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Stability Analysis of Cubic Duffing Oscillator (The Hard Spring Model)

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Abstract

In this paper, the stability analysis of periodic solution of cubic Duffing oscillator-the hard spring model were investigated. The eigenvalue classification were adopted and Mathcad software was used to demonstrate the behaviour of the solution. The major findings revealed the spiral stable equilibrium point of the system. The main contribution is that the hard spring model is the only nonlinear system that has an equivalent behavior of linear system in terms of the stability of the equilibrium point.

Keywords: Stability, Cubic Oscillator, Mathcad, Eigenvalue approach
Mathematics Subject Classification: 34A05, 33E05, 34C29.

1.0 Introduction

Stability is the fundamental tool for design and control. Every system in life is stable unless provoke by eternal force[1] and any system that is not stable is potentially chaotic[2].Different researchers have used different techniques to examine the stability of solutions of Duffing equations. For instance see[3-11]. On the study of the Duffing-type equation using critical point theory see [12,13]. On the stability of periodic solutions see [14,15]. The Duffing equation

$\ddot{x} + c\dot{x} + ax + bx^3 = h(t)$ (1.1)

where a,b,care real constants and h(t)is continuous is second order nonlinear differential equation that is widely used in physics, economics, engineering, and many other physical phenomena. The study is significant because of the physical applications of the results. It is significant to the Physicist who uses it to study propagation of wave in mobile phones, radios, television [16]. The signal processor will find its significance in modelling of the non-linear spring mass system [14] and modeling of the ultra-wide band (UWB) radio systems for detecting high speed wireless [17]. Also used in Fuzzy modeling and the adaptive control of uncertain chaotic system.

It is also significant to the Engineers who find its applicability in the areas of high damping door construction, crash analysis, construction of Traffic lights, modelling conservative double well oscillator which occur in magneto-elastic mechanical system [18] and Prediction of emission characteristics of saw dust particles [19]. It is also significant to the medical and life scientist who will find if applicable to the modelling of the brain [20], modeling and predicting hearth beats (pulse). The environmentalist will find it applicable in predicting earthquake occurrences [21] and other natural disasters such as tsunamis and heat waves. The hard spring system of equation (1.2) is used in modelling of mechanical systems. It can be used to model plant systems, where the effect of nonlinear stiffness on resonant behavior of plants is described by the Duffing oscillators with hardening nonlinearization. Its important can be seen in crash analyses, signal processing[22] and prediction of weather condition. Motivated by the above literature and ongoing research in this direction, the objective of this paper is to investigate the stability of periodic solution of Duffing's equation of the form:

$\ddot{x} + c\dot{x} + ax + bx^2 + \beta x^3 = h(t)$ (1.2)

With boundary conditions as:

$x(0) = x(2\pi)$

$\dot{x}(0) = \dot{x}\,(2\pi)$

The equivalent system for (1.2) is

$\dot{x}_1 = x_2$

$\dot{x}_1 = -cx_2 - ax - bx^2 - \beta x^3$

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Where a, b, c are real constants and $h: [0,2\pi] \to \mathbb{R}^n$ is continuous.
$c =$ represent the damping coefficient
$a =$ resonance coefficient or stiffness constant
$b =$ the nonlinear term and $h(t)$can be any of the following functions $\sin k(t)$or $\cos k\,(t)$.
$k =$ amplitude.
x^3 is called the Duffing's term and can be approximated as small as possible. The system can be treated as perturbed single Hamiltonian system. It is assumed that this term is responsible for the multiplicity of periodic solution [23].
β is the coefficient of nonlinearity. If $\beta > 0$ then equation (1.2)represent a hard spring which is the problem that motivated this research.

