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Application of Lyapunov Functions to Some Nonlinear Second Order Ordinary Differential Equations

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Abstract

In this paper, Lyapunov functions have been constructed for linear and nonlinear equations and applied to illustrate the usefulness of the functions in determination of stability or instability of nonlinear second order solutions of ordinary differential equations.

Keywords: Lyapunov functions; positive definite; positive semi-definite; negative definite; and negative semi-definite

1.0 Introduction

Lyapunov functions remain the most viable source for determining the qualitative properties of solutions of ordinary differential equations which are very useful in science and technology. It is also useful in social sciences, disease transmission and dynamics etc. [1 – 9].However, problems and complex computations are encountered in trying to construct appropriate Lyapunov functions for nonlinear second order differential equations or systems. Our method for construction of Lyapunov functions for nonlinear second order differential equations lies on the concept that given any real system

$$\dot{x} = Ax \tag{1.1}$$

$x \in R^n$, where A is a constant matrix. Assume that A has all its eigenvalues with negative real parts. Then it is known from the general theory that corresponding to any positive definite quadratic form $U(x)$ there exists another positive definite quadratic form $V(x)$ such that

$$\dot{V} = -U \tag{1.2}$$

along the solution paths of (1.1). This result in equation (1.2) has since been extended and is known to hold for positive semi definite quadratic $U(x)$ as well. Therefore our basis for construction of Lyapunov function in this paper would ultimately satisfy equation (1.2).

1.1 Definition of Some Terms

(1) A continuous function $V(x,t) = V(x_1, x_2, ..., x_n, t)$ is positive definite if $\lim_{\|x\| \to 0} V(x,t) = 0$ and there exists $\Phi(\|x\|)$ such that $V(x,t) > \Phi(\|x\|)$.

Example: $V(x_1, x_2) = x_1^2 + x_2^2$

(2) A continuous function $V(x,t) = V(x_1, x_2, ..., x_n, t)$ is positive semi definite if $\lim_{\|x\| \to 0} V(x,t) = 0$ and there exists $\Phi(\|x\|)$ such that $V(x,t) \geq \Phi(\|x\|)$.

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Example: $V\left(x_1,x_2\right)=\left(x_1^2-x_2^2\right)^2$.

(3) A continuous function $V\left(x,t\right)=V\left(x_1,x_2,...,x_n,t\right)$ is negative definite if $\lim_{\|x\|\to 0}V\left(x,t\right)=0$ and there exists $\Phi\left(\|x\|\right)$ such that $V\left(x,t\right)<-\Phi\left(\|x\|\right)$.

Example: $V\left(x_1,x_2\right)=-\alpha\left(x_1^2+x_2^2\right),\ \alpha>0$.

(4) A continuous function $V\left(x,t\right)=V\left(x_1,x_2,...,x_n,t\right)$ is negative semi definite if $\lim_{\|x\|\to 0}V\left(x,t\right)=0$ and there exists $\Phi\left(\|x\|\right)$ such that $V\left(x,t\right)\le-\Phi\left(\|x\|\right)$.

Example: $V\left(x_1,x_2\right)=-\alpha\left(x_1^2-x_2^2\right)$.

(5) A continuous function $V\left(x,t\right)=V\left(x_1,x_2,...,x_n,t\right)$ is indefinite if it assumes both positive and negative values in any origin in a Domain D, example

$$V\left(x_1,x_2,x_3\right)=\left(x_1^2+x_2^2-x_3^2\right).$$

2.0 One Method for construction of Lyapunov functions for some nonlinear second order differential equations or systems

The word "one" method and "some" in the title underscore the fact that there are many methods available and equations or systems covered by one method may not be covered by another method.

