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Optimization of Convex Functions in Infinite Dimensional Spaces

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Abstract

This paper considered the optimization of convex functions in infinite dimensional spaces. Requisite theorems were reviewed and concise proofs of the relevant results given. Some analogues of Bolzano-Weirestrass results in infinite dimensional spaces were studied using the Eberlein-Smul'yan theorem. The main thrust is on the application of Lax-Milgram theorem which guarantees the existence of a unique minimizer of a convex functional defined on the Sobolev spaces $H_0^1(\Omega)$. Finally example was given to illustrate the results. The main contribution is that every finite dimensional space problem has an analoguous infinite dimensional space version provided the right topology and assumptions are made.

Keywords: Convex Functions, Eberlein-Smul'yan Theorem, Lax-Milgram Theorem, Sobolev Spaces, Infinite Dimensional Spaces
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1.0 Introduction

It is a fact that both finite and infinite dimensional spaces play vital role in mathematical analysis and other areas of mathematics. The concept of optimization has been studied vigorously in the literature, be it in the case of finite dimensional spaces or otherwise. For instancein [1-4] there are known results concerning optimization of continuous functionalwhich now forms a must know for everystudents of classicalanalysis. On the optimization of convex functions, see [5-8]

Theorem 1.1 Let $f: [a, b] \to \mathbb{R}$be continuous then the following holds
i)$f$is bounded ie $\exists$m $\in \mathbb{R}$such that $|f(x)| \leq m \;\forall\; x \in [a, b]$
ii) There exists a point $c_1 \in [a, b]$such that $f(c_1) = \min f(x) \;\forall\; x \in [a, b]$
iii) There exists a point $c_2 \in [a, b]$such that $f(c_2) = \max f(x) \;\forall\; x \in [a, b]$
iv)$f[a, b] = |f(c_1), f(c_2)|$
v) $f$is uniformly continuous on $[a, b]$

This result carries through to finite dimensional space. Precisely, we have the following results

Theorem 1.2 (Weierstrass theorem) Let $D \subset \mathbb{R}^n$be a compact set (closed and bounded) and $f: D \to \mathbb{R}$be a continuous function. Then $f$attains a global maximum or a global minimum on Dthat is$\exists\; x_1$ and $x_2$such that
$f(x_1) \geq f(x) \geq f(x_2) \;\forall\; x \in D$

Observe that from these theorems, the function achieve its maximum and minimum on the given domain. The proof of this relies on the property of $[a, b]$or Dthat is the set is compact (closed and bounded). It is also known that in Bolzano Weierstrass theorem that every bounded sequence in $\mathbb{R}^n$has a convergent subsequence. This result is utilized rigorously in the proof of theorem (1.1) and (1.2). Attempts to move this result to infinite dimensional spaces have proved abortive because compact sets are rare to find in infinite dimensional spaces [9-17]. However, in some infinite dimensional spaces, analogue of Bolzano Weierstrass theoremexist. Examples are the Arzela Ascoli theorem and Eberlein Smu'lyan theorem. We state those analogues here without proof.

Theorem 1.3 (Arzela Ascoli theorem) Any uniformly bounded equicontinuous sequence $\{x_n\}$in a continuous $C[a, b]$ has a uniformly convergent subsequence [17]

Remark: This result is very useful for solution of ordinary differential equations.

Theorem 1.4 (Eberlein Smu'lyan theorem) A Banach space E is reflexive if and only if every norm bounded sequence in E has a subsequence which converge weakly to an element of E [12]

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Remark: This is very useful for solution of partial differential equations because all solutions of partial differential equations are found in Sobolev spaces which are reflexive spaces.
The objective of this paper is to study existence and uniqueness of solutions of convex optimization problem in the infinite dimensional space $H_0^1(\Omega)$.

