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Optimal Control of Rabies in a Meta-population Model

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Abstract

Rabies is a dangerous disorderin humans and animals central nervous system leading to convulsions, inability to move from one place to another and untypical behavior.The present effort is to characterize the mathematical model of Rabiesand theeffect of vaccine distribution with application to Rabies and bats in a meta-population model. Three classes of population wereconsidered for the purpose of the study namely; Susceptible, Infected, and the Removed class (SIR). The optimality system for the model was established, existence and uniqueness results for the system was also derived in an attempt to minimize the number of individuals that will be infected. Finally, the optimality system for the model was solved numerically using Runge-kutta fourth order scheme in particular. The result obtained shows that vaccination is a very efficient factor in minimizing the outbreak of the Rabies among the population.

Keywords: Optimal control, Rabies, bats, metapopulation model, vaccine distribution
AMS (MOS) Subject Classification Codes: 49-XX; 34K35; 35H05.

1.0 Introduction

Rabies is transmitted viabites or scratches or close contact with infected saliva, its infection without post-exposure vaccination leads to death within short period. The symptoms include weakness, discomfort, headache andfever which later results in delirium, abnormal behavior and insomnia.A literature reviewed showedthateffective post-exposure prophylaxis exists but is very expensive (often scarce), [1-6].

Vaccination can be refers to as the administration of antigenic material to stimulate an individual's immune system to develop adaptive immunity to a pathogenwhichis the most effective method of preventing infectious diseases. For example, vaccine for influenza, vaccine for the HPV,vaccine for chicken pox to mention a few[7].

Bats are numerous and found in distant as well as close proximity to humans in Nigeria. The handling, processing and consumption of bats of various species by children and adults may pose some danger to the public. Presently very few studies on zoonotic viral pathogens in bats have been documented in Nigeria, among these are the lyssaviruses and coronaviruses, [1, 2].

Research interest in bats has increased substantially following the identification of bats as important reservoirs of pathogens of both zoonotic and veterinary importance. Rabies models with variation control on their spread had been written detail in many papers. Studies conducted in Ibadan detected rabies virus (RABV) and LBV neutralizing antibodies in the sera of fruit bats,[3].The presence of numerous lyssaviruses in bat species has led to increasing research efforts towards search for lyssaviruses in bat populations globally. In Nigeria there is the need to establish an understanding of the epidemiology of rabies.

The factors responsible for the spread and the control strategies of rabiesusing mathematical model was considered in [8]. The mathematical model of rabies with similar controlling strategies in china was studied in [9]. The applicationof the principle of the optimal control theory to the study of Rabies was carried out in [10]. In addition the system of differential equations that modelled optimal control in SEIR model of rabies between dogs and humans with vaccination effect was studied in [11].

However, the present effort is to consider the application of optimal control theory to the vaccine distribution in a rabies metapopulation. For the purpose of this work the control result was illustrated by considering a population consisting of 8 subpopulation (a case study of Plateau State, Nigeria).

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2.0 Mathematical Formulation

Mathematical models is used in transforming world problem into mathematical programming problem that is solvable using mathematical packages. The population considered consisting of n sub populations which are connected together, see Figure 1. Sub population i is divided into three classes; the susceptible, S_i, that can be infected with rabies virus: the infected,I_i, individual bats that are currently infected with rabies and can transmit the virus, and R_i, corresponding to individuals that are vaccinated and become immune to infection by rabies virus.

Individuals in class $R_i$are removed from the system when they die, whileμ_R is the mortality rate, and after the waning period of the vaccine, they return to the susceptible class. For the purpose of this work mortality due to rabies for class I_i is included and since there is little evidence for naturally formed immune class of animals. Hence, once individual are infected, they die and are removed from the system. Where$a_{ij}$is the rate of geographic movement of uninfected individuals and$c_{ij}$is the rate of geographical movement of infected individuial. Note thata_{ij} may not be the same as a_{ji} and similarly for $c_{ij}$and c_{ji} because of the variation in the spatial orientation of sub populations.The movement coefficient for infected to be different or the same as for susceptible is allowed because of the controversy on whether infected animals change their character or movement. On the other hand, there are no reported alterations in behavior associated with vaccination. We assume that there is no significant change in the behavior of bats before the consumption of baits and after consumption, [10].

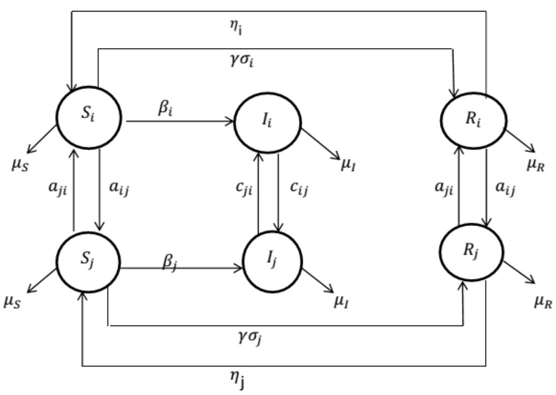


Figure 1: Flow diagram of the Model

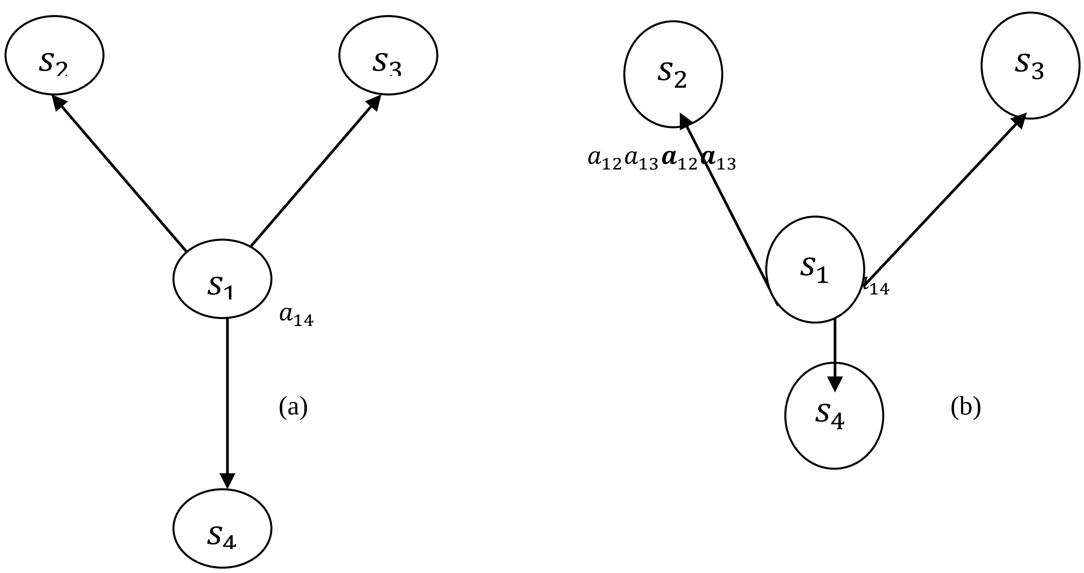


Figure 2: Rate of Geographic Movement

Figures 2 shows two examples of possible spatial configurations of four subpopulation from the viewpoint of subpopulation 1. In 2(a), the subpopulation 1 is located at the same distance from the other subpopulation 2, 3, and 4. In this case, the rates of geographic movement from S_1 to the other three, S_2, S_3 and S_4 are the same. However, if the distance between subpopulation 3 and subpopulation 1 is the largest, as shown in Figure 2(b), then the rate$a_{13}$is the smallest.