2.0 Preliminaries

Definition 2.1. (Stability):An equilibrium point x_e of a nonlinear system is said to be stable if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that $\bar{x} \in B(x_\varepsilon, \delta) \Longrightarrow \varphi(t, 0, \bar{x}) \in B(x_\varepsilon, \varepsilon)\ \ for\ all\ t \geq 0$
Note: The Lyapunov stability of x_e assumes a "simultaneous continuity", more precisely the equicontinuity at x_e of all the functions in the $\{\phi_t: \bar{x} \to \varphi(t, 0, \bar{x})\ for\ t \geq 0$
Definition 2.2. (Asymptotic Stability):The equilibrium point x_e is said to be asymptotically stable, if for all $\varepsilon > 0$, the exists a $\delta > 0$ such that,
(i) $\varphi(t; 0, \bar{x}) \in B(x_e, \varepsilon)$ for all $t \geq 0$
(ii) $\lim_{t\to\infty} \varphi(t; 0, \bar{x}) = x_e$
Definition 2.3. Consider the general non-linear differential equation of the form
$\dot{x} = f(t, x(t)) where\ f: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n\ is\ continuous$
The function f in definition 2.3 is said to be T-periodic if for every $(t, x) \in \mathbb{R} \times \mathbb{R}^n and\ some\ T > 0, f(t, x) = f(t + T, x) and f(t, x) \neq f(t + T^*, x)\ for\ all\ T^* < T$.
Definition 2.4. A solution x of definition 2.3 defined on R such that $x\,(t + T) = x\,(t)$ for all $t \in \mathbb{R}$ is called T- periodic solution or T- periodic harmonic solution.
Definition 2.5. Damping is an influence or effect upon an oscillatory system that prevents, restricts or stops an oscillation. If the damping is enough that the system just fails to oscillate then it is said to be critically damped. Any further influence results to over damping and less is similarly under damped.
Definition 2.6. (Asymptotic Stability):The equilibrium point x_e is said to be asymptotically stable, if for all $\varepsilon > 0$, the exists a $\delta > 0$ such that,
(i) $\varphi(t; 0, \bar{x}) \in B(x_e, \varepsilon)$ for all $t \geq 0$
(ii) $\lim_{t\to\infty} \varphi(t; 0, \bar{x}) = x_e$
Theorem 2.7. (Lyapunov stability) Let $x = 0$ be an equilibrium point of $\dot{x} = f(x)$. Let $V: D \subset \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable such that $V(0) = 0, V(x) > 0\ \forall x \in D\ \{0\}$ and $L_f V(x) \leq 0$ for all $x \in D$. Then $x = 0$ is asymptotically stable.

3.0 Result

3.1 Analysis of the "Hard Spring" System of Equation (1.2)

Equation (1.2) according to [24] is an example of a periodically forced oscillator with nonlinear elasticity.
Consider a Duffing Oscillator of the form

$$\ddot{x} + c\dot{x} + ax + bx^2 + \beta x^3 = h(t) \tag{3.1}$$

where the damping constant obeys $c \geq 0$
The equivalent system of (3.1) is

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -cx_2 - ax - bx^2 - \beta x^3 + h(t) \end{aligned}\right\} \tag{3.2}$$

This can also be called a simple model which yields chaos, as well as Vander pol oscillator.
For $a > 0$ the Duffing oscillator can be interpreted as a forced oscillator with a spring whose restoring force is written as

$$F = -ax - bx^2 - \beta x^3 + h(t) \tag{3.3}$$

when $\beta = 2$, this spring is called a hard spring and when $\beta < 0$, it is called a soft spring, although this interpretation is valued only for small x [25].
Equation (3.1), (3.2) and (3.3) can thus be rewritten as follows, when $\beta = 2$,

$$\ddot{x} + c\dot{x} + ax + bx^2 + 2x^3 = h(t) \tag{3.4}$$

With the equivalent system

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -cx_2 - ax - bx^2 - 2x^3 + h(t) \end{aligned}\right\} \tag{3.5}$$

Equation (3.4) is thus described the equation of a "hard spring" which is under consideration in this paper. The same is applicable to the system (3.5) which is referred to as the "hard spring" system.

For $a < 0$, equation (3.1) through system (3.5), describes the dynamics of a point mass in a double well potential and this can be regarded as a model for periodically forced steel beam which is deflected towards the two magnets as shown in Fig 1 [26,18]. It is assumed that the chaotic motions can be observed in this case.

