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A Mathematical Model of Competitive Advertising and Quality Using Differential Game Theory

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Abstract

This work incorporates quality into Sorger's sales-advertising dynamics, and uses differential game theory to model the relationship between advertising and quality in a duopolistic market. The dynamics is based on Sorger's model which is an extended version of Sethi's sales-advertising model. It obtains a Nash equilibrium for the advertising efforts, quality efforts and payoffs. It shows that a firm can use advertising to compensate for low product quality. The work further shows that a firm's advertising effort should reduce with the quality, and vice versa. In addition, a firm is safe to reduce his quality effort if he observes a reduction in his competitor's advertising effort. Similarly, a firm should increase his advertising effort if his competitor's quality effort increases.

Keywords: Advertising; Quality; Differential game; Sorger's model; Sethi's sales-advertising model.

1.0 Introduction

There are two central views on the relationship between advertising and quality. These are on the classical question of whether better quality products are more intensely advertised or not. One of these is the informative view [1, 2]**.** The other is the persuasive view [3]. There is a school of thought with the view that better quality products are more heavily advertised [4]. On the contrary, there is another school of thought with the opinion that low quality products are more heavily advertised [5]. In this regards both positive and negative relationships exist [6]. However, theoretical studies could not provide any common explanation on these conflicting or opposing relationships[7, 8].

Similar works to this paper have been considered on advertising expenditure [9-15]. Although concepts such as advertising and quality have been considered by individual models, there have been few attempts to consider them together using dynamic models on competition. This work will for the first time take advantage of Sethi model [16] by using Sorger's model [11] which is a special form of the Lanchester model [17]to incorporate quality into Sethi's advertising-sales dynamics of awareness in a differential game.

This work is based on the persuasive view which says that product of better quality is not more heavily advertised [5]. The idea is that if engaging in advertising can result in better preference for substitutable products of the same class and objective characteristics, then it (advertising) can lead to subjective product differentiation. In such a situation, a firm may use higher advertising to compensate for the effect of lower product quality.

2.0 The Model

To increase their individual payoffs, firms usually engage in advertising and quality spending. This will increase the market awareness of the firms' products. Firm $i,\ i = 1, 2$ strategizes on his advertising effort $a_i(t)$ and quality effort $q_i(t)$.

It is common in the advertising literature to assume the advertising cost function to be quadratic [12, 13].Thus we have that the advertising cost function C_a is given by

$$C_a(a_i) = a_i(t)^2 \quad (1)$$

Also in accordance with the existing literature [18, 19] the quality cost function C_q is given by

$$C_q(q_i) = q_i(t)^2 \quad (2)$$

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To model the dynamic effect of advertising on sale we will employ an extension which due to [11].This (Sorger's model) is given by

$$x'(t) = \beta_1 a_1(x,y)\sqrt{1-x} - \beta_2 a_2(x,y)\sqrt{x}\,, \qquad x(0) = x_0 \tag{3}$$

$$y(t) = \beta_2 a_2(x,y)\sqrt{1-y} - \beta_1 a_1(x,y)\sqrt{y}\,, \qquad y(0) = 1 - x_0 \tag{4}$$

This work will consider a duopolistic market structure in which two firms compete for a share of the market through advertising and quality. Let $x_i(t),\ i = 1,2$ be firm i's market share at time $t \geq 0$. Thus the awareness share dynamics of Firm 1 is given by

$$x_1'(t) = x'(t) = \beta\big(a_1(t) + q_1(t)\big)\sqrt{1 - x(t)} - \beta\big(a_2(t) + q_2(t)\big)\sqrt{x(t)}, \quad x(0) = x_0, \tag{5}$$

while the awareness share dynamics of Firm 2 is given by

$$x_2'(t) = y'(t) = \beta\big(a_2(t) + q_2(t)\big)\sqrt{1 - y(t)} - \beta\big(a_1(t) + q_1(t)\big)\sqrt{y(t)}, \ \ y(0) = y_0 \tag{6}$$

where β is the advertising effectiveness, x_0 is the initial proportion of the market share of Firm 1,y_0 is the initial proportion of the market share of Firm 2.

We observe that the firms' awareness shares are non-decreasing in advertising and quality. The dynamics of the firms' market shares are linearly affected by their advertising efforts $a_i(t)$ and quality efforts $q_i(t)$. These are the square roots of the advertising and quality expenditure $a_i(t)^2$ and $q_i(t)^2$ respectively. Thus the awareness share dynamics is a concave function of the advertising and quality expenditure.