Table 1: Nomenclature

Symbol	Definition
a_{ij}	The rate of geographic movement of noninfected from subpopulation i to subpopulation j
c_{ij}	The rate of geographic movement of infected from sub-population i to subpopulation j
β_i	The rate of transmission in subpopulation
$\mu_S, \mu_{I,}\mu_R$	The mortality rate in each class: S, I and R
σ	The rate of vaccine bait distribution (control)
γ	The efficacy of vaccine bait distribution
ŋ	The warning rate of the vaccination
S_i	The number of susceptible in subpopulation i
I_i	The number of infected in subpopulation i
R_i	The number of individuals immune to the disease in subpopulation i

The state system is given as

$$\frac{dS_i}{dt} = \beta_i S_i I_i - \gamma \propto_i S_i + \sum_{j=1, j\neq i}^{n} a_{ji} S_j - \sum_{j=1, j\neq i}^{n} a_{ij} S_i - \mu_S S_i + ŋ_i R_i$$

$$\frac{dI_i}{dt} = \beta_i S_i I_i + \sum_{j=1, j\neq i}^{n} c_{ji} I_j - \sum_{j=1, j\neq i}^{n} c_{ij} I_i - \mu_I I_i \qquad (1)$$

$$\frac{dR_i}{dt} = \gamma \alpha_i S_i + \sum_{j=1, j\neq i}^{n} a_{ji} R_j - \sum_{j=1, j\neq i}^{n} a_{ij} R_i - \mu_R R_i - ŋ_i R_i$$

$$S(0) = S_0, \qquad I(0) = I_{0,} \qquad R(0) = R_0.$$

Assumptions

i. The mortality rates for susceptible and immune classes are the same, i.e.,
ii. $\mu_S = \mu_R = \mu$
iii. The magnitude of the rates of geographic movement, a_{ij} and c_{ij}, reflects the distance between the subpopulations i and j
iv. If bats consume the baits containing the vaccine, they instantly become immune to the disease.
v. The vaccine has a waning period, after which the vaccinated bats return to the susceptible class.

2.1 Existence and Uniqueness of Solution

Theorem 1(See [12]**)**

Let D' denote the region

$$|t - t_0| \leq, \|x - x_0\| \leq b, \;\; x = (x_1, x_2, \dots, x_n), \qquad x_0 = (x_{10}, x_{20}, \dots, x_{n0}) \qquad (2)$$

and suppose $f(t,x)$ satisfies

$$\|f(t,x_1) - f(t,x_2)\| \leq k\|x_1 - x_2\| \qquad (3)$$

Whenever the pair (t,x_1) and (t,x_2) in$D$,$k$is a positive constants. Then, $\ni \delta > 0 : \ni$ a unique continuous vector solution $\underline{x(t)}$ in$|t - t_0| \leq \delta$.

since (3) is satisfied by

$\frac{\partial f_i}{\partial x_j}$, $i, j = 1,2, \dots n$.

We now return to our model equations (1). We are interested in the region

$$0 \leq \alpha \leq R \qquad (4)$$

whose partial derivations satisfy $\delta < \alpha < 0$, where, α and δ are positive constants.

Theorem 2

Let D' denote the region$0 \leq \alpha \leq R$. Then equation (1) has a unique solution.

Proof

Let

$$\begin{aligned}
f_1 &= ŋ_1 R_1 + a_{21} S_2 - (\gamma\sigma_1 + I_1\beta_1 + \mu + a_{12}) S_1 \\
f_2 &= (S_1\beta_1 - c_{12} - \mu_1) I_1 + c_{21} I_2 \\
f_3 &= \gamma\sigma_1 S_1 + a_{21} R_2 - (ŋ_1 + \mu + a_{12}) R_1 \\
f_4 &= ŋ_2 R_2 + a_{12} S_1 - (\gamma\sigma_1 + I_2\beta_2 + \mu + a_{21}) S_2 \\
f_5 &= (S_2\beta_2 - c_{21} - \mu_2) I_2 + c_{12} I_1 \\
f_6 &= \gamma\sigma_2 S_2 + a_{12} R_1 - (ŋ_2 + \mu + a_{21}) R_2
\end{aligned} \qquad (5)$$

Now, Let D' denote the region $0 \leq \alpha \leq 0$.

Then equation (1) have a unique solution. We show that

$$\frac{\partial f_i}{\partial x_j}, i, j = 1,2,3,4,5,6 \quad (6)$$

are continuous and bounded in D'.

Differentiate (5) partially with respect to S_1 , I_1, R_1, S_2, I_2 and R_2 respectivelywe have the following;

$$\left|\frac{\partial f_1}{\partial S_1}\right| = |-\gamma\sigma_1 - I_1\beta_1 - \mu - a_{12}| < \infty, \left|\frac{\partial f_1}{\partial I_1}\right| = |-S_1\beta_1| < \infty,$$
$$\left|\frac{\partial f_1}{\partial R_1}\right| = |\eta_1| < \infty, \left|\frac{\partial f_1}{\partial S_2}\right| = |a_{21}| < \infty, \left|\frac{\partial f_1}{\partial I_2}\right| = 0 < \infty, \quad (7)$$
$$\left|\frac{\partial f_1}{\partial R_2}\right| = 0 < \infty$$
$$\left|\frac{\partial f_2}{\partial S_1}\right| = |I_I\beta_1| < \infty, \left|\frac{\partial f_2}{\partial I_1}\right| = |S_1\beta_1 - c_{12} - \mu_1| < \infty, \left|\frac{\partial f_2}{\partial R_1}\right| = 0 < \infty,$$
$$\left|\frac{\partial f_2}{\partial S_2}\right| = 0 < \infty, \left|\frac{\partial f_2}{\partial I_2}\right| = |c_{21}| < \infty, \left|\frac{\partial f_2}{\partial R_2}\right| = 0 < \infty, \quad (8)$$
$$\left|\frac{\partial f_3}{\partial S_1}\right| = |\gamma\sigma_1| < \infty, \left|\frac{\partial f_3}{\partial I_1}\right| = 0 < \infty, \left|\frac{\partial f_3}{\partial R_1}\right| = |-\eta_1 - \mu - a_{12}| < \infty,$$
$$\left|\frac{\partial f_3}{\partial S_2}\right| = 0 < \infty, \left|\frac{\partial f_3}{\partial I_2}\right| = 0 < \infty, \left|\frac{\partial f_3}{\partial R_2}\right| = |a_{12}| < \infty, \quad (9)$$
$$\left|\frac{\partial f_4}{\partial S_1}\right| = |a_{12}| < \infty, \left|\frac{\partial f_4}{\partial I_1}\right| = 0 < \infty, \left|\frac{\partial f_4}{\partial R_1}\right| = 0 < \infty,$$
$$\left|\frac{\partial f_4}{\partial S_1}\right| = |-\gamma\,\sigma_2 - I_2\beta_2 - \mu - a_{21}| < \infty \quad (10)$$
$$\left|\frac{\partial f_4}{\partial S_1}\right| = |-S_2\beta_2| < \infty, \qquad \left|\frac{\partial f_4}{\partial R_2}\right| = |\eta_2| < \infty$$
$$\left|\frac{\partial f_5}{\partial S_1}\right| = 0, \left|\frac{\partial f_5}{\partial I_1}\right| = |c_{12}| < \infty, \left|\frac{\partial f_5}{\partial R_1}\right| = 0 < \infty, \qquad \left|\frac{\partial f_5}{\partial S_2}\right| = |I_2\beta_2| < \infty$$
$$\left|\frac{\partial f_5}{\partial S_2}\right| = |S_2\beta_2 - c_{21} - \mu_2| < \infty, \left|\frac{\partial f_5}{\partial R_1}\right| = 0 < \infty \quad (11)$$
$$\left|\frac{\partial f_6}{\partial S_1}\right| = 0 < \infty, \left|\frac{\partial f_6}{\partial I_1}\right| = 0 < \infty, \left|\frac{\partial f_6}{\partial R_1}\right| = |a_{12}| < \infty, \quad \left|\frac{\partial f_6}{\partial S_2}\right| = |\gamma\sigma_2| < \infty$$
$$\left|\frac{\partial f_6}{\partial I_2}\right| = 0 < \infty, \left|\frac{\partial f_6}{\partial R_2}\right| = |-\eta_2 - \mu - a_{21}| < \infty \quad (12)$$