2.0 Preliminaries

Definition 2.1 A bilinear form or functional B on a Hilbert space H is a mapping $B: H \times H \to \mathbb{R}$ such that $a(x, y)$ is linear in each of $x, y, w \in H$ ie for all $u_1, u_2 \in H$ and $c_1, c_2 \in \mathbb{R}$
$B(c_1u_1 + c_2u_2, w) = c_1B(u_1, w) + c_2B(u_2, w)$
$B(w, c_1u_1 + c_2u_2,) = c_1B(w, u_1) + c_2B(w, u_2)$
Definition 2.2 Let V be a normed linear spaces and $f: V \to \mathbb{R} \cup \{+\infty\}$ be an extended real valued function. Consider the optimization problem of the form
inf $f(v)$ where $v \in V$ (2.1)
A point $\bar{u} \in V$ is a local minimizer of f in v if there exists a positive constant $r > 0$ such that
$f(\bar{u}) \leq f(v)\ \forall\ v\ \in\ \mathrm{B}(\bar{u}, r) \cap V$ (2.2)
It is a global minimizer if equation (2.2) holds for all points $v \in V$. Solving an optimization problem like equation (2.1) I to find a global minimizer of f in V.
Definition 2.3 (Convex set) Let X be a real linear space and $C \subset X$. The set C is called convex if for each $x_1x_2 \in C$ and for each $t \in [0,1]$, we have
$tx_1 + (1-t)x_2 \in C$
Definition 2.4 Let D be a subset of real vector space and $f: D \to \mathbb{R} \cup \{+\infty\}$, then f is said to be convex if
(a) D is convex and
(b) for each $t \in [0,1]$ and for each $x_1x_2 \in D$ we have
$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2)$
Definition 2.5 Let $f: X \to \mathbb{R} \cup \{+\infty\}$ be a map. The effective domain of f is the set defined by
$D(f) \coloneqq \{x \in X: f(x) < +\infty\}$
Definition 2.6 A map $f: X \to \mathbb{R} \cup \{+\infty\}$ is called proper if $D(f) \neq \emptyset$
Definition 2.7 (Epigraph) The epigraph of f is the set defined by
$\mathrm{epi}(f) \coloneqq \{(x, \alpha) \in X \times \mathbb{R} : x \in D(f) \text{and } f(x) \leq \alpha\}$
Definition 2.8 (Section of f) Let $\alpha \in \mathbb{R}$, we have the following definition
$S_{f,\alpha} \coloneqq \{x \in X: f(x) \leq \alpha\} = \{x \in D(f): f(x) \leq \alpha\}$
Definition 2.9 A function $f: X \to \mathbb{R} \cup \{+\infty\}$ is lower semi-continuous at $\bar{x}$ if $\lim_{x \to \bar{x}} f(x) \geq f(\bar{x})$
Definition 2.10 Let E be a normed linear space and let J be the canonical embedding of E into E^{**}. If J is onto, then E is called reflexive. Thus a reflexive Banach space is one in which the canonical embedding is onto.
Definition 2.11 Let X be a reflexive space. A function $f: X \to \mathbb{R} \cup \{+\infty\}$ is called coercive if $\lim_{x \to +\infty} f(x) = +\infty$
Definition 2.12 The weak topology on E denoted by w is the smallest topology on E which makes the maps $\emptyset_f$ continuous.
Proposition 2.13 A mapping $f: X \to \mathbb{R} \cup \{+\infty\}$ is convex if and only if the epi(f) is convex
Proposition 2.14 If $f: X \to \mathbb{R} \cup \{+\infty\}$ is lower semi continuous at $\bar{x} \in X$ and $\{x_n\}$ is a sequence in X which converges strongly to $\bar{x}$, then $\lim_{n \to \infty} \inf f(x_n) \geq f(\bar{x})$
Theorem 2.15 Let $f: X \to \mathbb{R} \cup \{+\infty\}$ be any map. Then f is convex and lower semi continuous if and only if f is convex and weakly lower semi continuous.
Proof:f is convex and lower semi continuous $\Leftrightarrow$ epi(f) is convex and a closed set
$\Leftrightarrow$ is convex and weakly closed
$\Leftrightarrow$ f is convex and weakly lower semi continuous
Lemma 2.16 Let $K \subset E$ be convex and closed in the strong topology. Then K is closed in the weak topology ie if K is strongly closed and convex, then it is weakly closed.