Now, recall that this is a duopolistic market. Considering the competition between the two firms we have that $x_1(t) + x_2(t) = 1, t \geq 0$. Further since $x_1(0) + x_2(0) = 1$, it follows that $x_1(t) = 1 - x_2(t)$. This implies that the dynamics of the awareness of both firms can be described using that of any of the firms. Thus we use firm 1's dynamics. Hence we have

$$x'(t) = \beta\big(a_1(t) + q_1(t)\big)\sqrt{1 - x(t)} - \beta\big(a_2(t) + q_2(t)\big)\sqrt{x(t)}, \qquad x(0) = x_0 \tag{7}$$

This idea of using one firm's dynamics to describe that of more than one firm can be found in [12, 14]**.**Let m_1 and m_2 denote the profit margins of Firm 1 and Firm 2 respectively. The objective functions of Firm 1 and Firm 2 are given by

$$V_1(x) = \max_{a_1,q_1} \int_0^\infty e^{-rt}[m_1 x(t) - a_1(t)^2 + q_1(t)^2]dt \tag{8}$$

and

$$V_2(x) = \max_{a_2,q_2} \int_0^\infty e^{-rt}\big[m_2\big(1 - x(t)\big) - a_2(t)^2 + q_2(t)^2\big]dt \tag{9}$$

respectively, and are subject to (7), where r is the discount rate.

3.0 Players' Advertising Efforts, Quality Efforts and Payoffs

From (7) and (8)we have the HJB equation

$$rV_1(x) = \max_{a_1(t)\geq 0, q_1(t)\geq 0} \Big\{m_1 x(t) - a_1(t)^2 + q_1(t)^2 + V_{1x}\Big[\big(\beta a_1(t) + q_1(t)\big)\sqrt{1 - x(t)} - \big(\beta a_2(t) + q_2(t)\big)\sqrt{x(t)}\Big]\Big\} \tag{10}$$

Maximizing (10) wrt a_1 we have

$$-2a_1 + V_{1x}\beta\sqrt{1-x} = 0$$

$$\Rightarrow \quad a_1(x) = \frac{V_{1x}\beta\sqrt{1-x}}{2} \tag{11}$$

Also maximizing (10) wrt q_1 we have

$$2q_1 + V_{1x}\sqrt{1-x} = 0$$

$$\Rightarrow \quad q_1(x) = -\frac{V_{1x}\sqrt{1-x}}{2} \tag{12}$$

From (7) and (9) we have the HJB equation

$$rV_2(x) = \max_{a_2(t)\geq 0, q_2(t)\geq 0} \Big\{m_2\big(1 - x(t)\big) - a_2(t)^2 + q_2(t)^2 + V_{2x}\Big[\big(\beta a_1(t) + q_1(t)\big)\sqrt{1 - x(t)} - \big(\beta a_2(t) + q_2(t)\big)\sqrt{x(t)}\Big]\Big\} \tag{13}$$

Maximizing (13) wrt a_2 we have

$$-2a_2 - V_{2x}\beta\sqrt{x} = 0$$

$$\Rightarrow \quad a_2(x) = -\frac{V_{2x}\beta\sqrt{x}}{2} \tag{14}$$

Also maximizing (13) wrt q_1 we have

$$2q_2 - V_{2x}\sqrt{x} = 0$$

$$\Rightarrow \quad q_2(x) = \frac{V_{2x}\sqrt{x}}{2} \tag{15}$$

Putting(11), (12), (14) and (15) in (10) we have

$$rV_1 = m_1x(t) - \left(\frac{V_{1x}\beta\sqrt{1-x}}{2}\right)^2 + \left(-\frac{V_{1x}\sqrt{1-x}}{2}\right)^2$$
$$+V_{1x}\left[\left(\frac{V_{1x}\beta^2\sqrt{1-x}}{2} - \frac{V_{1x}\sqrt{1-x}}{2}\right)\sqrt{1-x(t)} - \left(-\frac{V_{1x}\beta^2\sqrt{x}}{2} + \frac{V_{1x}\sqrt{x}}{2}\right)\sqrt{x(t)}\right]$$
$$= m_1x + \frac{V_{1x}^2\beta^2(1-x)}{4} - \frac{V_{1x}^2(1-x)}{4} + \frac{(\beta^2-1)V_{1x}V_{2x}x}{2} \tag{16}$$