3.2 Stability Analysis of the Undamped and Unforced System

In this section, we examine the dynamics of the unforced system that is when $h(t) = 0$ in equation (3.4) and the system (3.5) when there is no damping that is when $c = 0$. The equations/systems mentioned above are conservative and all the orbits are described by an energy integral.

$$E(t) = \frac{1}{2}\dot{x}^2 + \frac{1}{2}ax^2 + \frac{1}{2}x^4 = constant \tag{3.6}$$

where the energy E(t) is a constant on each orbit. Each of such orbit is a closed oval, symmetric about the X-axis, the V-axis ($\dot{x}$) and the origin. Therefore in this case, the Duffing equation is a Hamiltonian system. The shape of $E(t)$ for $B = 2$ can be shown in Fig 1 and it can be seen that $E(t)$ is a single well potential for $a > 0$ and it is a double well potential for $a < 0$. The trajectory of $x = -(x, \dot{x})$ moves on the surface of $E(t)$keeping $E(t)$ constant

When $c > o, E(t)$ satisfies

$$\frac{d(E(t))}{dt} = -c\dot{x}^2 \leq 0 \tag{3.7}$$

Therefore the trajectory of x moves on the surface of $E(t)$ and so $E(t)$ decreases until x converges to one of the equilibria $\dot{x} = 0$

The amplitudes of the maximum x and the maximum velocity when $h(t)$ in equation (3.1) is decomposed into $c_1 \cos kt$ can easily be related to the energy

$$E(t) = \frac{1}{2}(V)^2 + \frac{1}{2}a(x)^2 + \frac{1}{2}(x)^4 \tag{3.8}$$

We see that for large $E(t)$, V goes like $\sqrt{E(t)}$ whereas x goes like $\sqrt[4]{E(t)}$, so that the orbit becomes increasingly elongated along the V-axis as $E(t)$ increases

Case I For $\beta > 0, a > 0, c > 0$ in equation (3.1) the only equilibrium is

$$\bar{x} \equiv (0,0) \tag{3.9}$$

and $E(t)$ satisfies

i)$E(t) = 0 \quad if \;\; and\; only\; if\; X = \bar{X}$

ii) $E(t) > 0 if\; X \neq \bar{X}$

iii) $E(t) < 0 \;\; if \;\; X \neq \bar{X}$

Therefore, $E(t)$ is a Lyapunov Function and $\bar{X}$ is globally asymptotically stable in the sense of Lyapunov.

Case II On the other hand for $\beta > 0, a < 0, c > 0$ in equation (3.1), there are three equilibria as indicated below in Fig 2

1. Two of which are at the bottom of $E(t)$ and
2. One of which is at its peak. In this case, almost all the initial conditions converge to one of the equilibria at the bottoms, except for initial conditions on the stable manifold of the equilibrium at the peak.

The equilibria of the Duffing oscillator for $h(t) = c_1 coskt \;\; or \; c_2 sink$ in the equation (3.1) when $h(t) = 0$ or specifically where $k = 0$ can be obtained by substituting $\dot{x} = 0$ to equation (3.1)

namely

$$x(a + \beta x^2) = 0 \tag{3.10}$$

Therefore the point $x = 0$is always an equilibrium.

Moreover, when$a + \beta x^2 > 0$, two equilibria

$$x = \pm\sqrt{{-a}/{\beta}} \quad \text{appear} \tag{3.11}$$

The stability of these equilibria can be understood by analyzing the equation.

In equation (3.1) for $h(t) = 0$ or $k = 0$ can be re-written as

$$\frac{d}{dt}\begin{pmatrix} x \\ \dot{x} \end{pmatrix} = \begin{pmatrix} \dot{x} \\ -c\dot{x} - ax - \beta x^3 \end{pmatrix} \tag{3.12}$$

and the Jacobian Matrix $DF(x)$ of the right hand side is calculated as

$$DF(x) = \begin{pmatrix} 0 & 1 \\ -ax - \beta x^2 & -c \end{pmatrix}. \tag{3.13}$$

Therefore, the eigenvalues of $DF(x)$ for the equilibrium $x = 0$ is

$$\lambda = \frac{-c \pm \sqrt{c^2 - 4a}}{2} \tag{3.14}$$

and it is found that this equilibrium is stable for $a \geq 0$ and unstable for $a < 0$.