The method which we shall discuss here has its basis on the following consideration. Given any real system in equation (1.1) in which A has all its eigenvalues with negative real parts. We can then construct a positive definite quadratic form V corresponding to any positive semi definite or positive definite form $U\left(x\right)$ of our choice such that equation (1.2) is satisfied. Then the V is the Lyapunov function for the linear constant coefficient system for equation (1.1). This V constructed for the linear constant coefficient equation is modified by taking into account the similarities between the linear constant coefficient system and nonlinear constant coefficient system. With this correlation, we obtain our trial V for the nonlinear constant coefficient system. The term "trial" is appropriate since the V obtained would be subject to satisfying condition (1.2), in addition to being positive definite. This procedure is closely connected with the method used [10]. See also the following [11 – 21].

Consider the second order differential equation

$$\ddot{x}+a\dot{x}+bx=0 \tag{2.1}$$

where a, b are constants with $a>0,\ b>0$.

The equation (2.1) is equivalent to the following two systems

$$\dot{x}=y,\ \dot{y}=-ay-bx \tag{2.2}$$

$$\dot{x}=-by-ax,\ \dot{y}=x \tag{2.3}$$

In each of the systems (2.2) or (2.3) the eigenvalues

$$\lambda^2+a\lambda+b=0$$

have negative real parts iff $a>0,\ b>0$. Thus our claim in respect of V,U satisfying (1.2) along the solution paths of (2.2) or (2.3) are valid.

Consider our Lyapunov function V by choosing the most general quadratic form of order two and picking the coefficients in this quadratic form to satisfy equation (1.2) along the solution paths of (2.2) or (2.3). Thus

$$2V=k_1x^2+k_2y^2+2k_3xy \tag{2.4}$$

where k_1,k_2,k_3 are constants yet to be determined. Since V must be positive definite, $k_1>0,k_2>0$. Now differentiating equation (2.4) with respect to t along the solution paths of (2.2), we obtain

$$\dot{V} = k_1xy + k_3y^2 - (k_2y + k_3x)(ay + bx)$$
$$= (k_3 - ak_2)y^2 - bk_3x^2 + (k_1 - bk_2 - ak_3)xy \qquad (2.5)$$

we are to determine $\dot{V}$ such that one of the following conditions hold:

(i) $$(i)\ \dot{V} \le -\alpha x^2,\ (ii)\ \dot{V} \le -\alpha y^2,\ (iii)\ \dot{V} \le -\alpha\left(x^2 + y^2\right) \qquad (2.6)$$

For some constant $\alpha > 0$. From equation (2.5) we obtain the following tabulations

Table 1:

Terms	Coefficients
xy	$k_1 - bk_2 - ak_3$
x^2	$-bk_3$
y^2	$k_3 - ak_2$

For realization of **case (i)** in equation (2.6), we require that

$ak_2 - k_3 = 0, k_1 - bk_2 - ak_3 = 0, k_3 > 0$.

Clearly $k_2 = a^{-1}k_3, k_1 = a^{-1}bk_3 + ak_3$, setting $k_3 = 1$, then

$$2V = \left(a^{-1}b + a\right)x^2 + a^{-1}y^2 + 2xy \qquad (2.7)$$

we can rewrite equation (2.7) in the form

$$2V = a\left(x + a^{-1}y\right)^2 + a^{-1}bx^2 \qquad (2.8)$$

From equation (2.8) our V is positive definite. Also $\dot{V}$ is given by $\dot{V} = -bx^2$.
Hence equation (1.2) is satisfied for case (i) in equation (2.6).
Case (ii) in equation (2.6) demands that

$ak_2 - k_3 > 0, k_1 - bk_2 - ak_3 = 0, k_3 = 0.$

Therefore $k_1 = bk_2$, substituting into equation (2.4), we have

$2V = bk_2x^2 + k_2y^2$, letting $k_2 = 1$, then

$$2V = bx^2 + y^2 \qquad (2.9)$$

Which is clearly positive definite. The time derivative of equation (2.9) along the solution paths of equation (2.2) or (2.3) is

$$\dot{V} = -ay^2 \equiv -\alpha y^2, \alpha > 0. \qquad (2.10)$$

Thus satisfying equation (1.2) for case (ii) in equation (2.6).
Finally for **case (iii)** in equation (2.6) the following conditions must be satisfied.