Putting (11), (12), (14) and 15) in (13) we have

$$rV_2(x) = m_2\big(1-x(t)\big) - \left(-\frac{V_{2x}\beta\sqrt{x}}{2}\right)^2 + \left(\frac{V_{2x}\sqrt{x}}{2}\right)^2$$
$$+V_{2x}\left[\left(\frac{V_{1x}\beta^2\sqrt{1-x}}{2} - \frac{V_{1x}\sqrt{1-x}}{2}\right)\sqrt{1-x(t)} - \left(-\frac{V_{2x}\beta^2\sqrt{x}}{2} + \frac{V_{2x}\sqrt{x}}{2}\right)\sqrt{x(t)}\right]$$
$$= m_2\big(1-x(t)\big) + \frac{V_{2x}^2\beta^2x}{4} - \frac{V_{2x}^2x}{4} + \frac{V_{1x}V_{2x}\beta^2(1-x)}{2} - \frac{V_{1x}V_{2x}(1-x)}{2} \tag{17}$$

Thus:

Proposition 3.1 In a duopolistic competition involving whose games are given by the control problems (7), (8) and (9), the players' advertising efforts are given by (11) and (14); their quality efforts are given by (12) and (15); and the payoffs are given by (16) and (17).

Considering (11) and (14)we observe that the firm's advertising efforts increase as the rate of increase of the payoffs and the advertising effectiveness increase. We also observe that the awareness share is pivotal to advertising spending. Obviously, the advertising efforts increase with reduction in the awareness and vice versa. Thus the firm is obliged to spend more on advertising when the awareness is low. This is to woo the consumers into buying the firm's product. However this spending tends to zero as the awareness tends to 1. That is advertising expenditure increased awareness. Of course, this is quite rational from the fact that it would be unwise to invest in advertising whereas almost the entire market is aware of the firm's product.

Observe that the quality effort increase with the rate of increase of a firm's payoff. It also depends on the awareness share.

Further considering (16) and (17)we observe that the players' margins are essential tools through which their payoffs can be decided. Obviously, a player with a large margin will have a large payoff.

Now, let

$$V_1(x) = C_1 + B_1x \tag{18}$$
$$\Rightarrow \quad V_{1x} = B_1 \tag{19}$$
$$V_2(x) = C_2 + B_2(1-x) \tag{20}$$
$$\Rightarrow \quad V_{2x} = B_2 \tag{21}$$

Using (18), (19) and (21) in (16) we have

$$r(C_1 + B_1x) = m_1x + \frac{B_1^2\beta^2(1-x)}{4} - \frac{B_1^2(1-x)}{4} + \frac{(\beta^2-1)(-B_2)B_1x}{2} \tag{22}$$

Equating the coefficients of x we have

$$B_1 = \frac{4m_1}{4r + (\beta^2-1)B_1 + 2B_2(\beta^2-1)} \tag{23}$$

Equating constants we have

$$C_1 = \frac{\beta^2B_1^2 - B_1^2}{4r} \tag{24}$$

Using (19), (20) and (21) in (17) we have

$$r\big(C_2 + B_2(1-x)\big) = m_2(1-x) + \frac{\beta^2(-B_1)^2x}{4} - \frac{(-B_2)^2x}{4} + \frac{\beta^2(-B_2)B_1(1-x)}{2}$$
$$-\frac{(-B_2)B_1(1-x)}{2} \tag{25}$$

Equating the coefficients of x in (25) we have

$$B_2 = \frac{4m_2}{4r + (\beta^2-1)B_2 + 2(\beta^2-1)B_1} \tag{26}$$

Equating the constants in (25) we have

$$C_2 = \frac{2m_2 - \beta^2B_1B_2 + B_1B_2 - 2rB_2}{2r} \tag{27}$$

Thus using (19) in (11) and (12) we

$$\Rightarrow a_1(x) = \frac{B_1\beta\sqrt{1-x}}{2} \tag{28}$$

and

$$\Rightarrow \quad q_1(x) = -\frac{B_1\sqrt{1-x}}{2} \tag{29}$$

respectively.

Also, using (21) in (14) and (15) we have

$$\Rightarrow a_2(x) = -\frac{B_2\beta\sqrt{x}}{2} \tag{30}$$

and

$$\Rightarrow \quad q_2(x) = \frac{B_2\sqrt{x}}{2}. \tag{31}$$

respectively.

Thus:

Proposition 3.2: Suppose linear functions (18) and (20) are used to approximate the players' payoffs in a duopolistic competition whose games are given by the control problems (7), (8) and (9). Then the advertising efforts are given by (28) and (30), the quality efforts are given by (29) and (31); and payoffs are given by (18) and (20), where B_1, B_2, C_1 and C_2 are given by (23), (24), (26) and (27) respectively.