Since the partial derivatives of the system equation (5) exists, finite and bounded. Hence, by the previous theorem, it has a unique solution.

3.0 Optimal Control of the System

In this section, the optimality condition of the model is established.

The following assumptions in the form of lemma are used to establish the existence of an optimality system.

Lemma 3.1 [12].

Consider the cost function defined by

$$J(u) = \int_{t_0}^{t_f} L[t, x(t), u(t)]dt \quad (13)$$

Subject to

$$\frac{dx(t)}{dt} = f(t, x(t), u(t), \dots, f_n(t, x, u) \quad (14)$$

Where $u = (\sigma_1, \sigma_2, \ \dots, \sigma_m)$ is the control system and

$x(t) = (x_1(t), x_2(t), \dots., x_n(t))$

is the state and measurable function.

$\sigma(t) = (\sigma_1(t), \sigma_2(t), \dots, \sigma(t))^T$

is called the control function.

The following assumptions are imposed

(i) The function $L: [t_0, t_f] \times R^n \times R^m \to R$ is measurable in t, continuous in (t, σ) and$\sigma \to L(t, x, \sigma)$is convex for every (t, σ), $(t, \sigma) \in [t_0, t_f] \times R^n, U(t) = U(t) = U \subset R^m$ is closed and convex.

(ii) The Hamiltonian function $H(t,x,p) = supp - L(t,x,v)/v \in U$ satisfies the growth condition
$H(t,x,p) \leq \mu(t,p) + \|x\| (\sigma(t) + \gamma(t)\|p\|)$
$\forall t \in [t_0, t_f], x \in R^n, p \in R^m$where $\sigma, \gamma \in L^1[t_o, t_f], \mu(t,p) \in L^1[t_o, t_f], \forall p \in R^m$
And $supp(t,p), \|p\| \leq \delta \in L^1[t_0, t_f] \forall \delta > 0 L: R^n \times R^m \to R = (-\infty, +\infty)$ is lower semi-continuous, bounded from below on bounded subsets and $L(x_1, x_2) \geq g_1(x_1) +$
$g_2(x_2) \forall (x_1, x_2) \in R^n \in R^m$
(iii) $f(t,x,u) = f_0(t,x) + f_1(x,t)\sigma, \forall (t,x,\sigma) \in [t_0, t_f] \times R^n \times R^m$
where f_0, f_1 are continuous in $x$measurable in$t$and
$f_0(t,\bar{x}).x \leq (\eta_0(t)\|t\| + \beta_0(t)\|t\|), t \in [0,T], x \in R^n,$
$\|F_1(t,\bar{x})\| \leq \alpha_1(t), \|f_0(t,\bar{x})\| \leq \alpha_\delta(t), t \in (t_0, t_f), \|\bar{x}\| \leq \delta,$
where,$\beta_0, \alpha_\delta \in L^1[t_0, t_f] \forall \delta > 0$
(iv)The Integrand of the functional is concave in the admissible control set and
is bounded by$c_2 - c_1|u|^3, c_1 > 0$ and β > 1 .
Thus, the assumptions above imply there exist an optimal control, $\sigma^* = ({\sigma^*}_1, {\sigma^*}_2, \ldots\ldots, {\sigma^*}_n)$ for above cost functional.

Theorem 3.1

Consider the control problem (13) with state equations (14). There exist $\sigma^* = {\sigma^*}_i \in \cup$ such that$Min J(\sigma_i) = J({\sigma^*}_i)$.

Proof

Since U is closed and convex, the state equation is bilinear in $o_i$and the RHS of (13) and (14) are continuous which can be rewritten as $f(t,P,U) = \overrightarrow{\beta_i}(t,P) + \overrightarrow{\beta_i}(t,P)U$, by lemma (3.1) and the boundedness of solution give
$|f(t,P,U)| \leq \propto_i (1 + \|P\| + \|U\|) \forall t_o \leq t_f, P \in R^m, U \in R^n$
Where $P = (S,I,R)^T and U = (\sigma_i), i = 1,2,\ldots.,n.$
For the convexity of the integrand of the objective functional to be verified, it is showed the
$L(t,P(1-\lambda)U + \lambda V \leq (1-\lambda)L(t,P,U) + \lambda L(t,P,V)$
Where

$$L(t,P,U) = \left[I_i + \frac{\alpha}{2}\sigma_i^T\sigma_i\right], 0 < \lambda 1 \quad (15)$$

$$L(t,P(1-\lambda)U + \lambda V) = \delta_i^T C_i + \alpha[(1-\lambda)\sigma_i + \lambda v_i)^T(1-\lambda)\sigma_i + \lambda v_i] \backslash$$
$$= \delta_i^T \alpha\left[\left((1-\lambda)\sigma_i^T + v_i^T\right)(1-\lambda)\sigma_i + \lambda v_i\right] \quad (16)$$
$$= \delta_i^T C_i + \propto [(1-\lambda)^2\sigma_i^T\sigma_i + \lambda(1-\lambda)\sigma_i^T v_i + \lambda v_i^T) + \lambda^2 v_i^T]$$
$$L(t,P(1-\lambda)U + \lambda V) = \delta_i^T C_i + \propto \sigma_i^T\sigma_i + \alpha\,[(\lambda^2 - 2\lambda)\sigma_i^T o_i$$
$$+\lambda^2 v_i^T v_i + \lambda(\sigma_i^T v_i + v_i^T o_i) - \lambda^2(\sigma_i^T)] \quad (17)$$

