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On the Commutativity of Some Differential Operators with the Airy Tracy-Widom Integral Operator

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Abstract

The eigen vectors of a Tracy-Widom integral operator K can by found by using the fact that, a differential operator L satisfying KL=LK has the same eigenvectors as K. In the papers, we consider the possibility of finding a self adjoint differential operator which commutes with a Tracy-Widom integral operator K. We observe that a TW integral operator can be written using commutators, via multiplication and Hilbert transform operators, and show that the Hilbert transform commutes with differentiation by expressing both these operators in their Fourier transform state. Using that fact and some commutator formulae, we expand the commutator of K and a typical self-adjoint differential operator, and find that for this to be zero, K must also be zero.

1.0 Introduction

The Airy function in the physical science, is a special function named after the British Astronomer George Biddell Airy (1801-92).The Airy function y satisfies

$$\frac{d^2y}{dx^2} - xy = 0$$

Airy function is a solution to Schrödinger's equation for a particle confined with a triangular potential well and for a particle in a one-dimensional constant force field.

We shall initially define some terms [1-11]. Let U,V, be two vector spaces. Any mapping from U to V shall be called anoperator, say (A+B)x = Ax + Bx
and (φA)x = φ Ax, such that ~~for~~ all A, B: U ⟶ VxU and A, BЄ K.

Suppose U and V defined ~~ab~~oveare equipped with norms. Then a linear operator from U to V is called bounded if there exist a constant C> 0 such that||A x|| ≤ C ||x||.

If a matrix value in random variable – that is, a matrix all of whose elements are random variables. Many important properties of physical system can be represented mathematically as matrix problems. For examples the thermal conductivity of a lattice can be computed from the dynamical matrix of particle-particle interactions within the lattice.

2.0 Section One

We shall begin this section with a well known theorem;

Fredrich's Theorem[7]

Let L be an operator defined on a dense subspace D of a Hilbert space
H The following conditions are satisfied.

i. L (D) ⊆ D
ii. (Lu, v) = (u, Lv) for u,v∈D) —
iii. (Lu, v) ≥ 0 for u∈ D —

Then L is a self adjoint extension.
Tracy – Widom operation not commuting with differential operator, and
Self adjointness for differential operator.

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~~So~~me differential operators are unbounded, so we need to be careful when discussing what is meant by self adjointness, so that spectral theory can be applied. A densely defined linear operator on a Hilbert space H is a pair comprising a dense linear subspace, which we call Dom (A) and a linear map A: Dom (A) $\longrightarrow$ H. If E is a linear subspace of H which contains Dom (A) and A is a map E: $\longrightarrow$ H which satisfiesA f = Af for all f Є Dom (A), then we say that A is an extension of Dom(A). The adjoint A* of a operator A satisfies the condition (Ax, y) = (x, A*y) for all x∈ Dom (A) y ∈ Dom(A*) where we define Dom (A*) to be the set of all y∈H for which (Ax,y) = (x, z) with z ∈ H.

Suppose *I*be interval on the real line. We consider integral operators, KA,B with kernel, KA,B (x,y) of Tracy-Widom type, where KA,B (x,y) = $\frac{A(x)B(y)-A(y)B(x)}{x-y}$ and which act on a function fЄ$L^2(I)$ in the usual way:

KA,B f(x) = $\int K_{A,B}$ (x,y) f(y) dy.

When the dependence on the function A and B is clear we omit the subscripts. In several important examples (e.g. Bessel Kernel, sine kernel, Airy Kernele.t.c) the eigenvector and eigen values of an operator of this form can be found via commuting self adjoint differential operator. By this, we mean differential operator L on some suitable space of functions which satisfy the condition.

$\int_I L_x K(x,y)\, f(y)\, dy = \int_I K(x,y)\, L_y\, f(y)dy$

~~∀~~ x, in which L_x means that L acts on the x variable and so on. The following general theorem on commuting operators means that if we can find such a differential operator, its eigenvector will be the same as those of the Tracy-Widom operator K.

