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A Non-Dual Approach to Sensitivity Analysis based on an Auxiliary Problem with One Artificial Variable

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Abstract

In linear programming, the dual simplex method is one of the common methods for restoring optimality when changes render current optimal solutions infeasible. In this work a simple procedure based on the concept of an auxiliary problem, and using only one artificial variable is proposed as an alternative to the dual simplex method. It is demonstrated by means of examples that there is significant reduction in the number of iterations before the new optimal solution.

Keywords: Artificial variable, auxiliary problem, feasible solutions, non-dual approach, sensitivity analysis.

1.0 Introduction

In the words of [1], sensitivity analysis is concerned with explaining the effect on an optimal solution to a linear program, of post-optimality variations in the constraints and coefficients of the initial LP model. The dual simplex method is a well known technique for restoring optimality in such situations. A number of researchers [2, 3, 4] have taken time to explore non-dual approaches to sensitivity analysis. Another is the work of [5], which presented a non-dual approach to restoring optimality when changes in the right hand side vector renders current optimal solution infeasible. A non-dual approach was also proposed [6] for situations where some basic variable become negative due to changes in the right-hand-side parameters. Similarly, [7] proposed the use of interior point method for sensitivity analysis in linear programming and semi-definite programming. Others are [8] and [9] who separately and independently proposed models that simultaneously caters for all rows in an optimal tableau that requires post-optimal analysis.

In continuation, this paper attempts to present a non-dual approach that restores optimality using the first phase of the simplex method, with only one artificial variable, and with emphasis on two peculiar situations, namely

a) When new constraints are introduced which makes the current optimal solution infeasible.

b) When there are changes on the right hand side parameters beyond their initial feasible ranges so that some basic variables become negative.

2.0 Methodology

Consider the LPP (already is standard form).

Maximize $Z = C^T X$

Subject to $AX = b$ and $X \geq 0$ … … … (1)

Where A is an m x n coefficient matrix, b an m-component vector, c an n-component vector and X an n-component vector of decision variables.

Case I: Introduction of New Constraints

Let $I \in C\{1, 2, \ldots, n\}$ be the indexed set of basic vectors and let the order of I be m.

Let $A_I = \{A_i\}$ such that $i \in I$ is an invertible matrix,

and Let $X_1 = (X_i)$ for $i \in I$, $C_I = (C_i)$ for $i \in I$

Let $A^* = \{A_k\}$ such that $k \notin I$. Similarly define X^* and C^*.

Using the above, (1) can be re-written as

Maximize $Z = C_I^T A_I^{-1} b + \left(C^{*T} - C_I^T A_I^{-1} A^*\right) X^*$ Subject to

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$$\begin{cases} X_I + A_I^{-1} A^* X^* = A_I^{-1} b \\ X_I \geq 0, \quad X^* \geq 0 \end{cases} \qquad \ldots \quad \ldots \quad \ldots \qquad (2)$$

Case II: A Change in the Right hand Side

If one or more components of the right hand side of the constraint in (2) turns out to be strictly negative, then the basic solution $X^* = 0$ and $X_1 = A_I^{-1} b$ is not feasible.

Let us now consider the phase I of the simplex method with a single artificial variable t. The auxiliary problem of phase I is

Maximize w = -t

Subject to
$$\begin{cases} X_I + A_I^{-1} A^* X^* - \delta t = A_I^{-1} b \\ X_I \geq 0, \; X^* \geq 0, \; t \geq 0 \end{cases} \qquad \ldots \quad \ldots \qquad (3)$$

where $\delta = (\delta i), \; i \in I$ is an m-component column vector satisfying.

$$\delta_i = \begin{cases} 0 \; if \; ith \; component \; of \; A_I^{-1} b \; is \; -ve \\ 1 \; if \; ith \; component \; of \; A_I^{-1} \; is \; not \; -ve \end{cases} \qquad \ldots \quad \ldots \quad \ldots \quad . \qquad (4)$$

A basic feasible solution for (3) is then obtained by replacing the most negative basic variable by t, and eliminating t from the other equations. Next is to follow the usual phase I procedure of the simplex method. We shall now illustrate the above cases with two linear programmes.