And

$$(1-\lambda)L(t,P,U) + \lambda L(t,PV) = I_i + \propto \sigma_i^T\sigma_i - \lambda(\propto \sigma_i^T\sigma_i - \propto v_i^T v_i \quad (18)$$

Thus, it is showed that

$$\alpha[(\lambda^2 - 2\lambda)\sigma_i^T + \lambda^2 v_i^T + 2\lambda(1-\lambda)\sigma_i^T v_i^T] \leq \lambda(-\propto \sigma_i^T + \alpha_i^T \quad (19)$$

Since

$$\Longrightarrow \quad (\sigma_i^T v_i = \sigma_i^T\sigma_i, \sigma_i^T\sigma_i = \sigma_i^2, \qquad v_i^T\sigma_i = v_i^2)$$
$$\alpha(\lambda^2 - \lambda)(\sigma_i^T\sigma_i + v_i^T v_i) + 2\lambda(1-\lambda)(\propto \sigma_i^T v_i) \leq 0 \quad (20)$$
$$\equiv -\propto (\sqrt{\lambda(1-\lambda)\sigma_i} - \sqrt{\lambda(1-\lambda)v_i})^T\left(\lambda(1-\lambda)\sigma_i - \sqrt{\lambda(1-\lambda)v_i}\right) \leq 0$$

This implies

$$-\propto \left\|(\sqrt{\lambda(1-\lambda)\sigma_i} - \sqrt{\lambda(1-\lambda)v_i})\quad\right\|^2 \leq 0 \quad (21)$$

Since $\alpha > 0$,the above inequality holds. Hence the theorem holds.
In conclusion, it is showed that
$L(t,P,U) \leq q_2 + q_1\|U\|^\beta$, $q_i > 1$, $\forall L(t,P,U) = \delta_i^T C_i + \propto \sigma_i^T \leq q_2 - q_1\|U\|^2$
Whereq_2 depends on the upper bound t_2, $q_1 > 0$ and since $\propto > 0$. Hence by Lemma(3.1) there exist an optimal control for the model.
We consider this system on the time interval [0,T]. The control set defined as
$U = \{\sigma = (\sigma_{i,\ldots,}\sigma_n)|\sigma_i$ is Lebesgue measurable,
$0 = \leq \sigma_i(t) \leq \sigma_{max} a$, $e$for $i = 1,2,\ldots,n\}$
We choose the upper bound for σ to be 1, to represent the amount of vaccine distribution resulting from the highest level of vaccine bait distribution currently used, roughly about 150 baits km^2. The combined coefficient $\gamma \propto_i (t)$ represents the rate of removal of susceptible from subpopulation i due to vaccination. We wish to minimize the total number of infected and the cost associated with vaccination.

We consider the problem,

Minimize $J(\sigma) = \sum_{i=1}^{n} \int_0^T (I_i + \frac{\propto}{2}\sigma_i^2)dt$ (22)

Subject to

$$\frac{dS_i}{dt} = -\beta_i S_i I_i - \gamma\alpha_i S_i + \sum_{j=1, j\neq i}^{n} a_{ji}S_j - \sum_{j=1, j\neq i}^{n} a_{ji}S_i - \mu_S S_i + \eta_i R_i$$

$$\frac{dI_i}{dt} = \beta_i S_i I_i + \sum_{j=1, j\neq i}^{n} c_{ji}I_j - \sum_{j=1, j\neq i}^{n} C_{ji}I_j - \mu_I I_i$$

$$\frac{dR_t}{dt} = \gamma\alpha_i S_i + \sum_{j=1, j\neq i}^{n} a_{ji}R_j - \sum_{j=1, j\neq i}^{n} a_{ji}R_j - \mu_R R_i - \eta_i R_i \quad (23)$$

$$S(0) = S_0 I(0) = I_0, \quad R(0) = R_0, \quad \sigma \in U.$$

Where $\propto > 0$ is the weight factor in the cost of control. We choose a quadratic cost on the control for analysis convenience for this prototype problem. One can modify this to consider a combination of quadratic and linear cost or other convex functions, $A\sigma_i + B\sigma_i^2 \;\; where A > 0, B > 0.$

3.1 Necessary Condition

To obtain the necessary conditions we shall proof the following theorems

Theorem 3.2

There is an optimal control $\sigma \in U$that minimizes(22) subject to (23).

Using pontryagins's Minimum/Maximum principle, the necessary condition for optimality is derived. The Hamitonian is established with adjoint variables λ_{1i} , λ_{2i}, and λ_{3i} for $i = 1, 2, \ldots, n$.

$$H(t, S, I, R, \sigma) = \sum_{i=1}^{n} (I_i + \frac{\propto}{2}\sigma_i^2 + \lambda_{1i}S_i^{'} + \lambda_{2i}I_i^{'} + \lambda_{3i}R_i^{'} \quad (24)$$

Where λ_{1i} is multiplied by the RHS of the $S_i$and similarly for λ_{2i} and λ_{3i}.

Theorem 3.3 Given an optimal control $\sigma = (\sigma_1, \sigma_2 \ldots, \sigma_n)$ in U and corresponding state solutions $S = (S_1, S_2 \ldots, S_n), I = (I_1, 1_2 \ldots, 1_n)$ and $R = (R_1, R_2 \ldots, R_n \;\;)$, there exist $\lambda_1 = (\lambda_{11}, \lambda_{12} \ldots, \lambda_{1n}), \lambda_2 = (\lambda_{21}, \lambda_{22} \ldots, \lambda_{2n}), and \lambda_3 = (\lambda_{31}, \lambda_{32} \ldots, \lambda_{3n}) satisfying$

$$theadjointsystem:$$

$$\lambda_{1i}^{'} = \frac{\partial H}{\partial S_i} = \lambda_{1i}\left(\beta_i I_i + \gamma\sigma_i + \sum_{k=1, k\neq i}^{n} a_{ik} + \mu_S\right) - \lambda_{2i}\beta_i I_i - \lambda_{3i}\gamma\sigma_i - \sum_{k=1, k\neq i}^{n} \lambda_{1k}a_{ik}$$

$$\lambda_{2i}^{'} = \frac{\partial H}{\partial I_i} = -1 + \lambda_{2i}\left(\beta_i S_i + \sum_{k=1, k\neq i}^{n} c_{ik} + \mu_I\right) - \lambda_{Ii}\beta_i S_j - \sum_{k=1, k\neq i}^{n} \lambda_{2k}c_{ik} \quad (25)$$

$$\lambda_{3i}^{'} = \frac{\partial H}{\partial R_i} = \lambda_{3i}\left(\sum_{k=1, k\neq i}^{n} a_{ik} + \mu_R\right) - \sum_{k=1, k\neq i}^{n} \lambda_{3k}a_{ik} + \eta_i$$

and

$$\lambda_{1i}(T) = \lambda_{2i}(T) = \lambda_{2i}(T) = 0 \quad for i = 1, 2, \;. \;. \;., n \quad (26)$$

