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On Contraction and Fixed Point of the Solution of an Evolution Equation in Banach Space

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Abstract

We show boundedness and contraction of a solution to an evolution equation of the type $u(t) = u_0 s(t) + \int_0^t s(t-u)\, f(u)du$ ***in Banach space, using the definition of contraction mapping and Banach Steinhaus theorem. The result shows that solution to the evolution equation has a unique fixed point.***

1.0 Introduction

In this paper, we study the contractiveness and boundedness of a solution of an evolution equation of the form:

$$\dot{u}(t) - A(t)u(t) - f(t) = 0,\ u(0) = u_0 > 0, \qquad (1.1)$$

on a Banach space X, where A(t) is a linear operator and f is a non-autonomous continuous function from [0,T] on X. $u(0) = u_0$is referred to as the initial value problem, or Cauchy problem.$L[X] = L([0,T], X)$is the Lebesque equivalent class of measurable function

$$\|f\|_p = (\int |f|^p d\mu)^{\frac{1}{p}}\ p \in [1, \infty).$$

Many authors have worked on solution of evolution equation producing sound results for instance see [1-4]. On the other hand, the existence and uniqueness of classical solution of the form

$$\dot{u}(t) = A(t)u(t) \quad t \in [0,T], \quad u(0) = u_0, \qquad (1.2)$$

had been studied in [5-9]. Motivated by the above literature, the objective of this paper is to show boundedness, contraction and fixed point of the solution of nonhomogeneous abstract Cauchy problem in (1.1)

For a given evolution family $S(t,u)_{t\geq u}$,the evolution semigroup $\{S(t)\}_{t\geq 0}$ is defined on a time space E of function $f: \mathbb{R} \to X$ by the function

$$(S(t)f)(u) = S(u, u-t)f(u-t), \quad u \in \mathbb{R}, t \geq 0. \qquad (1.3)$$

When $E = L_P(\mathbb{R}, X), p \in [0, \infty)$ or $E = C^0(\mathbb{R}, X)$ then the space of continuous functions vanishes at $\{\infty\}$ and $\{S(t)\}_{t\geq 0}$ is strongly continuoussemi group.

2.0 Preliminaries

Definition 2.1 Let X be a Banach space. A one parameter family $S(t), t \in [0, \infty)$ of bounded linear operator from X into X is a semigroup of bounded linear operators on X if

i)$S(t+u) = S(t)S(u)$for every $t, u \geq 0$

ii)$S(0) = I$where$I$ is an identity operator on X

iii)$\lim_{t\to 0}\|S(t) - S(0)\|_X = 0$ is uniformly continuous

Definition 2.2 Let X, Ybe normed spaces and $T: X \to Y$ be linear, then T is called bounded linear map if ∃ some constant $M \geq 0$ such that $\|T(x)\| \leq M\|x\| \forall\ x \in X. M$ is called bound of T

Definition 2.3 Let

i)Mbe a closed nonempty set in Banach space X over K

ii) The operator $T: M \to M$ be K −contractive ie by definition $\|Tu - Tv\| \leq K\|u - v\|\ \forall\ u, v \in M$ and fixed $K, K \in [0,1)$ then the following holds

i) Existence and Uniqueness: The equation $Tu = u,\ \ u \in M$ has exactly one solution.

ii) Convergence of the iteration method: For each given $x_0 \in M$, the sequence $\{x_n\}$ constructed as $x_{n+1} = Tx_n$ converges to the unique solution x.

