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Relating Bernoulli Numbers and Euler Numbers from their Generating Functions, And Through Trigonometry

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Abstract

Bernoulli Numbers and Euler Numbers are two classes of special numbers that have wide applications in the resolution of power series, and in the evaluation of moments and cumulants of a statistical distribution. In this work, the method of generating functions is employed to show that both types of numbers can be expressed in terms of trigonometry. This trigonometrical characterization is further explored to establish a connection between Bernoulli Numbers and Euler Numbers. The work also attempts to provide approximate trigonometric expressions for both Bernoulli and Euler Numbers.

Keywords: Bernoulli Numbers, Euler Numbers, Generating functions, trigonometry and Hyperbolic functions.

1.0 Introduction

Both Bernoulli numbers and Euler Numbers are of great importance in statistics and applied mathematics [1]. In determining the moments of a random variable, power series expansion of the appropriate generating function may be employed [2]. However, in some cases the expansion may be excessively tasking, and one alternative may be to express them in terms of a known power series whose coefficients are Bernoulli numbers or Euler numbers. The analytic method of generating functions has been used variously in [3], [4] and [5] to show that both types of numbers can be given in terms of logarithmic representation. In this work an attempt is made to further express this special numbers in terms of trigonometry, and this trigonometrical characterization is then used to establish a connection between Bernoulli Numbers and Euler Numbers.

2.0 Bernoulli Numbers and Polynomials

According to [1], and of course [6], Bernoulli numbers came to be with the work of Jacob James Bernoulli (1654 - 1705) which was titled *Ars Conjectandi*. It was in connection with a purely algebraic problem but whose scope of importance was later extended into probability theory and engineering. Euler had one of his finest triumphs when he established the connection between Bernoulli numbers and the Rieman Zeta function, for cases where the arguments are even [7]. This he gave as

$$Z_e(2n) = \sum_{y=1}^{\infty}\frac{1}{y^{2n}} = (-1)^{n-1}\frac{(2\pi)^{2n}}{2(2n)!}\beta_{2n} \qquad - \quad - \quad - \quad - \qquad (1)$$

Bernoulli numbers were shown by [8], [9] and [10] with Bernoulli polynomials, $\{\beta_n(x)\}$, as follows:
Let $S_{n-1}(x) = 1^{n-1} + 2^{n-1} + 3^{n-1} + \ldots + (x-1)^{n-1}, \quad n > 1$ - - - (2)
We seek a set of polynomials that, for each n, will satisfy

$$S_{n-1}(x) = \frac{1}{n}\{\beta_n(x) - \beta_n(0)\} \qquad - \quad - \quad - \quad - \qquad (3)$$

Replacing x by x + 1 in (2), results in
$S_{n-1}(x+1) = 1^{n-1} + 2^{n-1} + 3^{n-1} + \ldots + (x-1)^{n-1} + x^{n-1}$
i.e. $S_{n-1}(x+1) = S_{n-1}(x) + x^{n-1}$ - - - - (4)

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i.e $\frac{1}{n}\{\beta_n(x+1) - \beta_n(0)\} = \frac{1}{n}\{\beta_n(x) - \beta_n(0)\}$ + x^{n-1}

Hence $X^{n-1} = \sum \frac{1}{n}\{\beta_n(x+1) - \beta_n(x)\}$ - - - -- (5)

The method of generating functions as used by [2], [3] and [4] to derive the Bernoulli polynomials, $\beta_n(x)$, is presented as follows:

Let g(x, t) = $\sum \beta_n(x)\frac{t^n}{n!}$ - - - - - (6)

be a generating function for $\beta_n(x)$, with $\beta_0(x) = 1$.
Replace x by x + 1 to get

g(x + 1, t) = $\sum \beta_n(x+1)\frac{t^n}{n!}$ - - - - - - (7)

Subtract (6) from (7)

g(x + 1, t) - g(x, t) = $\sum \{\beta_n(x+1) - \beta_n(x)\}\frac{t^n}{n!}$

$= \sum_{n=1}^{\infty} n\, x^{n-1}\frac{t^n}{n!}$, from (5)