Furthermore, the controls

$$\sigma_i = \min \{\max \{0, \frac{\gamma S_i(\lambda 1_i - \lambda_{3i}}{\propto}\} \sigma\max\} \; for \; i = 1, 2, \;\; \ldots, n \quad (27)$$

Proof:

Suppose $\sigma = (\sigma_1, \sigma_2 \ldots, \sigma_n)$ is an optimal control and $S = (S_1, S_2 \ldots, S_n), I = (I_1, 1_2 \ldots, 1_n)$ and $R = (R_1, R_2 \ldots, R_n)$, are corresponding solutions. Using the result of Pontryagin's Maximum Principle, there exists adjoint variables$(\lambda_{3i}, \lambda_{2i} and \lambda_{1i})$ satisfying:

$$\lambda_{1i}^{'} = -\frac{\partial H}{\partial S_i}$$

$$\lambda_{2i}^{'} = -\frac{\partial H}{\partial I_i} \quad (28)$$

$$\lambda_{3i}^{'} = \frac{\partial H}{\partial R_i}$$

For $i = 1, 2, \;. \;. \;., n.$

Where H is the Hamiltonian, with the transversality condition

$$\lambda_{1i}(T) = \lambda_{2i}(T) = \lambda_{2i}(T) = 0 \tag{29}$$

For example, $\lambda_{1i}^{'} fori = 1, 2, \dots, n$ is given by

$$\lambda_{1i}^{'} = -\frac{\partial H}{\partial S_i}$$

$$\lambda_{1i}\left(\beta_i I_i + \gamma\sigma_i + \sum_{k=1, k\neq i}^{n} a_{ik} + \mu_S\right) - -\lambda_{2i}\beta_i I_i - \lambda_{3i}\gamma\sigma_i - \sum_{k=1, k\neq i}^{n} \lambda_{1k} a_{ik} \tag{30}$$

The general form for the optimality condition is given by

$$\frac{\partial H}{\partial \sigma_i} = \propto \sigma_i - \gamma S_i(\lambda_{1i} - \lambda_{3i}) = 0 \quad at \sigma_i^* \tag{31}$$

On the set $t: 0 < \sigma_i^*(t) < \sigma_{max}, i = 1, 2, \dots., n.$ by Solving (31) for $\sigma_i^*(t) fori = 1,2, \dots n$ on the interior of the control set, we have

$$\sigma_i^*(t) = \frac{\gamma S_i(\lambda 1_i - \lambda_{3i})}{\propto} \tag{32}$$

Using the control bounds, we obtain

$$\sigma_i = \min\{\max\{o, \frac{\gamma S_i(\lambda 1_i - \lambda_{3i})}{\propto}\}, \sigma max\} \; fori \; 1,2, \dots., n \tag{33}$$

Since the solution of the state and adjoint system are L^∞ bounded, the right hand side of these ODEs are Lipcithz in the state and adjoint variables, which guarantee the uniqueness of the optimality system consisting (23), (25), and (27).

4.0 Numerical Solution

In this section, Numerical solution are presented via maple 18, we used the Forth Order Runge-Kutta method to solve the state system, the solution to the optimal control problem are presented using the data collected from Plateau State. Collection of bats was based on the availability of bat roost, consent from relevant authorities, and the cooperation of local hunters and community inhabitants who were knowledgeable of bat 55 roosts and available to assist in bat capture. Bats were collected from 8 different locations within 4 Local government areas in plateau state, Nigeria,[13].The data are presented in Table 2

Table 2: Bats sampled and Geographic Positions of their Roast sites, [13]

S/no	Bat species (family)	Bat common name	Name of location bat roost sites	Number of hat/species (population	Number of bats tested positive	Geopositioning of bat roast
1	Eidolonhelvum (Pteropodidae)	African straw coloured bats or African fruit bat	National Commission for Museums and Monuments, Jos. roosting on trees	732	32	$9^0$54'48N, 8^o 53'10E, Altitude 1223m
2	Rhinopoma microphylum(Rhinopomatida)	Mousetailed bat	Leptur, roosting in a cave	12	4	$9^0$26'48N, 9^o 2'0E, Altitude 966m
3	Chaerophon pumila (Molossidae)	Little freetailed bats	Dawaki, roosting in hospital roof and house wall	99	5	$9^0$5'1N,9^o 4'2E, Altitude 759m
4	Rhinolophus landeri (Rhinolophidae)	Lander's horseshoe bat	Pandam Game Reserve, roosting in an ancient well.	196	10	$8^0$8'12N, 8^o 57'52E, Altitude 162m
5	Nycteris macrotis (Nycteridae)	Largeeared slitfaced bat	Pandam, roosting in a culvert	18	3	$8^0$2'9N, 8^o 9'0E, Altitude 145m
6	Epomophorus franquetti (Pteropodidae)	Franquet Epaulet	Sabon layi, roosting on trees in the Chief's compound	124	5	$8^0$4'59N, 9^o 7'45E, Altitude 173m
7	Epomorphorus Gambianus (Pteropodidae)	Gambian Epaulet	Sabon layi, roosting on trees beside the bridge	5	2	$8^0$39'N, 9^o 8'9E, Altitude 158m
8	Laviafrons (Megadermatid)	Yellow winged bat	Lakushi,roostingon thorny trees in a swampy forest-like area	100	3	$8^0$4'4N, 9^o 9'3E, Altitude 152m

To illustrate our control results, we consider a population consisting of 8 subpopulation (a case study of 4 different local government of Plateau State, Nigeria.)

All eight (8) sub populations are connected to each other and each movement coefficient only depends on the distance between two sub populations. For each sub populations $i = 1,2, \dots.,8$, our state system (23) consists of 24 ordinary differential equations. And 24 ordinary differential equations of adjoint variables which were numerically solved.

Table (2) shows the details of bats sample, the geographical positions of their roost, Number of population and infected bats in each sub population. The following is the default setting for the initial conditions and parameters.

$\alpha = 100, \beta_i = 0.01, \gamma_i = 0.1, \mu_S = \mu_R = 0.00236, U_I = 0.1818$

The rate of $a_{ij}, a_{is}$is guessed based on deductions from [10].

Then parameter (γ) , the efficacy of vaccine distribution is difficult to find in the literature, here we evaluated 2 different values (i.e. 0.1, 0.4).

The numbers in each class (susceptible (Solid line),Infected (dotted line) and removed(dashed line)).We tried different examples with different values for $\sigma$and γ in table 3.