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Definition 2.4 Let X be a Banach space and T a bounded operator on $X. T$ is not necessarily linear operator. $T$is said to be a contractive operator if there exist $K < 1$ such that $\|Tu - Tv\| \leq K\|u - v\| \quad \forall\, u, v \in X.$
Theorem 2.5 Let $T(t)$ and $S(t)$ be uniformly continuous semigroup of bounded linear operator. If
$\lim_{t\to 0} \frac{T(t)-T(0)}{t} = \lim_{t\to 0} \frac{S(t)-S(0)}{t} = A,$ (2.1)
then$T(t) = S(t)$ for $t \geq 0$
Proof: see [10]
Theorem 2.6 (Banach-Steinhaus) LetX, Y be Banach spaces and $\{T_n\}_{n=1}^{\infty} \subset B(X,Y)$. Let $\{T_n x\}_{n=1}^{\infty}$ for each $x \in X$ converges to Tx then
i) $\sup_{n\geq 1}\|Tx\| < \infty$
ii) $T \in B(X,Y)$
iii) $\|T\| \leq \lim_{n\to\infty} inf\|Tn\|$
Theorem 2.7 (Uniform Boundedness theorem) Let$X$ be a Banach space and Y a normed space. If $\mathcal{A} \subseteq B(X,Y)$ such that for each $x \in X$, $Sup\{\|Ax\|: A \in \mathcal{A}\} < \infty$ then $Sup\{\|A\|: A \in \mathcal{A}\} < \infty$
Theorem 2.8 Let T be an operator on X such that the kth power of T is a contraction operator. Then the equation $Tf = f$ has a unique solution in X.
Proof: Take $T^k f = f$ has a unique solution. In fact, we can obtain the solution by finding
$\lim_{n\to\infty} T^{kn} f_0 = f$ (2.2)
For an arbitrary initial function f_0. In particular, we observe that by letting $f_0 = Tf$
$\lim_{n\to\infty} T^{kn} f_0 = \lim_{n\to\infty} T^{kn}(Tf) = f$ (2.3)
Recall that if $T^n f = f$, then we have that $T^{kn} f = f$ so that
$\lim_{n\to\infty} T^{kn}(Tf) = \lim_{n\to\infty} TT^{kn} f = \lim_{n\to\infty} Tf = Tf = f$
To show that the solution is unique, we have
$Tf = f$and$Tg = g$ (i)
$T^k f = f$and$T^k g = g$ (ii)
For (i)$\|f - g\| = [\![Tf - Tg]\!] \leq \alpha\|f - g\| \Longrightarrow (1-\alpha)\|f - g\| \leq 0 \Longrightarrow f = g$
For (ii) $\|f - g\| = [\![T^k f - T^k g]\!] \leq \alpha\|f - g\| \Longrightarrow (1-\alpha)\|f - g\| \leq 0 \Longrightarrow f = g$
Therefore the operator has a unique fixed point.
Theorem 2.9 (Gronwalls-Bellman's Inequality) Let U and$f$be continuous and nonnegative function defined on $I = [a, b]$ and Let K be a non-negative constant. The inequality
$u(t) \leq K + \int_a^t f(s)u(s)ds, t \in I = [a, b]$ (2.4)
Then
$u(t) \leq Kexp(\int_a^t f(s)ds), t \in I$ (2.5)
Proof:
Define a function $w(t)$ by the right side of equation (2.4) then we observe that
$w(a) = K, u(t) \leq w(t)$
and
$\dot{w}(t) = f(t)u(t) \leq f(t)w(t), \quad t \in I$ (2.6)
Now multiply equation (2.6) by exp $(-\int_a^t f(s)ds)$ then, we have

$$\dot{w}(t) \exp\left(-\int_a^t f(s)ds\right) - w(t)f(t)\exp\left(-\int_a^t f(s)ds\right)$$

$= \frac{d}{dt}\left(w(t)\exp\left(-\int_a^t f(s)ds\right)\right)$
which implies that
$\frac{d}{dt}\left(w(t)\exp\left(-\int_a^t f(s)ds\right)\right) \leq 0$ (2.7)
Integrating both sides of equation (2.7) over a to t we have

$$w(t)\exp\left(-\int_a^t f(s)ds\right) - w(a) \leq 0$$

Hence equation (2.5) is obvious.

3.0 Main Results

We will denote X a Banach space.and $L[X] = L([0,t], X)$ as the equivalent class of Lebesgue measurable function and $\|.\|$ a norm as in.

Theorem 3.1 Let f be a bounded real valued function,$f: X \longrightarrow X$and $u\epsilon \mathbb{R}^n$,$\mathrm{M} \geq 1, \mathbb{R} \subset X$ and$\omega\epsilon\mathbb{R}$ then if $\|S(t)\|_{L(X)} \leq \mathrm{M}e^{\omega t}$(semigroup operator) and A ϵ $\mathbb{R}^n \times \mathbb{R}^n$,then the evolution equation (1.1) satisfies the following;

i). Boundedness.

ii). Contraction, hence has a unique fixed point.