From (28) we observe that as in the case of (11)a_1 increases with B_1. Now (23) shows that increase in B_1 depends on increase in m_1 and reduction in r (foresightedness). The reverse is the case with q_1 as can be seen in (29). q_1 decreases with m_1. By extension similar explanations also go for (30) and (31)respectively. That is while a_2 increases with m_2, q_2 reduces.

Now, we observe from (18) and (20) that B_1 and B_2 respectively are very important to the players. This is because as B_1 increase Firm 1's payoff increases. This also applies to B_2 in the case of Firm 2.

Now observe from (23)that as m_1 increases, B_1 increases and thus V_1 increases. Thus as m_1 increases, V_1 also increases.

4.0 Awareness (Market) Share

Using (28) to (31) in (5) we have

$$x'(t) = \left(\frac{B_1\beta^2\sqrt{1-x}}{2} + \frac{B_1\sqrt{1-x}}{2}\right)\sqrt{1-x} - \left(-\frac{B_2\beta^2\sqrt{x}}{2} - \frac{B_1\sqrt{x}}{2}\right)\sqrt{x}$$

$$= \frac{\beta^2B_1 + B_1}{2} - \frac{\beta^2B_1 + B_1 - \beta^2B_2 - B_2}{2}x \tag{32}$$

We use the integrating factor

$$I.F. = \exp\left(\int \frac{\beta^2B_1 + B_1 - \beta^2B_2 - B_2}{2}dt\right) = \exp\left(\frac{\beta^2B_1 + B_1 - \beta^2B_2 - B_2}{2}\right)t \tag{33}$$

Multiplying (32) by (33) we have

$$\exp\left(\frac{\beta^2B_1 + B_1 - \beta^2B_2 - B_2}{2}t\right)x'(t) + \exp\left(\frac{\beta^2B_1 + B_1 - \beta^2B_2 - B_2}{2}t\right)\left(\frac{\beta^2B_1 + B_1 - \beta^2B_2 - B_2}{2}x\right)$$

$$= \exp\left(\frac{\beta^2B_1 + B_1 - \beta^2B_2 - B_2}{2}t\right)\left(\frac{\beta^2B_1 + B_1}{2}\right). \tag{34}$$

Integrating we have

$$x(t) = \frac{\beta^2B_1 + B_1}{\beta^2B_1 + B_1 - \beta^2B_2 - B_2} + \frac{C}{\exp\left(\frac{\beta^2B_1 + B_1 - \beta^2B_2 - B_2}{2}t\right)} \tag{35}$$

At $t = 0$, $x = x_0$. Thus

$$C = x_0 - \frac{\beta^2B_1 + B_1}{\beta^2B_1 + B_1 - \beta^2B_2 - B_2} \tag{36}$$

Using (36) in (35) we have

$$x(t) = \frac{(\beta^2+1)B_1}{(\beta^2+1)B_1 - (\beta^2+1)B_2} + \frac{[(\beta^2+1)B_1 - (\beta^2+1)B_2]x_0 - (\beta^2+1)B_1}{(\beta^2+1)B_1 - (\beta^2+1)B_2}$$

$$\times \exp\left(-\frac{\beta^2B_1 + B_1 - \beta^2B_2 - B_2}{2}t\right) \tag{37}$$

As $t \to \infty$ we have that

$$x(t) = \frac{(\beta^2+1)B_1}{(\beta^2+1)B_1 - (\beta^2+1)B_2} \tag{38}$$

Recall that $y(t) = 1 - x(t)$. Thus from (37) we have

$$y(t) = 1 - \left\{\frac{(\beta^2+1)B_1}{(\beta^2+1)B_1 - (\beta^2+1)B_2} + \frac{[(\beta^2+1)B_1 - (\beta^2+1)B_2]x_0 - (\beta^2+1)B_1}{(\beta^2+1)B_1 - (\beta^2+1)B_2}\right.$$
$$\left.\times \exp\left(-\frac{\beta^2 B_1 + B_1 - \beta^2 B_2 - B_2}{2}t\right)\right\}$$
$$= \frac{(\beta^2+1)B_2}{(\beta^2+1)B_2 - (\beta^2+1)B_1} + \frac{[(\beta^2+1)B_2 - (\beta^2+1)B_1]x_0 + (\beta^2+1)B_1}{(\beta^2+1)B_1 - (\beta^2+1)B_2}$$
$$\times \exp\left(-\frac{\beta^2 B_2 + B_2 - \beta^2 B_1 - B_1}{2}t\right) \tag{39}$$

As $t \to \infty$, we have that

$$y(t) = \frac{(\beta^2+1)B_2}{(\beta^2+1)B_2 - (\beta^2+1)B_1} \tag{40}$$

From the forgoing we have:

Proposition 3: In a duopolistic competition whose games are given by the control problems (7), (8) and (9), the awareness share of a given player is given by (36) which becomes (37) in the long run; while that of his competitor is similarly given by (38) which eventually becomes (39) in the long-run.