Table 3: Table of values for examples

Example	$\sigma(t)$	γ
	0	–
2	0.1	0.4
3	0.5	0.1
4	0.5	0.4
5	1.0	0.4

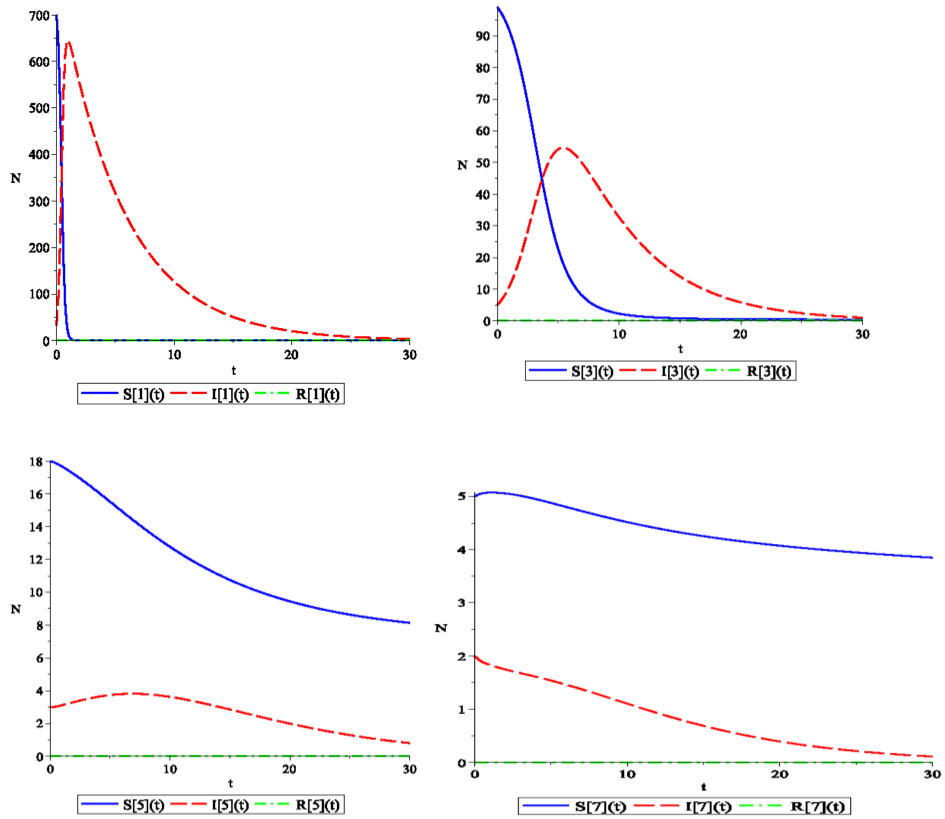
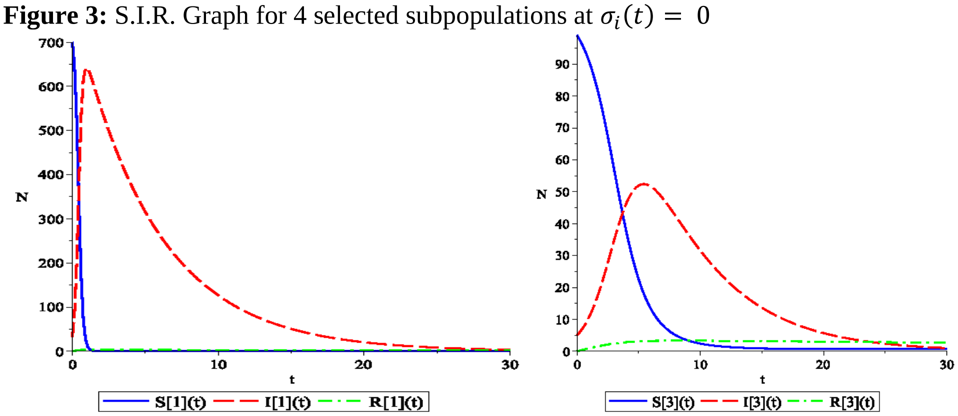


Figure 3: S.I.R. Graph for 4 selected subpopulations at $\sigma_i(t) = 0$



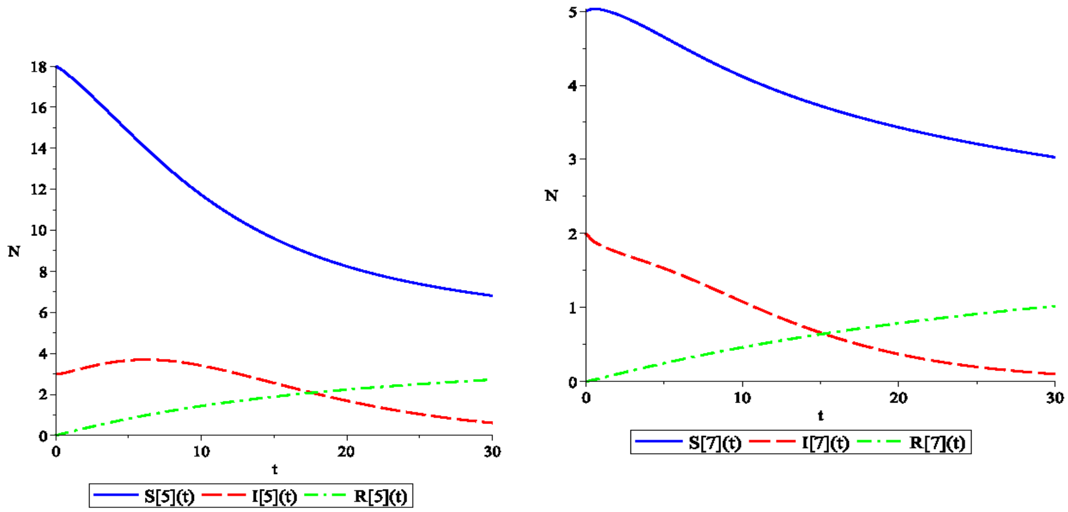


Figure 4: S.I.R. Graph for 4 selected subpopulations at $\sigma_i(t) = 0.1, \gamma = 0.1$