3.0 Some Results in Optimization

Definition 3.1$f: X \to \mathbb{R} \cup \{+\infty\}$ is strictly convex if for each $x_1, x_2 \in D(f), x_1 \neq x_2$ and for each $t \in (0,1)$ we have$f(tx_1 + (1-t)x_2) < tf(x_1) + (1-t)f(x_2)$. Example: $f: X \to \mathbb{R} \cup \{+\infty\}$ defined by $f(x) = k$ (constant map). This f is convex but not strictly convex.
Theorem 3.2 Let $f: X \to \mathbb{R} \cup \{+\infty\}$ be strictly convex and proper. If f has a minimum at $x_0 \in X$, then x_0 is unique.
Proof: Suppose that $a \in X,\ b \in X$ are such that $a \neq b$ and $f(a) = f(b) = f(x_o) \leq f(x)\ \forall\ x\ \in X$.
Then $f(a) = f(x_0) \leq f\left(\frac{1}{2}a + \frac{1}{2}b\right) < \frac{1}{2}f(a) + \frac{1}{2}f(b) = f(x_0) = f(a)$. Contradiction, so $a = b$

Theorem 3.3 Let X be a reflexive Banach space and let K be a closed convex bounded and nonempty subset of X. Let $f: X \to \mathbb{R} \cup \{+\infty\}$ be lower semi continuous and convex. Then there exists $\bar{x} \in K$ such that $f(\bar{x}) \leq f(x)\ \forall\ x\ \in K$ ie $f(\bar{x}) = \inf f(x) = \min f(x)$

Proof:f is lower semi continuous and convex $\Longrightarrow f$ is weakly lower semi continuous. Put $m := \inf f(x) \forall\ x\ \in K$. First suppose $m = -\infty$. Then for $n \in \mathbb{N}, \exists\ x_n\ \in K$ such that

$$f(x_n) < -n \qquad (3.1)$$

Boundedness of K implies $\{x_n\}$ is bounded and by Eberlein-Smul'yan theorem $\exists\ \{x_{n_k}\}$ subsequence of $\{x_n\}$ such that $x_{n_k} \to x \in X$. But K is convex and closed implies that K is weakly closed. Hence $x \in K$. By weak lower semi-continuity of f, we have

$$f(x) \leq \lim\inf f\left(x_{n_k}\right) < -\infty$$

By equality of equation (3.1) and this is impossible since $f(x) \in\ \mathbb{R} \cup \{+\infty\}$. Hence $m \in \mathbb{R}$. We now use the definition of "inf". Let $n \in \mathbb{N}$ and take$\varepsilon_n = \frac{1}{n}$, then $\exists x_n\ \in K$ such that

$$m \leq f(x_n) < m + \frac{1}{n}$$

The sequence $\{x_n\}$ in K implies $\{x_n\}$ is bounded and so $\exists\ \{x_{n_k}\}_{k\in\mathbb{N}}$ subsequence of $\{x_n\}$ and $\bar{x} \in K$ such that $x_{n_k} \to \bar{x}$. Since f is weakly lower semi-continuous we have

$$f(\bar{x}) \leq \lim_{k\to\infty}\inf\,(m + \frac{1}{n_k}) = m$$

Thus $(\bar{x}) \leq m = \inf f(x) \forall\ x\ \in K$. So $m \leq f(\bar{x}) \leq m \Longrightarrow f(\bar{x}) = m$

Theorem 3.4 Let X be a real reflexive Banach space and Let $f: X \to\ \mathbb{R} \cup \{+\infty\}$ be a convex, proper and lower semi-continuous function. Suppose $\lim_{\|x\|\to\infty} f(x) = +\infty$, then $\exists\ \bar{x} \in X$ such that $f(\bar{x}) \leq f(x) \forall\ x\ \in X$ ie $f(\bar{x}) = \inf f(x)$

Proof: Applying contraction mapping principle, Since f is proper $\exists\ x_0 \in X$ such that $f(x_0) \in \mathbb{R}$ ie $f(x_0) \neq +\infty$
Let $K := \{x \in X : f(x) \leq f(x_0)\}$ We now show that K is closed, convex, nonempty and bounded set. But K is a section with $\alpha := f(x_0)$. So it is convex and closed since f is convex and lower semi-continuous.