Considering (36) and (37)as well as (38) and (29)we observe that advertising and quality efforts have limited effects on the awareness share. The knowledge of this will save the players from unnecessary advertising and quality expenditure. Thus it is necessary for a player to bear in mind that his long-run awareness is central to his success in the game.

5.0 Illustration

Note that the advertising effectiveness $\beta \in [0,1]$. Thus we let $\beta = 0.2$. Since the two players are considered to be distinct we let the values of their margins to be different. Thus $m_1 = 0.3$ and $m_2 = 0.6$. The game is to be played on an infinite horizon, and being that the players are rational they are considered to be foresighted. This implies a low discount rate. As such we take $r = 0.07$.

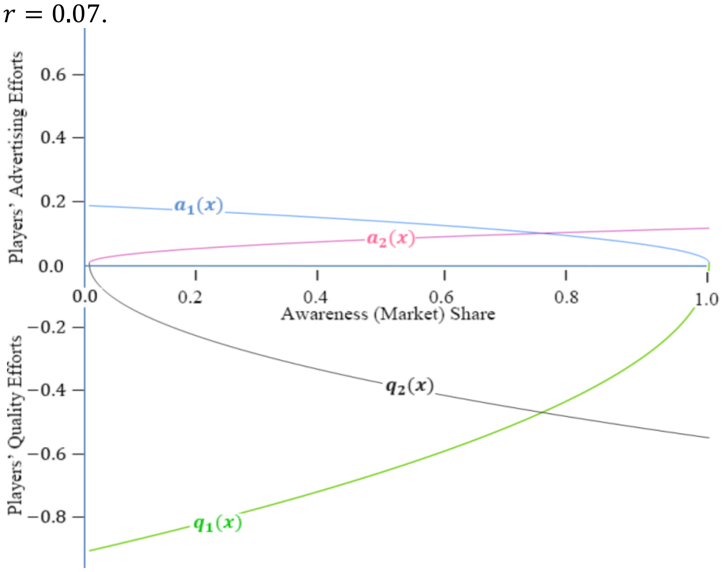

Figure 1: Relationship between the players' awareness shares, advertising efforts and quality efforts

From Figure 1 we observe that as Firm 1's awareness share increases, his advertising effort reduces while the quality effort increases. This supports our earlier claim. On the other hand Firm 2's advertising effort increases while his quality effort reduces. Thus an increase in Firm 1's awareness share leads to increase in his advertising effort, but reduction in Firm 2's quality effort. That is a reduction in Firm 1's advertising effort leads to a reduction in Firm 2's quality. Similarly, Firm 2's advertising effort increases with an increase in Firm 1's awareness. This is to gain a "reasonable" portion of the market which has been taking by Firm 1. That is a firm has the tendency to engage more in advertising if his competitor's market share is dominantly large (or equivalently if his own market share is small). Further we observe that Firm 2's quality effort reduces with his (increasing) advertising effort.

In essence a firm still has the tendency to retain his market share if he should reduce his quality on observation that his competitor has reduced his advertising effort. That is equilibrium is maintained in this give and take situation.

6.0 Conclusion

The work modeled the relationship between awareness share, quality and advertising in a duopolistic market setting using differential game theory. It successfully incorporated quality effort – which indicates the effect of quality – into Sethi's sales-advertising dynamics, thus making a significant contribution to the existing advertising-quality literature.

From the model we observed that at equilibrium, as a player's awareness share increases, his advertising effort reduces, while the quality effort increases. That is quality effort should reduce with (increasing) advertising effort, or on the other hand advertising effort should reduce with quality effort. Thus a firm whose product is of a lower quality can use advertising to counter-balance the effect the low quality on his market share. It further shows that as a firm's quality effort increases, his competitor's advertising effort increases, and vice versa. Thus a firm can use any of advertising or quality to increase his awareness share which will eventually increase his payoff.

It is a known fact that a lot of market settings are based on more than two players. Thus an extension to an oligopolistic market setting can be more ideal. Further, concepts such as price can improve the advertising-quality literature.

7.0 References

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