$= t\sum_{n=1}^{\infty} t^{n-1}\, x^{n-1}/(n-1)!$

$= t\sum_{n=1}^{\infty} (xt)^m/m! = t\,e^{xt}$

i.e g(x + 1, t) - g(x, t) = te^{xt} - - - - (8)
Suppose g(x,t) is taken to be proportional to e^{xt}
i.e. g(x, t) = A(t) e^{xt}

Then we can show from (8) that A(t) = $\frac{t}{e^t - 1}$

And g(x, t) = $\frac{t\,e^{xt}}{e^t - 1} = \sum_{n=0}^{\infty}\beta_n(x)\frac{t^n}{n!}$ - - - (9)

The Bernoulli numbers are now obtained from $\beta_n(0)$ and are given as the coefficients of the generating function g(0, t) = $\frac{t}{e^t - 1} = \sum_{n=0}^{\infty}\beta_n(x)\frac{t^n}{n!}$ - - - - - - - (10)

Direct expansion gives

$$\frac{t}{e^t - 1} = \frac{1}{\sum_{r=1}^{\infty} t^{r-1}/r!} \qquad = \left[1 + \sum_{r=2}^{\infty} t^{r-1}/r!\right]^{-1}$$

Hence $\left[1 + \sum_{r=2}^{\infty} t^{r-1}/r!\right]^{-1} = \sum_{n=0}^{\infty}\beta_n\frac{t^n}{n!}$ - - - - - (11)

Expand the left hand side of (11) and comparing coefficients of $t^n/n!$ on both sides to get

$$\beta_0 = 1,\ \ \beta_1 = -1/2,\ \ \beta_{2r+1} = 0 \ \ for\ all\ r \ge 1 \ \ \beta_2 = 1/6,\ \ \beta_4 = -1/30,\ \ \beta_6 = \frac{1}{42},\ \ \beta_8 = -\frac{1}{30}$$

$$\beta_{10} = \frac{5}{66}, \quad \beta_{12} = -\frac{691}{2730}, \quad \beta_{14} = 7/6$$

and so on

If we make some analytic dissection of (10) by substituting the values of β_0 and $\beta 1$, the expression can be re-written in the form

$$\frac{t}{e^t - 1} - 1 + \frac{t}{2} = \sum_{n=2}^{\infty} \beta_n \frac{t^n}{n!}$$

i.e $$\frac{t}{e^t - 1} - 1 + \frac{1}{t} + \frac{1}{2} = \sum_{n=2}^{\infty} \beta_n \frac{t^{n-1}}{n!}$$

$$\frac{1}{2}\frac{Cosh(t/2)}{Sinh(t/2)} - \frac{1}{t} = \sum_{n=2}^{\infty} \beta_n \frac{t^{n-1}}{n!}$$

Or $$\frac{d}{dt}\left\{\log\left[\frac{Sinh(t/2)}{t/2}\right]\right\} = \sum_{n=2}^{\infty} \beta_n \frac{t^{n-1}}{n!}$$

Integrating with respect to t gives

$$\log\left[\frac{Sinh(t/2)}{t/2}\right] = \sum_{n=2}^{\infty} \beta_n \frac{t^n}{n!n} \qquad - \quad -- \quad - \quad - \quad - \qquad (12)$$

This is a representation of Bernoulli numbers in terms of the logarithmic and hyperbolic functions.

3.0 Euler Polynomials and Numbers

According to [11] and [12], the Euler numbers are most simply introduced through associated polynomials, $\{E_n(x)\}$, called the Euler polynomials. These polynomials are unique and satisfy the relation

$$\frac{E_n(x+1) + E_n(x)}{2} = x^n \qquad - \quad - \quad - \quad - \qquad (13)$$

for any real x. Equations (13) and (5) have some rough resemblance. By same argument as in (5) we can deduce the generating function for the Euler polynomials as

$$g(x, t) = \frac{2e^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} \qquad - \quad - \quad - \quad - \quad - \qquad (14)$$