3.0 Illustrations

(a) Introduction of new constraints

Consider the linear programme

Maximize Z = $30 X_1 + 20 X_2$

Subject to $2X_1 + X_2 \leq 80$

$X_1 \leq 30$

$X_1, X_2 \geq 0$

With X_3 and X_4 as the non-negative slack variables, the optimal tableau is presented in Table I

Table I

BV	Z	X_1	X_2	X_3	X_4	B
Z	1	10	0	20	0	1600
X_2	0	2	1	2	0	80
X_4	0	1	0	0	1	30

Supposed two new constraints are introduced post – optimally as

$$-X_1 + \frac{1}{2} X_2 \leq 0, \text{ and } X_2 \leq 30 \qquad \ldots \quad \ldots \quad \ldots \qquad (5)$$

The basic solution in Table I becomes infeasible. Let X_5 and X_6 represent the slack variable on the two constraints. Eliminate X_2 and X_4 from the said constraints to get Table II.

Table II

BV	Z	X_1	X_2	X_3	X_4	X_5	X_6	B
Z	1	10	0	20	0	0	0	1600
X_2	0	2	1	1	0	0	0	80
X_4	0	1	0	0	1	0	0	30
X_5	0	-2	0	1 ½	0	1	0	-40
X_6	0	-2	0	-1	0	0	1	-50

Introduce an artificial variable, and use the objective function of the auxiliary problem to get Table III

Table III

BV	W	Z	X_1	X_2	X_3	X_4	X_5	X_6	T	B
W	1	0	0	0	0	0	0	0	1	0
Z	0	1	10	0	20	0	0	0	0	1600
X_2	0	0	2	1	1	0	0	0	0	80
X_4	0	0	1	0	0	1	0	0	0	30
X_5	0	0	-2	0	- ½	0	1	0	-1	-40
X_6	0	0	-2	0	-1	0	0	1	-1	-50

The variable t enters the basis while X_6 leaves, updating yields Table IV.

Table IV

BV	W	Z	X_1	X_2	X_3	X_4	X_5	X_6	T	B
W	1	0	-2	0	-1	0	0	1	0	-50
Z	0	1	10	0	20	0	0	0	0	1600
X_2	0	0	2	1	1	1	0	0	0	80
X_4	0	0	1	0	0	1	0	0	0	30
X_5	0	0	0	0	½	0	1	0	-1	10
T	0	0	2	0	1	0	0	-1	1	50

X_1 enters, while t leaves the basis to get Table V.

Table V

BV	W	Z	X_1	X_2	X_3	X_4	X_5	X_6	T	B
W	1	0	0	0	0	0	0	0	1	0
Z	0	1	0	0	15	0	0	5	-5	1350
X_2	0	0	0	1	0	0	0	1	-1	30
X_4	0	0	0	0	- ½	1	0	½	- ½	5
X_5	0	0	0	0	½	0	1	-1	0	10
X_1	0	0	1	0	½	0	0	- ½	½	25

To obtain the first tableau for phase II, we delete the row corresponding to w and the column corresponding to t in Table V to get Table VI.

Table VI

BV	Z	X_1	X_2	X_3	X_4	X_5	X_6	b
Z	1	0	0	15	0	0	5	1350
X_2	0	0	1	0	0	0	1	30
X_4	0	0	0	- ½	1	0	½	5
X_5	0	0	0	½	0	1	-1	10
X_1	0	1	0	½	0	0	- ½	25

Table VI, which is the first for phase II is already optimal. So the sensitivity analysis ends here with $X_2 = 30$, $X_1 = 25$, and $Z = 1350$.