<u>Case III</u> On the other hand, the eigenvalues of the equilibria

$$x = \pm \sqrt{{-a}/{\beta}}$$

Are $\lambda = \frac{-c \pm \sqrt{c^2 - 8a}}{2}$ (3.15)

and are found that these equilibria are stable for $\beta > 0$ and $a < 0$ and unstable for $\beta < 0$ and $a < 0$

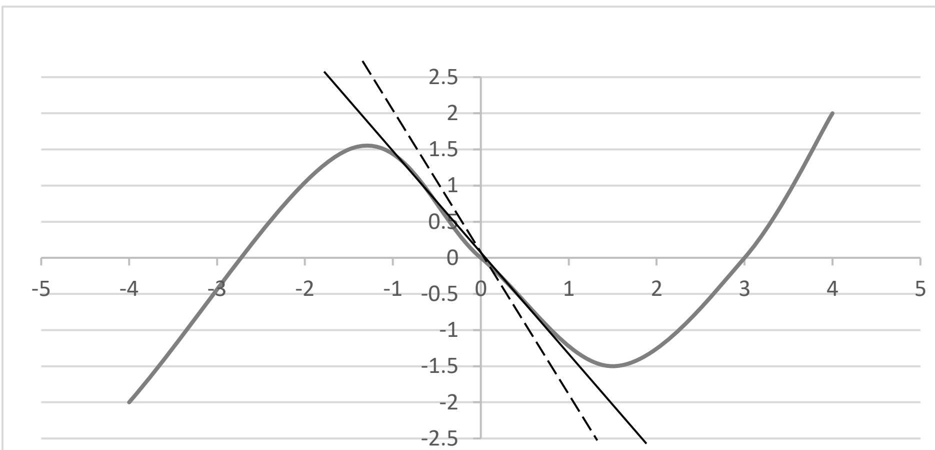


Fig 1: For $a > 0$, the Duffing oscillator can be interpreted as a forced oscillator with a nonlinear spring whose restoring force is written as $F = -ax - \beta x^3$

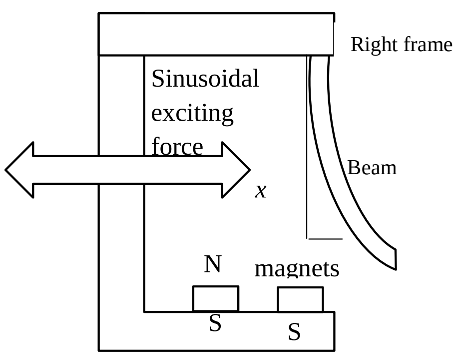


Fig 2: For $a < 0$, the Duffing oscillator can be regarded as a model of a periodically forced steal beam which is deflected towards two magnets.

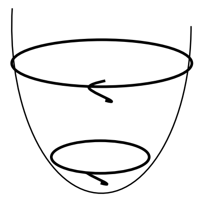
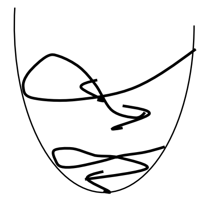
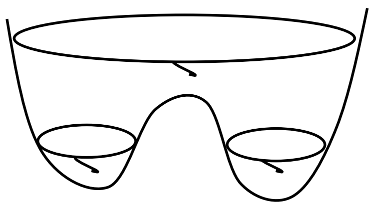
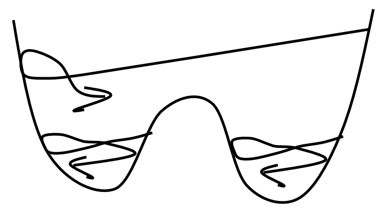
$\beta > 0$	$\delta = 0$	$\delta > 0$
$a > 0$		
$a < 0$		

Fig 3: The shape of $E(t)$ and schematic trajectories of Duffing equation is in the $\left(x, \dot{x}\ E(t)\right)$ space for $\beta > 0$