It was verified by [12] that when n is odd, $\frac{1}{2}$ is a zero of $E_n(x)$, and consequently defined the Euler numbers as $E_n = 2^n E_n\left(\frac{1}{2}\right)$. Hence taking x = $\frac{1}{2}$ and replacing *t* by *2t* and *n* by *2n* in (14) gives

$$g(\frac{1}{2}, t) = \frac{2e^t}{e^{2t} + 1} = \sum_{n=0}^{\infty} E_{2n} \frac{t^{2n}}{(2n)!} \qquad - \quad - \quad - \quad - \quad - \qquad (15)$$

If we replace *t* by *it* in (15) we have

$$\frac{2e^{it}}{e^{2it} + 1} = \sum_{n=0}^{\infty} E_{2n} \frac{(it)^n}{(2n)!}$$

i.e. $$\sec(t) = \sum_{n=0}^{\infty} (-1)^n \frac{E_{2n} t^{2n}}{(2n)!} \qquad - \quad - \quad - \quad - \qquad (16)$$

In general Euler numbers increase very rapidly in magnitude and their algebraic signs alternate. Expanding both sides of (16), and comparing coefficients will reveal that $E_0 = 1$, $E_2 = -1$, $E_4 = 5$, $E_6 = -61$, $E_8 = 1385$, $E_{10} = -50521$, $E_{12} = 2702765$, $E_{14} = -199360981$ and so on.

4.0 Relationship between the Bernoulli Numbers and the Euler Numbers

From the sets of numbers generated for Bernoulli and Euler numbers, it seems difficult to establish a connection. But an approach via trigonometry may prove useful and less rigorous. Now, recall that

Sin t . Sec t = tan t - - - - - - (17)

The Maclaurin series for *sin t* and *sec t* are both absolutely convergent for

$|t| < \pi/2$, hence *tan t* also converges absolutely within the same radius.

Thus using (16) in (17) we get

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}t^{2n-1}}{(2n-1)!}\cdot\sum_{m=0}^{\infty}\frac{(-1)^{m}E_{2m}t^{2m}}{(2m)!}=\sum_{r=1}^{\infty}\frac{(-1)^{r}2^{2r}\left(1-2^{2r}\right)\beta_{2r}t^{2r-1}}{(2r)!} \quad \ldots \quad .. \qquad (18)$$

If we put $n+m = r$ on the left hand side, (18) reduces to

$$\sum_{r=1}^{\infty}\left[\sum_{m=0}^{r-1}\frac{E_{2m}}{(2r-2m-1)!(2m)!}\right](-1)^{r-1}t^{2r-1}=\sum_{r=1}^{\infty}\frac{2^{2r}\left(2^{2r}-1\right)\beta_{2r}}{(2r)!}(-1)^{r-1}t^{2r-1} \quad - \qquad (19)$$

Due to the uniqueness of the Maclaurin series representation [13], whenever it exists, we can equate coefficients of both sides of (19) in powers of t.

Hence, from (19) $$\frac{2^{2r}\left(2^{2r}-1\right)\beta_{2r}}{(2r)!}=\sum_{m=0}^{r-1}\frac{E_{2m}}{(2r-2m1)!(2m)!} \quad .. \quad \ldots \qquad (20)$$

which implies $$\beta_{2r}=\frac{1}{2^{2r-1}\left(2^{2r}-1\right)}\sum_{m=0}^{r-1}(r-m)\binom{2r}{2m}E_{2m} \quad - \quad - \qquad (21)$$

Equation (21) shows that Bernoulli numbers (excluding β_1) can be obtained from a knowledge of Euler numbers, and β_1 can then be obtained through backward substitution.

5.0 Conclusion

Both Bernoulli numbers and Euler Numbers are of great importance in statistics and applied mathematics. To determine the moments of a random variable, power series expansion of the appropriate generating function may be employed. However, in some cases the expansion may be excessively tasking. The alternative method which we have presented here is to express their generating functions in terms of a known power series whose coefficients are the Bernoulli numbers or Euler numbers. This analytic method of generating functions has been used here to show that both types of numbers can be expressed in terms of trigonometry, and that this trigonometrical characterization can be used to establish the connection between Bernoulli Numbers and Euler Numbers. It is hoped that this has provided for a clearer understanding of this connection and hence, will make for easier application of both.

6.0 References

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