(b) A change in the right hand side

We shall illustrate with the program

Maximize $Z = 9X_1 - 23X_2 - 5X_3 + 24X_4 - 3X_5$

Subject to

$-2X_1 + X_2 - 2X_4 + X_5 = -4$

$X_1 - X_2 + X_4 = 2$

$X_1 + 2X_2 + X_3 - 2X_4 = 6$

$X_1, X_2, X_3, X_4, X_5 \geq 0$

After a few iterations the optimal tableau is obtained as in Table VII

Table VII

BV	Z	X_1	X_2	X_3	X_4	X_5	B
Z	1	0	0	0	0	2	-2
X_1	0	1	0	0	1	-1	2
X_2	0	0	1	0	0	-1	0
X_3	0	0	0	1	-3	3	4

From the initial tableau we have

$$I = \{1,2,3\} \qquad A_I = \begin{pmatrix} -2 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 2 & 1 \end{pmatrix} \qquad A_I^{-1} = \begin{pmatrix} -1 & -1 & 0 \\ -1 & -2 & 0 \\ 3 & 5 & 1 \end{pmatrix}$$

$b_1 = -4$, $b_2 = 2$, $b_3 = 6$

Now suppose $b_1 = 1$ post-optimally. The right hand side of the optimal tableau becomes

$$A_I^{-1} b = A_I^{-1} \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} = A_I^{-1} \left[\begin{pmatrix} -4 \\ 2 \\ 6 \end{pmatrix} + \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} \right] = \begin{pmatrix} -3 \\ -5 \\ 19 \end{pmatrix}$$

The new basic solution becomes infeasible because a basic variable is negative. To proceed, we introduce an artificial variable t, and the objective function to the auxiliary problem to obtain Table VIII.

Table VIII

BV	W	Z	X_1	X_2	X_3	X_4	X_5	t	B
W	1	0	0	0	0	0	0	1	0
Z	0	1	0	0	0	0	2	0	-7
X_1	0	0	1	0	0	1	-1	-1	-3
X_2	0	0	0	1	0	0	-1	-1	-5
X_3	0	0	0	0	1	-3	3	0	19

t enters, while X_2 leaves the basis.

Table IX

BV	W	Z	X_1	X_2	X_3	X_4	X_5	t	B
W	1	0	0	1	0	0	-1	0	-5
Z	0	1	0	0	0	0	2	0	-7
X_1	0	0	1	-1	0	1	0	0	2
T	0	0	0	-1	0	0	1	1	5
X_3	0	0	0	0	1	-3	3	0	19

X_5 enters while t leaves the basis to get

Table X

BV	W	Z	X_1	X_2	X_3	X_4	X_5	t	B
W	1	0	0	0	0	0	0	1	0
Z	0	1	0	2	0	0	0	-2	-17
X_1	0	0	1	-1	0	1	0	0	2
X_5	0	0	0	-1	0	0	1	1	5
X_3	0	0	0	3	1	-3	0	-3	4

If we delete the columns for w and t, and the row for w, we shall obtain the first tableau of phase II as Table XI.

Table XI

BV	Z	X_1	X_2	X_3	X_4	X_5	b
Z	1	0	2	0	0	0	-17
X_1	0	1	-1	0	1	0	2
X_5	0	0	-1	0	0	1	5
X_3	0	0	3	1	-3	0	4

Table XI happens to be optimal for the sensitivity analysis.

4.0 Discussion

The two cases treated above can also be handled by the dual simplex method [10]. However, the method presented here uses the first phase of the simplex method with just one artificial variable, and attains optimality at the first step of phase 2. The procedure is therefore, simple and converges to a solution more rapidly than the dual simplex procedure. Furthermore, it is based on the concept of an auxiliary problem with only one artificial variable.

5.0 Conclusion

This work has presented a simple procedure for sensitivity analysis which is based on the concept of an auxiliary problem, and uses only one artificial variable as an alternative to the dual simplex method. It has been shown by means of examples that the number of iterations before the new optimal solution is drastically reduced.

6.0 References

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