3.3 Stability Analysis of the Undamped and Unforced System Using the Eigenvalue Approach

Considering the Duffing equation of the form (3.1)
The first equivalent systems of (3.1) is given by

$$\dot{x} = y$$
$$\dot{y} = -cy - ax - bx^2 - 2x^3 + h(t) \quad (3.16)$$

For the unforced case, equation (3.3) is reduced to

$$\dot{x} = y$$
$$\dot{y} = -cy - ax - bx^2 - 2x^3 \quad (3.17)$$

At fixed points, $\dot{x} = y = 0$
So that $y = 0$ and $\dot{y} = -ax - bx^2 - 2x^3$
$= x(-ax - bx - 2x^2) = 0$
Giving us $x = 0, x_1 = \frac{-b+\sqrt{b^2-8a}}{4}$, and $x_2 = \frac{-b-\sqrt{b^2-8a}}{4}$ which correspond to (0,0), $(x_1, 0)$ and $(x_2, 0)$ at fixed point. Analysis of the stability of the fixed points can be done by linearizing equation (3.17) which gives

$$\ddot{x} = \dot{y}$$
$$\ddot{y} = -c\dot{y} - (a + 2bx + 6x^2)\dot{x} \quad (3.18)$$

The matrix equation for (3.18) can be written as

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -(a + 2bx + 6x^2) & -c \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} \quad (3.19)$$

Examining the stability at the point (0,0) gives

$$\begin{bmatrix} 0 - \lambda & 1 \\ -(a + 2bx + 6x^2) - \lambda & -c - \lambda \end{bmatrix}$$
$$\lambda c + \lambda^2 + a + 2bx + 6x^2 + \lambda = 0$$
$$\lambda^2 + \lambda(1 + c) + g = 0 \quad (3.20)$$
$$\lambda = \frac{-(1 + c) \pm \sqrt{(1 + c)^2 - 4g}}{2}$$

Where $\lambda_1 = {}^1\!/_2 \left(-(1 + c) + \sqrt{(1 + c)^2 - 4g}\right)$and $\lambda_2 = {}^1\!/_2 \left(-(1 + c) - \sqrt{(1 + c)^2 - 4g}\right)$ are the roots of equation (3.20). The above can be written as

$\lambda_1 = {}^1\!/_2\left(-\delta + \sqrt{\delta^2 - 4g}\right)$ and $\lambda_2 = {}^1\!/_2\left(-\delta - \sqrt{\delta^2 - 4g}\right)$ where $\delta = 1 + c$ and
$g = a + 2bx + 6x^2$
The coefficient of β shows that the Duffing equation is highly damped representing a hard spring. This hard spring is represented in equation (3.18) where the coefficient of x^2 is 6.

At the origin, $\lambda_{\pm}{}^{(0,0)} = {}^1\!/_2\left(-\delta \pm \sqrt{\delta^2 - 4g}\right)$ (3.21)

With $\delta = 0, \lambda_{1,2} = {}^1\!/_2\left(\pm\sqrt{-4g}\right)$
For this, we consider the following cases:
(1) When $g = 0, \quad \lambda_{1,2} = 0$ and this implies that b, x, a are all zero
(2) When $g > 0, \lambda_{1,2} = \pm i\sqrt{g}$ which corresponds to critical points that are centres for which stability is ensured.
(3) When $g < 0, \lambda_{1,2} = \pm\sqrt{g}$ which corresponds to saddles giving rise to instability [8].
With $\delta > 0, \lambda_{1,2} = {}^1\!/_2\left(-\delta \pm \sqrt{\delta^2 - 4g}\right)$
For this, we consider the following cases:
(1) When $g = 0, \lambda_{1,2} = 0, -\delta$
(2) When $g > 0, \lambda_{1,2} = {}^1\!/_2\left(-\delta \pm \sqrt{\delta^2 - 4g}\right)$
For the discriminate, we have the following cases:
When $\delta^2 < 4g, \lambda_{1,2} = {}^1\!/_2\left(-\delta \pm i\sqrt{\delta^2 - 4g}\right)$ showsthat the roots are imaginary which to lead to spiral and asymptotic stability.
When $\delta^2 > 4g, \lambda_{1,2} = {}^1\!/_2\left(-\delta \pm \sqrt{\delta^2 - 4g}\right)$ shows that the roots are real which leads to saddles and instability.
When $\delta^2 = 4g, \lambda_{1,2} = \pm\sqrt{g}$shows that the roots are real corresponding to saddles and instability.
When $g < 0, \lambda_{1,2} = {}^1\!/_2\left(\delta \pm \sqrt{\delta^2 + 4g}\right)$
For the discriminate, we consider the following cases:
(1) When $\delta^2 + 4g < 0, \lambda_{1,2} = {}^1\!/_2\left(\delta \pm i\sqrt{\delta^2 + 4g}\right)$ which corresponds to spirals and asymptotic stability.
(2) When $\delta^2 + 4g > 0, \lambda_{1,2} = {}^1\!/_2\left(\delta \pm i\sqrt{\delta^2 + 4g}\right)$ which leads to instability.
(3) When $\delta^2 + 4g = 0, \lambda_{1,2} = \pm\left(i\sqrt{g}\right)$ which leads to centres and instability.
Interestingly, for special case when $c = 0$ with no forcing term equation (3.17) becomes