Claim: K is bounded. Suppose this is not the case, then for each $n \in \mathbb{N}, \exists\ x_n\ \in K$ such that $\|x_n\| > n$. Thus since $x_n\ \in K$, we have that

$$f(x_n) \leq f(x_0) \qquad (3.2)$$

This implies that $\lim_{n\to+\infty} \|x_n\| = +\infty$ and so by hypothesis

$$\lim_{n\to+\infty} f(x_n) = +\infty \qquad (3.3)$$

Contradicting inequality (3.2). Hence K is bounded. By theorem (3.4), we have that $\exists\ \bar{x} \in K \subset X$ such that $\forall\ x\ \in K f(\bar{x}) \leq f(x)$. Now, let $x \in X\backslash K$. Then $f(x) > f(x_0)$. But $x_0 \in K$ So $f(\bar{x}) \leq f(x_0)$.
Hence $f(x) > f(\bar{x}) \forall\ x\ \in X$ ie $f(\bar{x}) \leq f(x) \forall x\ \in X$.

4.0 Main Result

Lemma 4.1 (Riesz-Frechet) For every continuous linear form $f$on**the** Hilbert space H,**there** exist a unique $y \in H$such that**for** all $x \in H$,the following identity holds

1. $f(x) = \langle x, y\rangle$
2. $\|y\| = \|f\|$

Lemma 4.2 Let V be a given Hilbert spaces with scalar product (.,.) and corresponding norm $\|.\|$. Furthermore, let there ba a giving mapping $a: V \times V \to \mathbb{R}$ with the following properties:
(i) For an arbitrary $u \in V$, both $a(u, .)$ and $a(., u)$ define linear functional on V
(ii) There exists a constant $M > 0$such that $|a(u, v)| \leq M\|u\|\|v\|$for all u, v $\in V$
(iii) There exists a constant $r > 0$ such that $a(u, u) \geq r\|u\|^2$ for all $u \in V$. A mapping $a(.,.)$ satisfying (i) and (ii) is called a continuous bilinear form. The essential property (iii) is called V-ellipticity of connectivity.

Theorem 4.3 (Lax-Milgram) Let $a: V \times V \to \mathbb{R}$ be a continuous, α elliptic bilinear form. Then for each $f \in V^*$, the variational equation

$$a(u, v) = f(v) \text{ for all } v \in V \qquad (4.1)$$

has a unique solution $u \in V$. Furthermore, a priori estimate

$$\|u\| \leq \frac{1}{r}\|f\| \qquad (4.2)$$

Proof: First we show that the solution of equation (4.1) is unique. Suppose that $u \in V$ and $\bar{u} \in V$ are both solutions. Then the linearity of $a(.v)$ implies that $a(\bar{u} - u, v) = 0$ for all $v \in V$. Choosing $v := \bar{u} - u$, we get $a(v, v) = 0$ which by $\alpha -$ellipticity implies that $v = 0$ as desired. Note that α-ellipticity, however is stronger than the condition"$a(v, v) = 0$implies $v = 0$". To prove the existence of a solution to (4.1), we use Banach fixed point theorem. Therefore, we need to choose a contractive mapping that has as a fixed point a solution of equation (4.1). For each $y \in V$, the assumption (i) and (ii)

for bilinear form guarantee that $a(y,.) - f \in V^*$. Hence Riesz's theorem ensures the existence of a solution $z \in V$ of
$(z,v) = (y,v) - r[a(y,v) - f(v)]\ \forall\, v \in V$ (4.3)
for each real $r > 0$. Now we define the mapping $T_x: V \to V$by $T_r y := z$and study its properties especially contractivity. The relation (4.3) implies
$(T_r y - T_r w, v) = (y - w, v) - ra(y - w, v)$ for all $v, w \in V$ (4.4)
Given $p \in V$, by applying Riesz's theorem again we define an auxillary linear operator
$S: V \to V$ by $(S_P, v) = a(p,u)\ \forall\, v \in V$ (4.5)
Property (ii) of the bilinear form implies that
$\|S_P\| \le M\|p\|\ \forall\, p \in V$. (4.6)
The definition of operator S means that equation (4.4) can be written as $(T_r y - T_r w, v) = \Big(y - w - rS\big((y - w), v\big)\Big)\ \forall\, v, w \in V$. This allows us to investigate whether $T_r$is contraction:
$\|T_r y - T_r w\|^2 = (T_r y - T_r w, \quad T_r y - T_r w)$
$= (y - w - rS(y - w), \quad y - w - rS(y - w))$
$= \|y - w\|^2 - 2r(S(y - w), y - w) + r^2(S(y - w), S(y - w))$
By equations (4.5) and (4.6) this yields:
$\|T_r y - T_r w\|^2 \le \|y - w\|^2 - 2ra(y - w, y - w) + r^2M^2\|y - w\|^2$. Finally invoking the V-ellipticity of $a(.,.)$ we get $\|T_r y - T_r w\|^2 \le (1 - 2ry + r^2M^2)\|y - w\|^2\ \forall\, y, w \in V$. Consequently, the operator
$T_r : V \to V$ is contractive if $0 < r < \frac{2r}{M^2}$. Choose $= \frac{r}{M^2}$. Now Banach fixed point theorem tells us that there exist $u \in V$ with $T_r u = u$ since$r > 0$. The definition (4.3) of T_r implies that
$a(u,v) = f(v)\ \forall\, v \in V$ (4.7)