Proof:

The evolution equation (1.1) in Banach space X has a general solution by variation of constant or Duhamalformular as

$$u(t) = u_0 S(t) + \int_0^t S(t-u)f(u)du, \quad (3.1)$$

where$S(t)$ is a semigroup operator, $S(t) = e^{At}$. To show that equation (3.1) is bounded, we have for $\mathrm{K} > 0$, then $\|S(t)\|_{L(X)} \leq \mathrm{K}$. where $K = \mathrm{M}e^{\omega t}$ *and* $t > 0$.

Before we prove the first condition, we consider the theorem below.

Theorem 3.2 Let $\{s(t); t > 0\}$ be a $C_0 - semigroup$, then $\exists\ M \geq 1$ *and* $w \in R \ni \|S(t)\|_{L(X)} \leq M\, e^{w(t)}$ for each t≥ 0

Proof:

Claim: it suffices to show that there exists $\alpha > 0$ *and* $M \geq 1$ such that

$$\|S(t)\|_{L(X)} \leq M \; for\; eacht \in [\,0, \alpha]. \quad (3.2)$$

Proof of Claim: Suppose by contradiction that equation(3.2) is not so. Then there exists at least one C_0-semigroup $\{S(t): t \geq 0\}$ *with* the property that, for each $\alpha > 0$ and each

$\mathrm{M} \geq 1, \exists\ t_{\alpha,m} \in [0, \propto]$, such that

$$\|S(t)\|_{L(X)} > M. \quad (3.3)$$

Taking $\alpha = \frac{1}{n},\ M = n\; and\; take\; t_{\alpha,m} = tn\; for\; n \in \mathbb{N}^*, thus\; we\; have$

$$\|S(t)\|_{L(X)} > n, \quad (3.4)$$

where$t_n \in \left[0, \frac{1}{n}\right]$ *for each* $n \in N^*$.

By property of semi group for each $t \in X$

$$\lim_{n\to\infty} S(t_n)t = t \quad (3.5)$$

This means that the family of the semigroup $\{S(t)$: n$\in$ N$^*\}$ of linear bounded operators is proof wise bounded. That is for each $t \in X$, the set$\{S(t)$:n$\in N^*\}$ is bounded. By uniform boundedness principle or Banach-Steinhaus theorem, it follows that the family is bounded in the uniform operator norm $\|\cdot\|_{L(X)}$ which contradicts equation (3.4). This contradiction can be read off only if equation (3.2) holds.

Now let $t \geq 0, then\; there\; exists\; n \in \mathbb{N}^* and\; \sigma \in [0, \alpha] \ni\ t = n\alpha + \sigma, thus\; we\; have$

$$\|S(t)\|_{L(X)} = \|S^n(\alpha)S(\sigma)\|_{L(X)} \quad (3.6)$$

$$\leq \|S^n(\alpha)\|_{L(X)}\|S(\sigma)\|_{L(X)} \quad (Banach\; Algebra)$$

$$\leq MM^{n.}$$

But $n = \frac{t-\sigma}{\alpha} < \frac{t}{\alpha}$, thus

$$\|S(t)\|_{L(X)} \leq MM^{\frac{t}{\alpha}} = Me^{tw},$$

where$w = \frac{1}{\alpha} log_e M$.

Hence

$$\|S(t)\|_{L(X)} \leq Me^{tw}. t \in [0, \alpha]. \quad (3.7)$$

Now, from equation (3.2) we have $u(t) = \underline{u}_0 S(t) + \int_0^t S(t-u)f(u)du$.

Taking the norm of both sides we have

$$\|u(t)\|_{L(X)} = \left\|\underline{u}_0 S(t) + \int_0^t S(t-u)f(u)du\right\|_{L(X)}$$

Where $L(X) = L([0, t], X)$

$$\|u(t)\|_{L(X)} \leq \left\|\underline{u}_0 S(t)\right\|_{L(X)} + \left\|\int_0^t S(t-u)f(u)du\right\|_{L(X)}$$

$$= \underline{u}_0\|S(t)\|_{L(X)} + \left\|\int_0^t S(t-u)f(u)du\right\|_{L(X)}$$

$$\leq \underline{u}_0\|S(t)\|_{L(X)} + \int_0^t \|S(t-u)f(u)\|du_{L(X)}$$

$$\leq \underline{u}_0\|S(t)\|_{L(X)} + \int_0^t \|S(t-u)\|_{L(X)}\|f(u)\|du_{L(X)}$$