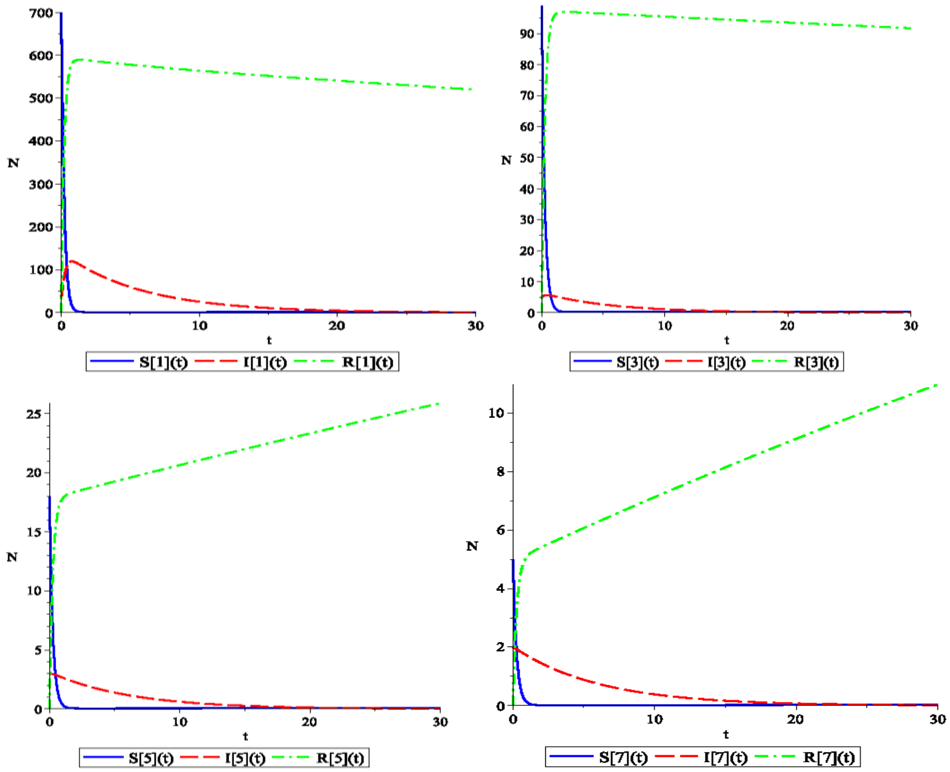


Figure 5: S.I.R. Graph for 4 selected subpopulations at $\sigma_i(t) = 0.1, \gamma = 0.4$

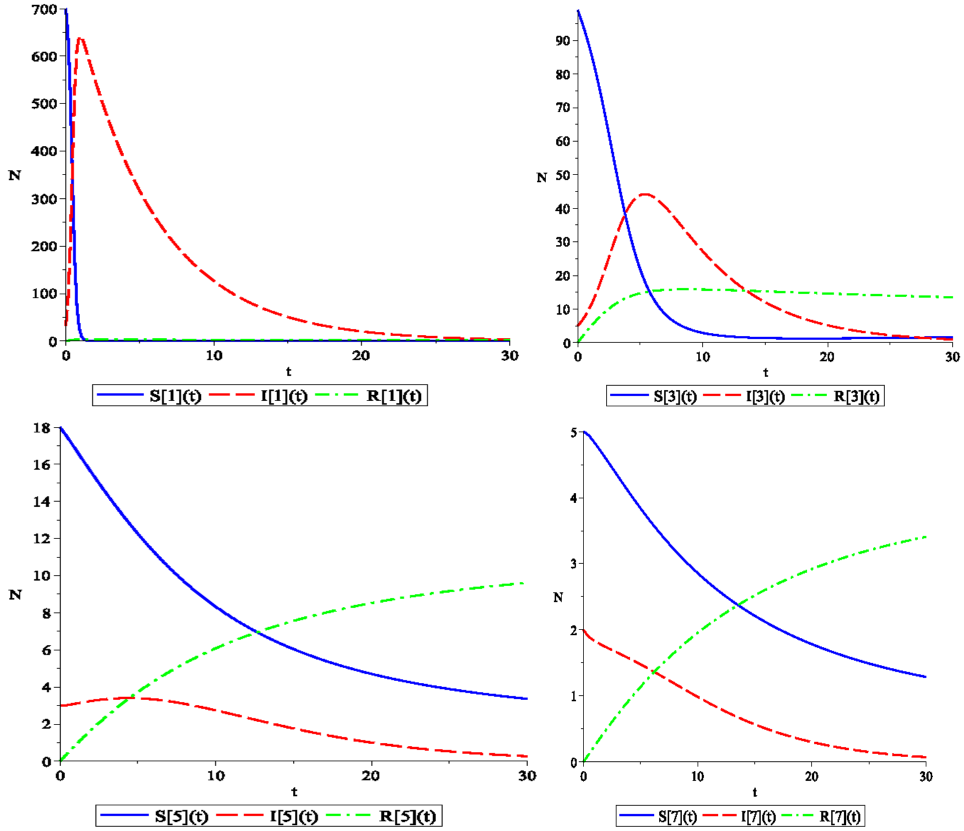


Figure 6: S.I.R. Graph for 4 selected subpopulations at $\sigma_i(t) = 0.5, \gamma = 0.1$

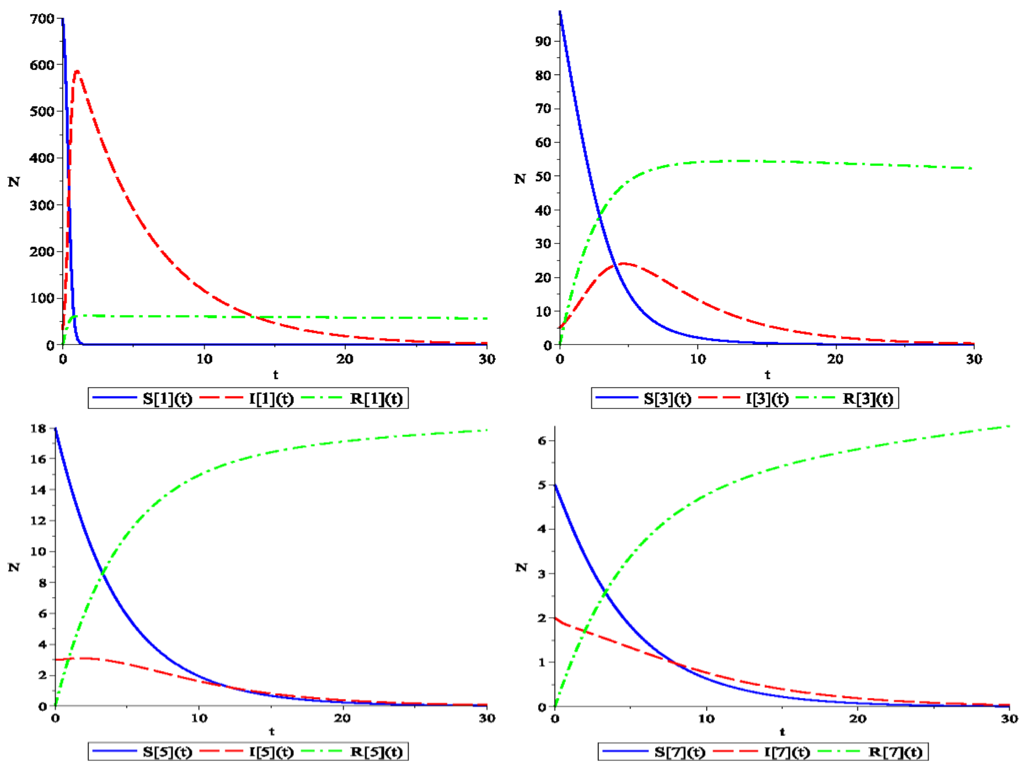


Figure 7: S.I.R. Graph for 4 selected subpopulations at $\sigma_i(t) = 0.5, \gamma = 0.4$