5.0 Overview of Sobolev Spaces

Let Ω be an open set in $\mathbb{R}$. Let k be a natural number and let $1 \le p < \infty$, the Sobolev space $W^{k,p}(\Omega)$ is defined to be the set of all functions f on Ω such that every index α with $|\alpha| \le k$ is denoted by
$W^{k,p}(\Omega) := \{u \in \mathcal{L}^p(\Omega) : D^\alpha u \in \mathcal{L}^p(\Omega)\ \forall\, \alpha$ with $|\alpha| \le k\}$. The natural number k is called the Sobolev space [16]
Definition 5.1 The $W^{1,2}(\Omega)$ is defined as $W^{1,2}(\Omega) := \{u \in \mathcal{L}^p(\Omega) : D^\alpha u \in \mathcal{L}^p(\Omega)\ \forall\, \alpha$ with $|\alpha| \le 1\}$. Usually we denote $W^{1,2}(\Omega) = H^1(\Omega)$. $H^1(\Omega)$ is a separable Hilbert space when it is equipped with inner product functions in the Sobolev space H. The definition of $H^1(\Omega)$ is based on the inner product
$\langle f, g\rangle_{H^1} = \int f(x)g(x) + \nabla f(x) + \nabla g(x)dv$ over Ω (5.1)
where $\Omega \subset \mathbb{R}^n$and $f, g : \Omega \to \mathbb{R}$.
Note that for n= 1.The inner product simply
$\langle f, g\rangle_{H^1} = \int_a^b f(x)g(x) + f'(x)g'(x)dx$ (5.2)
Based on the inner product of equation (5.2) the $H^1 -$ norm
$\|f\|_{H^1} = \sqrt{\langle f, g\rangle_{H^1}}$ (5.3)
is defined and we obtain the Sobolev space.
Definition 5.2$H^1(\Omega) = \{f: \Omega \to \mathbb{R} : \|f\|_{H^1} < \infty$ of all function Ω for which $H^1 -$ norm is finite.
Definition 5.3 The Sobolev space $W_0^{k,p}(\Omega)$ is defined as the completion of $C_0^\infty(\Omega)$ in the norm of $W_0^{k,p}(\Omega)$ie $\|\cdot\|_{W^{k,p}(\Omega)}$ and $W_0^{k,p}(\Omega) = C_0^\infty(\Omega)$
Definition 5.4 The space $C_0^\infty(\Omega)$ is the space of infinitely often differential real function with compact (closed and bounded) support in $\Omega$is denoted by $C_0^\infty(\Omega) = \{v : v \in C_0^\infty(\Omega), \sup(v) \subset \Omega\}$ where $\text{supp}(v) = \{x \in \Omega : v(x) \ne 0\}$
Definition 5.5$H_0^1(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $(H_0^1(\Omega), \|\cdot\|_{H^1})$. $H_0^1(\Omega)$ is a Hilbert space for the reduced norm$|\cdot|_{H^1,\Omega}$. It can be considered as subspace of $H^1(\Omega)$ comprising of all those functions that vanish (in certain cases).
Note: $H_0^1(\Omega) \equiv H^1(\Omega)$ because their norms are equivalent.