$$\dot{x} = y$$
$$\dot{y} = -ax - bx^2 - 2x^3 \quad (3.22)$$

The above can be integrated by quadrature, differentiating (3.8) and plugging in (3.9) gives
$\dot{x} = \dot{y} = -ax - bx^2 - 2x^3$
Multiplying both sides by $\dot{x}$ gives $\dot{x}\dot{x} - ax\dot{x} - b\dot{x}x^2 - 2\dot{x}x^3 = 0$ (3.23)
Equation (3.23) can be written as

$$\frac{d}{dt}\left[\frac{1}{2}\dot{x}^2 - \frac{1}{2}ax^2 - \frac{1}{3}bx^3 - \frac{1}{2}x^4\right] = 0 \quad (3.24)$$

So we have a variant of motion

$$h = \frac{1}{2}\dot{x}^2 - \frac{1}{2}ax^2 - \frac{1}{3}bx^3 - \frac{1}{2}x^4 \quad (3.25)$$

solving for $\dot{x}^2$ gives $\dot{x}^2 = \left(\frac{dx}{dt}\right)^2 = 2h + ax^2 + \frac{2}{3}bx^3 + x^4$

$$\frac{dx}{dt} = \sqrt{2h + ax^2 + \frac{2}{3}bx^3 + x^4}$$

$$t = \int dt = \int \frac{dt}{\sqrt{2h + ax^2 + \frac{2}{3}bx^3 + x^4}} \quad [27]$$

Note that the invariant of motion h satisfies $\dot{x} = \frac{\partial h}{\partial \dot{x}} = \frac{\partial h}{\partial y}$ where

$$\frac{\partial h}{\partial x} = ax + bx^2 + 2x^3 = \dot{y} \quad (3.26)$$

So the equation of Duffing oscillator are given by the Hamiltonian system [27]

$$\dot{x} = \frac{\partial h}{\partial y} \quad \text{and} \quad \dot{y} = -\frac{\partial h}{\partial x}$$

Table 1: The Stability Analysis and Numerical Solutions of Duffing's Equation at Different Values a, b, c

$a = 0.1$	$x_{12} = \frac{-0.2 \pm \sqrt{-0.76}}{4}$	-0.05	*Spiral*	*Asymptotically stable*
$b = 0.2$		-0.05	*Spiral*	*Asymptotically stable*
$c = 0.1$	$x_{21} = \frac{-0.2 \pm \sqrt{-0.76}}{4}$	0.168	*Saddle*	*Unstable*
		-1.07	*Spiral*	*Stable*
$a = 0.6$	$x_{12} = \frac{-0.2 \pm \sqrt{-4.71}}{4}$	-0.0075	*Spiral*	*Asymptotically stable*
$b = 0.3$		-0.0075	*Spiral*	*Asymptotically stable*
$c = 0.5$	$x_{21} = \frac{-0.2 \pm \sqrt{4.71}}{4}$	0.468	*Saddle*	*Unstable*
		-0.618	*Spiral*	*Stable*
$a = 0.01$	$x_{12} = \frac{-0.2 \pm \sqrt{-0.0796}}{4}$	-0.02	*Spiral*	*Asymptotically stable*
$b = 0.02$		-0.02	*Spiral*	*Asymptotically stable*
$c = 0.03$	$x_{21} = \frac{-0.2 \pm \sqrt{0.0796}}{4}$	0.066	*Saddle*	*Unstable*
		-0.076	*Spiral*	*Stable*
$a = -0.2$	$x_{12} = \frac{-0.2 \pm \sqrt{-1.56}}{4}$	-0.2	*Spiral*	*Asymptotically stable*
$b = 0.2$		-0.2	*Spiral*	*Asymptotically stable*
$c = 0.03$	$x_{21} = \frac{-0.2 \pm \sqrt{1.56}}{4}$	0.262	*Saddle*	*Unstable*
		-0.362	*Spiral*	*Stable*