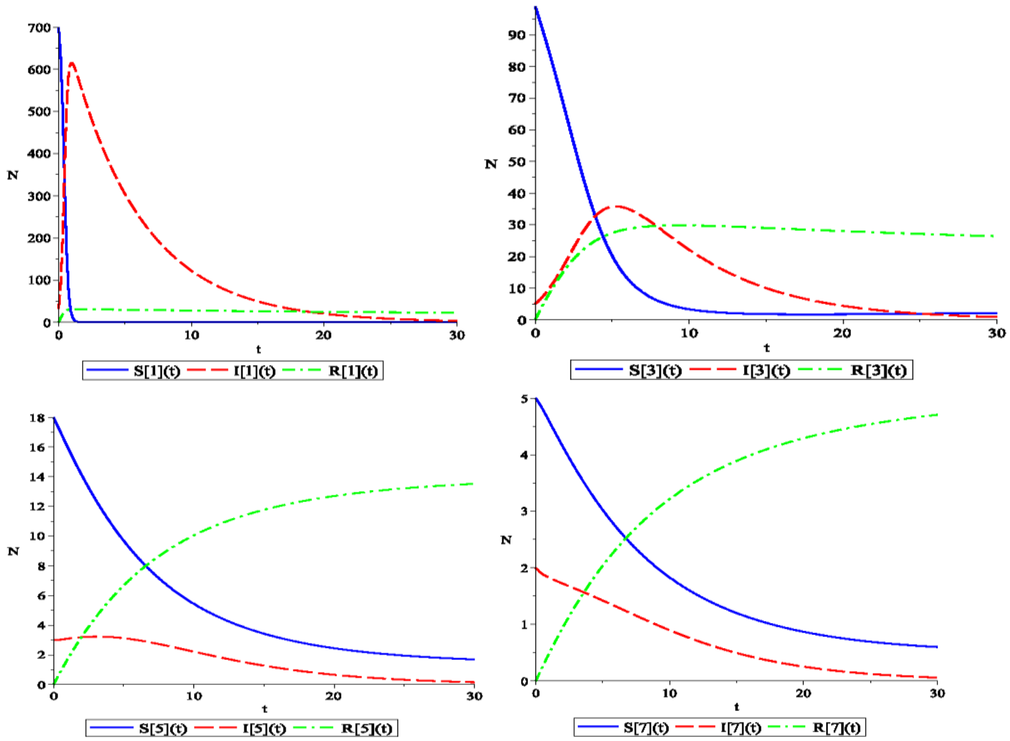


Figure 8: S.I.R. Graph for 4 selected subpopulations at $\sigma_i(t) = 0.1, \gamma = 0.1$

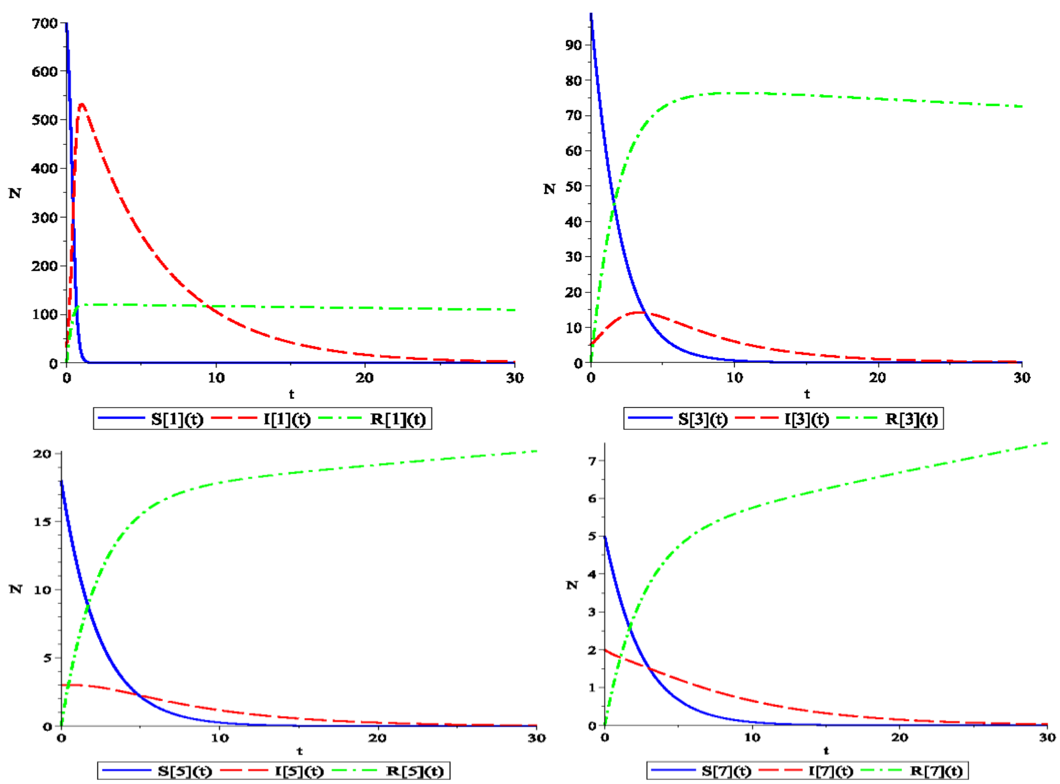


Figure 9: S.I.R. Graph for 4 selected subpopulations at $\sigma_i(t) = 0.1, \gamma = 0.4$

4.1 Interpretation of Results

The general shapes of the graphs were very similar with small changes in magnitude. If the coefficient β_i are too large, the disease spread very fast. We choose $\beta_i = 0.001$for illustration (Note for a more realistic case, one could choose different β_i in different sub-populations to represent different conditions). With the following default setting for the initial conditions and parameters.

$\propto = 100,\quad \beta_i = 0.1,\ \gamma_i = 0.1,\ \mu_S = \mu_R = 0.00236,\ U_I = 0.1818$
The rate of $a_{ij},\ a_{is}$ was guessed based on deductions from [10].
If γ is lower than 0.4, this results in a corresponding lower level of vaccine control. From the graphs, if the coefficient β_i are too large, the disease spread very fast.
Different examples with different values for σ and γ were evaluated which was shown in Table 3. The simulations show clearly that when there is no control (vaccine), there is no remove class. Similarly when there is control, the remove class surface, you will also note the efficacy of the vaccine is very important as it was shown that when the efficacy was high, the remove class was higher and when the efficacy is low, the remove class reduced drastically.

5.0 Conclusion

Optimal control theory approach was used to characterizes the rabies metapopulation model.The existence and uniqueness of the model was established, and finally solved using numerical method (Runge-Kutta fourth order in Paticular). The result obtained shows that vaccination is a very efficient factor in reducing the number of infected individuals and increasing the number of recovered individuals.And because of human consumption of bat meat in many parts of Nigeria, There is need to optimally control the number of infectious so as to reduce the spread of the diseases among bats, domestic animals and wildlife generally. Henceforth, there is need for continued surveillance nationwide to have better understanding of the epidemiology of viral agents of economic and public health importance in bats and, other wild animals so as to aid in their prevention, control and eradication.Equipping of existing laboratories in Research institutes and Universities in the nation for modern, speedy and quality diagnosis is much recommended

6.0 References

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