Proposition[9]: Let A and B be compact self-adjoint operators on a Hilbert space H, and suppose that AB=BA. There exist an orthonormal basis (ϕ_j) of H Э $A\phi_j = \Lambda\, \phi_j$and $B\phi_j = N_j\phi_j$ for some scalars Λand N_j.

We provide an example to show the commutativity of some operators with the Airy-Widom operators

Example:

An Airy Kernel arises when we scale the eigen values of a random matrix at the soft edge of the spectrum in the Gaussian unitary ensemble. It can be written as the square of the Hankel operator with Kernel A (x + y):

$$\frac{A_j(x)\, A_j(y) - A_j(x)\, A_j(y)}{x-y} = \int_0^x A_i(x+t)A_i(y+t)dt.$$

Let L be a differential operator defined on the space D = $C_0^\infty\ (R+)$ of smooth function on R+ with compact support by

L = $-\frac{d}{dx}\left(\frac{xd}{dx}\right) + x^2$

Note that, L is symmetric. To see this, take the f , g ЄD, integrating them by parts and using the fact that f(x), g(x) $\longrightarrow$0 as x approaches infinity, together give

(Lf, g)(x) = $\int_0^\infty (-(xf'(x))' + x^2 f(x))g(x)dx$

= $[-xf'(x)\ g(x)]_0^\infty + \int_0^\infty x\, f'(x)g'(x)dx$

+ $\int_0^\infty x^2 f(x)\ g(x)\ dx$

=$\int_0^\infty xf'(x)g'(x)dx + \int_0^\infty x^2 f(x)g(x)dx$

While

(f, Lg)(x) = $\int_0^\infty f(x)(-(xg'(x)]' + x^2\ g(x))dx$

= $[-xf(x)\ g'(x)]_0^\infty + \int_0^\infty xf'(x)g'(x)dx +$

$\int_0^\infty x^2 f(x)g(x)dx$

= $\int_0^\infty xf'(x)g'(x)dx + \int_0^\infty x^2 f(x)g(x)dx$

= (Lf, g)

A similar argument shows that L is a positive operator and hence L is a self adjoint, by Fredrich's theorem. Also L commutes with the Hankel integral operator with Kernel A_j (x+y). Recall that A_i (x) = xA_i(x) and then

$L\Gamma^1 f(x) = L\int_0^\infty A(x+y)f(y)dy = \int_0^\infty(-(xAi(x+y)' + x^2 Ai(x+y))\, f(y)dy$

=$\int_0^\infty -x(x+y)Ai(x+y) - Ai'(x+y) + x^2 Ai(x+y))\, f(y)dy$

=$\int_0^\infty (-xyAi(x+y) - Ai'(x+y)f(y)dy,$

While integration by parts yield

$\Gamma^1 L(x) = \int_0^\infty Ai\,(x+y)\Big(-yf'(y)\Big)' + \, y^{\,2} f(y))dy$

$=[Ai\,(x+y)yf'(y)]_0^\infty + \int_0^\infty Ai'(x+y)yf'(y)dy + \int_0^\infty Ai\,(x+y)y^2 f(x)dy$

$=\int_0^\infty Ai'(x+y)yf'(y)dy + \int_0^\infty Ai\,(x+y)\,y^{\,2} f(y)dy$

$=[Ai'(x+y)\,yf(y)]_0^\infty - \int_0^\infty\Big(Ai'(x+y) + \,y\,Ai''(x+y)\Big)f(y)dy$

$+\int_0^\infty Ai'(x+y)y^{\,2} f(y)dy$

$= -\int_0^\infty Ai'(x+y)f(y)dy - \int_0^\infty y\,(x+y)Ai\,(x+y)f(y)dy$

$+\int_0^\infty y^{\,2} Ai\,(x+y)f(y)dy$

$= L\Gamma\ f(x).$

where we used the fact that Ai(x) → 0 as x →∞ to show that the boundary terms are zero.It is then clear that L commutes with the Airy Tracy-Widom operator on the half line.

3.0 Conclusion

The main result in this paper tell us that L operator commute with the Airy Tracy-Widom operator.

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