3.3.1 Numerical Solution Of Duffing's Equation Using Mathcad

$t_0 := 0 \qquad t_1 := 150$ Solution interval endpoints

$ic := \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ Initial condition vector

$N := 1500$ Number of solution values on $[t_0, t_1]$

$D(t, X) := \begin{bmatrix} X_1 \\ -a.X_0 - b.(X_0)^2 - 2.(X_0)^3 - c.X_1 \end{bmatrix}$ Derivative function

$S := rkfixed(ic, t_0, t_1, N, D)$

$T := S^{\langle 0 \rangle}$ Independent variable values

$X_1 := S^{\langle 1 \rangle}$ Solution function values

$X_2 := S^{\langle 2 \rangle}$ Derivative function values

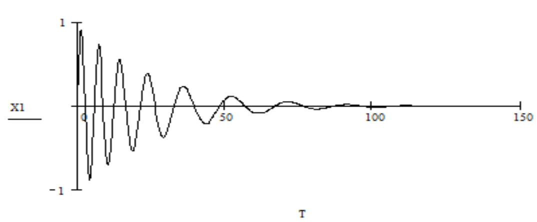


Figure 4: Trajectory profile of Duffing equation for values $a = 0.1,\ b = 0.2,\ c = 0.1$

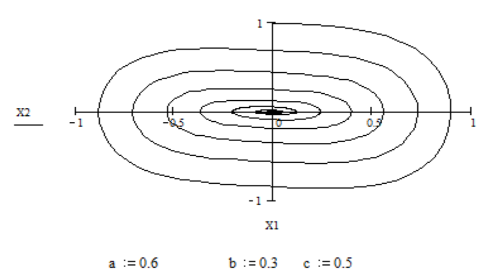


Figure 5: Phase portrait of Duffing's equation showing asymptotic stability of solution as a spiral sink.

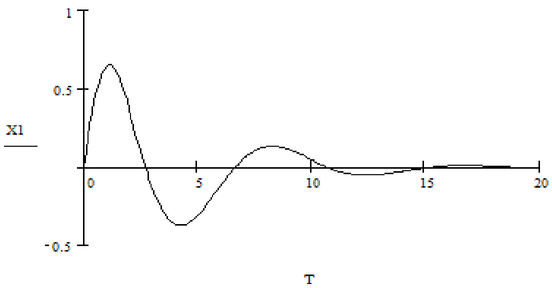


Figure 6: Trajectory Profile of Duffing Equation for values $a = 0.6,\ b = 0.3,\ c = 0.5$

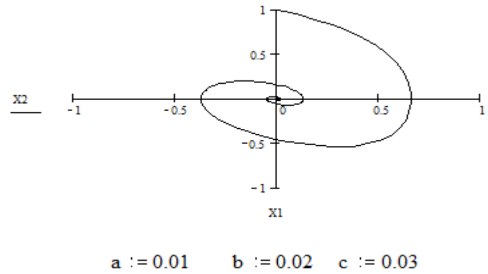


Figure 7: Phase portrait of Duffing's equation depicting asymptotic stability of solution as a spiral sink

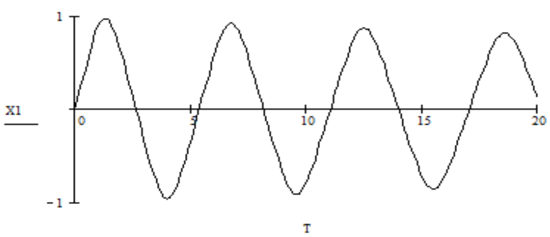


Figure 8: Oscillatory profile of Duffing equation for values $a = -0.2\ \ b = 0.2\ \ c = 0.03$