$$\leq \underline{u}_0\|S(t)\|_{L(X)} + \int_0^t \|S(t-u)\|_{L(X)}\, Mdu.$$

By theorem (3.2) we have

$$\|u(t)\|_{L(X)} \leq \underline{u}_0 Me^{wt} + \int_0^t \|S(t-u)\|_{L(X)}\, Mdu, t \in [0,\alpha]$$

$$\leq \underline{u}_0 Me^{wt} + M\int_0^t Me^{w(t-u)} du$$

$$= \underline{u}_0 Me^{wt} + M^2 \int_0^t e^{w(t-u)}\, du$$

$$= \underline{u}_0 Me^{wt} + M^2 e^{wt} \int_0^t e^{-wu}\, du$$

$$= \underline{u}_0 Me^{wt} + M^2 e^{wt} [-\frac{1}{w} e^{-wu}]_0^t$$

$$= \underline{u}_0 Me^{wt} + M^2 e^{wt} [\frac{1}{w}(1-e^{-wt})]$$

$$= \left(\underline{u}_0 M + \frac{M^2}{w}\right) e^{wt} - M^2 e^{w(t-t)}$$

$$= \left(\underline{u}_0 M + \frac{M^2}{w}\right) e^{wt} - M^2\,, t \in [0,\alpha].$$

Hence,

$$\|u(t)\|_{L(X)} \leq \left(\underline{u}_0 M + \frac{M^2}{w}\right) e^{wt} - M^2.$$

To show that (3.2) is a contraction, it suffices to show that

$\|u(t_1) - u(t_2)\|_{L(X)} \leq \beta\|t_1 - t_2\|_{L(X)}$, where $\beta \in [0,1)$.

Now,

$$u(t_1) - u(t_2) = \underline{u}_0 S(t_1) + \int_0^t S(t_1-u)f(u)du - \underline{u}_0 S(t_1) - \int_0^t S(t_2-u)f(u)du$$

$$= \underline{u}_0 S(t_1) - \underline{u}_0 S(t_1) + \int_0^t S(t_1-u)f(u)du - \int_0^t S(t_2-u)f(u)du$$

$$\|u(t_1) - u(t_2)\|_{L(X)} = \left\|\underline{u}_0 S(t_1) - \underline{u}_0 S(t_1) + \int_0^t S(t_1-u)f(u)du - \int_0^t S(t_2-u)f(u)du\right\|$$

$$\leq \left\|\underline{u}_0 S(t_1) - \underline{u}_0 S(t_2)\right\|_{L(X)} + \left\|\int_0^t S(t_1-u)f(u)du - \int_0^t S(t_2-u)f(u)du\right\|_{L(X)}$$

$$= \underline{u}_0\|S(t_1) - S(t_2)\|_{L(X)} + \left\|\int_0^t S(t_1-u)f(u)du - \int_0^t S(t_2-u)f(u)du\right\|_{L(X)}$$

$$\leq \underline{u}_0\|S(t_1) - S(t_2)\|_{L(X)} + \int_0^t \|S(t_1-u)f(u) - S(t_2-u)f(u)\|_{L(X)}\, du$$

$$\leq \underline{u}_0\|S(t_1) - S(t_2)\|_{L(X)} + \int_0^t \|S(t_1-u) - S(t_2-u)\|_{L(X)}\|f(u)\|_{L(X)} du$$

$$\leq \underline{u}_0 \alpha\|t_1 - t_2\|_{L(X)} + \int_0^t \|S(t_1-u) - S(t_2-u)\|_{L(X)}\, Mdu$$

By theorem (2.9), we have

$\|u(t_1) - u(t_2)\|_{L(X)} \leq \alpha\underline{u}_0\|t_1 - t_2\|_{L(X)}e^{[\int_0^t \|S(t_1-u)-S(t_2-u)\|_{L(X)}du]}$

$= \beta\|t_1 - t_2\|$

$\beta \in [0,1)\ and\ t \in [0, \alpha]$

Hence,

$\|u(t_1) - u(t_2)\|_{L(X)} \leq \beta\|t_1 - t_2\|_{L(X)}$

And $u(t)$ is a contraction. Now since $u(t)$ is contraction, it has a unique fixed point ie $u(t) = t$

4.0 Conclusion

We have shown boundedness and contraction of a solution to an evolution equation(3.1) in Banach space. And also shown thatthe construction of solution to a given evolution equationinto a contraction operator yields equation (3.1) to a unique fixed point.

5.0 References

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