0 References

[1] High, R.(2000): Important Factors in Designing Statistical Power Analysis Studies. URL: http://cc.uoregon.edu/cnews/summer2000/statpower.html. Accessed on 12th June, 2013.

[2] Greene, W.H. (2003): Econometric Analysis, 5th ed. Upper Saddle River, NJ: Prentice Hall.

[3] Park, H.M. (2010): Hypothesis Testing and Statistical Power of a Test**.** The Trustees of Indiana University, University Information Technology Services Center for Statistical and Mathematical Computing: 1-41. http://www.indiana.edu/~statmath

[4] Johnson, D. H. (1999): The Insignificance of Statistical Significance Testing. Journal of Wildlife Management, **63**(3):763-772.

[5] Neyman, J. and Pearson, E. (1933):On the Problem of the Most Efficient Tests of Statistical Hypotheses. Philosophical Transactions of the Royal Society of London, Series A, 231: 289–337. doi:10.1098/rsta.1933.0009.

[6] Vexler, A and Gurevich, G. (2011): A note on optimality of hypothesis testing. Mathematics In Engineering, Science And Aerospace, **2**(3): 243-250.

[7] Cohen, J.(1988): Statistical Power Analysis for the Behavioral Sciences. (2nd ed.) ISBN 0-8058-0283-5. Lawrence Erlbaum Associates, New Jersey.

[8] Thomas, L.(1997): Retrospective Power Analysis. Conservation Biology, **11**(1):276–280.

[9] Aberson, C.L.(2010): Applied Power Analysis for the Behavioral Science. ISBN 1-18487-2835.2.

[10] Vexler, A., Tsai, W-M., Gurevich, G. and Yu, J. (2012): Two-Sample Density-Based Empirical Likelihood Ratio Tests Based on Paired Data, with Application to a Treatment Study of Attention-Deficit/Hyperactivity Disorder and Severe Mood Deregulation. Statistics in Medicine (Accepted).

[11] Birnbaum, A.(1962): On the Foundations of Statistical Inference. Journal of the American Statistical Association, **57**:269-306.

[12] Berger, J.O. and Wolpert, R.L.(1984): The Likelihood Principle. IMS Lecture Notes, Vol. 6. Institue of Mathematical Statistics, Haywood, CA. MR0773665.

[13] Berger, J.O., Liseo, B. and Wolpert, R.L.(1999): Integrated Likelihood Methods for Eliminating Nuisance Parameters (with discussion). Statistical Science,**14**: 1–28. MR1702200.

[14] Polcer, III. and Albert, J.(2009): Confidence Intervals for the Ratio of Two Exponential Means with Applications to Quality Control. Student Research Conference Select Presentations, Paper 16.http://digitalcommons.wku.edu/selpres/16

[15] Vexler, A. and Tarima, S. (2011): An optimal approach for hypothesis testing in the presence of incomplete data. Annals of the Institute of Statistical Mathematics, **63**(6): 1141-1163.DOI: 10.1007/s10463-010-0270-0.

[16] Casella, G. and Berger, R.L.Statistical Inference, Second edition. Duxbury Press. United Kingdom, (2001). ISBN-10 : 0534243126

[17] Marshall, A. and Olkin, I. (2007): The Exponential Distribution. Life Distributions, Springer Series in Statistics, Part IV:291-307.DOI:10.1007/978-0-387-68477-2_8.

[18] Epstein, B. (1958): The Exponential Distribution and its Role in Life Testing. Technical Report no. 2,Wayne State University, Michigan: 1-23.

[19] Laherrere, J. and Sornette, D.(1998): Stretched Exponential Distributions in Nature and Economy: Fat Tails with Characteristic Scales. European Physical Journal, B2: 525-539.10.1007/s100510050276

[20] Yahya, W.B. and Ulm, K.(2009): Survival Analysis of Breast and Small-Cell Lung Cancer Patients using Conditional Logistic Regression Model. International Journal of Ecological Economics and Statistics, **14**(S09):15-30.

[21] Keatinge, C.(2012): Modeling Losses with the Mixed Exponential Distribution. http://casualtyactuaries.com/pubs/proceed/proceed99/99578.pdf(Accessed on 7 March, 2012)

[22] Wilks, S.S. (1938):The Large-Sample Distribution of the Likelihood Ratio for Testing Composite Hypotheses. The Annals of Mathematical Statistics9,: 60–62. doi:10.1214/aoms/1177732360.

[23] Ellis, P.D. (2010): The Essential Guide to Effect Sizes: An Introduction to Statistical Power, Meta-Analysis and the Interpretation of Research Results. United Kingdom: Cambridge University Press.

[24] R Development Core Team, R: A language and environment for statistical computing. R foundation for Statistical computing, Vienna, Austria, (2007), ISBN 3-900051-07-0, URL http://www.R-project.org