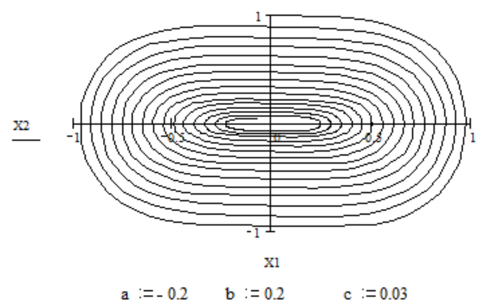


Figure 9: Phase portrait of Duffing's equation depicting a centre of a non-stable node

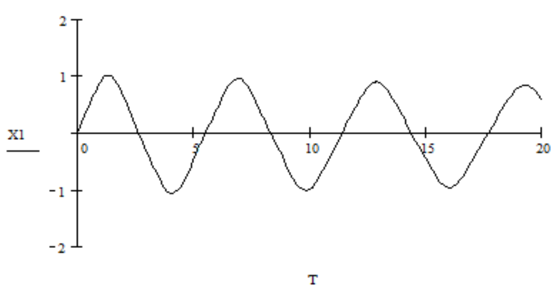


Figure 10: Oscillatory profile of Duffing equation for values $a = -0.2\ \ b = 0.2\ \ c = 0.03$

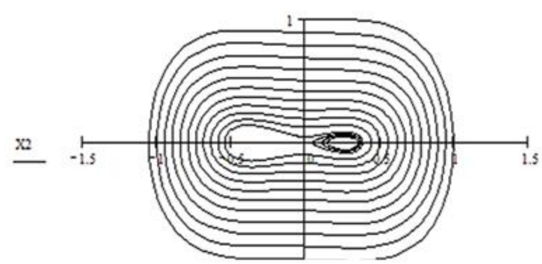


Figure 11: Phase portrait depicting the centre as an unstable node (parameters).

4.0 Discussion

The numerical tool (MATHCAD) in this paper was used to demonstrate the numerical behavior of the solution.In Figure 4, the value of $c = 0.1$ shows that the damping coefficient is high. At this point the system will oscillate but with small amplitude which returns to equilibrium as fast as possible. This degree of damping also describes the behaviour of the system. In Figure 5, the MATHCAD described the behavior of the Duffing equation when $a = 0.6, b = 0.3$ and $c = 0.5$ is periodic. We observed asymptotically stable behavior at both saddle and spiral points which are three equilibrium points. In Figure 6, the value of $c = 0.5$ shows that the system will oscillate but not as fast as the oscillation in Figure 4. This is due to the increase in the damping coefficient from $c = 0.1$ to $c = 0.5$.

In Figure 7, the dynamics of Duffing equation were shown when $a = 0.6, b = 0.3$ and $c = 0.5$. We observe asymptotically stable at spiral point. This shows that the order is revolving round. At this point, the spiral sink toward the equilibrium point. In Figure 8, the oscillatory profile of Duffing equation was shown. The damping coefficient in this case is low that is $c = 0.03$hence forcing the systemto oscillate with a decrease amplitude. The respond in this case is a sinusoid. Damping is a frictional force so it generate heat and dissipate energy. In this case, the system is undamped having spiral node. In Figure 9, the MATCAD were obtained for the values $a = -0.2, b = 0.2$ and $c = 0.03$. In this case the phase line tend to converge toward the equilibrium point due to the decrease in amplitude. This makes the phase portrait of Duffing equation depicting a centre of a non-stable node. In Figure 10, the system will oscillate with decrease in amplitude due to decrease in the damping coefficient. In this case, the system is said to be underdamped. In Figure 11, the dynamics of Duffing equation were shown for $a = 0.2, b = 0.2$ and $c = 0.03$.In this case, the phase line tends to converge to the centre as an unstable node.

The hard spring system can be linked to the spring in the shock absorber. This second order differential equation is of a form known as a conservative equation. It admits a conservation law which is an energy equation. In this hard spring, the only equilibrium point is the rest state which is the origin.

5.0 Conclusion

With the use of eigenvalue approach and numerical tool (Mathcad), we investigated the stability properties of the cubic Duffing oscillator and demonstrated the behavior of the solution. Fig 4 to Fig 11 discussed abovedescribes different behaviour of the solution which was gotten either by increase or decrease of the damping coefficient. As a result of this changes, spiral stable equilibrium point of the system was observed for most parameters.

6.0 References

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