6.0 Applications

The achievements of Lax and Milgram result in [15]was to specify conditions for this weak formulations to have a unique solution that depends continuously upon the specified data $f \in V^*$

6.1 Optimization in real Hilbert Spaces

We now consider the special case in which the real reflexive space is a real Hilbert space. Let H be a real Hilbert space, we know that H is reflexive. We now consider the following examples of mapping defined on H.
Example 1 (Projection onto closed, convex subset H)
Let K be a nonempty, closed convex subset of H. Let $f : H \to \mathbb{R} \cup \{+\infty\}$ be defined for arbitrary $x \in H$ and fixed $u \in H$ by

$$f(x) := \begin{cases} \|x-u\| & if\ x \in K \\ +\infty & elsewhere. \end{cases}$$

Then f is convex, lower semi-continuous, proper and $\frac{f(x)}{\|x\| \to +\infty} = +\infty$

Thus, there exists $x_0 \in K$ such that $f(x_0) = \min_{x \in K} f(x)$

Verification:

a. Convexity of f follows from that of $\|\cdot\|$

b. Lower semi-continuity of f follows from that of $\|\cdot\|$ since the $\|\cdot\|$ is continuous and every continuous function is lower semi-continuous.

c. To show that f is proper. From the problem, since K is non-empty, let $x_0 \in K$, then $f(x_0) = \|x_0 - u\| < \infty$.

d. To show that f is coercive, ie $\lim_{\|x\| \to \infty} f(x) = +\infty$

Case 1: Suppose x is not in K, then the result is trivial because the image of f is infinity no matter how x behaves

Case 2: If $x \in K$, then $f(x) = \|x - u\|$
$\geq \|x\| - \|u\| \to \infty\ as\ \|x\| \to \infty \Longrightarrow f(x) \to \infty\ as\ \|x\| \to \infty$.

Example 2. Let $\Omega \subset \mathbb{R}^n$be a bounded open set. Let us consider the Dirichlet problem

$-\Delta u = f\ in\ \Omega$

$u = 0\ on\ \partial\Omega$

Where $f: \Omega \to \mathbb{R}$ is a given function and $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_n^2}$

The solution space could be taken to be the Sobolev space $H_0^1(\Omega)$ with dual $H^*(\Omega)$ and the former a subspace of L^p space ie $V = L^2(\Omega)$. The bilinear form associated to $-\Delta$ is $L^2(\Omega)$ inner product of the derivatives

$B(u,v) = \int \nabla \mathrm{u}\ \nabla \mathrm{v(x)dx}$ (6.1)

A classical solution of equation (6.1) is a function $x \in C^2(\Omega)$ which satisfies equation. Let u be a classical solution of the Dirichlets problem. If we multiply equation (6.1) by $\phi \in C_0^\infty(\Omega)$ and integrate along Ωwe get

$-\int \nabla u\phi = \int f\phi$ (6.2)

From Green's theorem, we have

$$\int \nabla u\, \nabla \phi\ = -\int \nabla u\phi + \int \phi_{\partial\Omega}^{\partial u}$$

Using the fact that $\phi = 0$ on $\partial\Omega$, we deduce that

$\int \nabla u\ \nabla \phi = \int f(\phi)$ (6.3)

Now since $u \in C^2(\Omega)$ and $u = 0$ and $\partial\Omega$, it follows that $u \in H_0^1(\Omega)$ and

$$\int \nabla u\ \nabla v = -\int f(v)\ \forall v \in H_0^1(\Omega)$$

Then we say that u is a weak solution of equation (6.1) if $u \in H!(\Omega)$ and satisfies $\int \nabla u\ \nabla v = -\int f(v)$ $\forall\ v \in\ H!(\Omega)$ which means that the solution is $\min_{u \in H_0^1(\Omega)} J(u) := \frac{1}{2}\int |\nabla u|^2\, dx - fudx$. It is worthy to note that J is convex, continuous and coercive (coercivity of J is a consequences of Pointcare-Friedrich inequality). Thus the existence of a unique minimizer is ensured by Lax Milgram Theorem.

7.0 Conclusion

We have been able to study some analogue of the Bolzano-Weierstrass results in infinite dimensional spaces. We reviewed some results in optimization using precisely, the Eberlin-Smul'yan theorem which showed a major outcome.

Furthermore, we studied optimization in the classical reflexive Banach space $H_0^1(\Omega)$using one of the results in functional Analysis, the Lax-Milgram theorem which were illustrated with examples.

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