$k_1 - bk_2 - ak_3 = 0, ak_2 - k_3 > 0, k_3 > 0.$

Choose ε, such that $k_2 = a^{-1}\left(k_3 + \varepsilon k_3\right)$. Now $k_1 = a^{-1}b\left(k_3 + \varepsilon k_3\right) + ak_3$ substituting these values of k_1, k_2, k_3 into equation (2.4), we obtain

$$2V = \left[a^{-1}b\left(k_3 + \varepsilon k_3\right) + ak_3\right]x^2 + a^{-1}\left(k_3 + \varepsilon k_3\right)y^2 + 2k_3xy \qquad (2.11)$$

Setting $k_3 = 1$ then

$2aV = \left[b\left(1 + \varepsilon\right) + a^2\right]x^2 + \left(1 + \varepsilon\right)y^2 + 2axy$.

Choosing $\varepsilon = 1$ then

$2aV = 2bx^2 + 2y^2 + a^2x^2 + 2axy$.

Let $aV = V_1$ then

$$2V_1 = 2bx^2 + y^2 + \left(ax + y\right)^2 \tag{2.12}$$

Since $a > 0, b > 0$ equation (2.12) is positive definite. The time derivative of equation (2.12) along the solution paths of (2.2) or (2.3) is

$$\dot{V}_1 = -y^2 - bx^2 \tag{2.13}$$

Thus satisfying equation 1.2 for case (iii) in equation (2.6) and in general we have shown that V satisfies the conditions in (i), (ii) and (iii) in equation (2.6).

Next, we discuss on how to obtain Lyapunov functions for nonlinear second order differential equations using the similarities between the constant coefficient system of the linear and nonlinear second order systems.

Consider the nonlinear second order equation

$$\ddot{x} + g\left(\dot{x}\right) + bx = 0 \tag{2.14}$$

Here b is a constant and g depends on $\dot{x}$ and is continuous. We remark that the scalar equation (2.14) is equivalent to the system

$$\dot{x} = y,\ \dot{y} = -g\left(y\right) - bx. \tag{2.15}$$

Observe that (2.15) is exactly the same as equation (2.2) if $g\left(y\right)$ is replaced by ay

Also we have found suitable Lyapunov functions for that system in equation (2.2) which are

$$2V_2 = bx^2 + y^2$$

and

$$2V_3 = a\left(x + a^{-1}y\right) + a^{-1}bx^2$$

Since in the constant coefficient equation a does not appear explicitly in V_2, this suggests that we should take

$$2V_2 = bx^2 + y^2 \tag{2.16}$$

as a trial Lyapunov function for the system (2.15) since $b > 0$ in equation(2.16), it is clearly positive semi definite. Also the time derivative along the solution paths of (2.15) is

$$\dot{V}_2 = -yg\left(y\right) \tag{2.17}$$

Therefore we hypothesize that g is such that $yg\left(y\right) \geq 0$ for all y .Then V_2 is the Lyapunov function for equation (2.14) since V_2 is positive definite and the time derivative is negative semi definite.

Next consider again the scalar equation

$$\ddot{x} + f\left(x\right)\dot{x} + h\left(x\right) = 0 \tag{2.18}$$

Where f and h are continuous functions depending only on x .taking equation (2.18) in system form

$$\dot{x} = y, \dot{y} = -f\left(x\right)y - h\left(x\right) \tag{2.19}$$

Note that equation (2.19) is exactly the same as equation (2.2) if

$$\left.\begin{array}{l} f\left(x\right) \text{is replaced by a} \\ \text{h}\left(\text{x}\right) \text{is replaced by bx} \end{array}\right\} \tag{2.20}$$

For the linear system (2.2) we have the following as appropriate Lyapunov functions:

$$\left.\begin{array}{l} 2V_4 = bx^2 + \left(y + ax\right)^2 \\ 2V_5 = bx^2 + y^2 \end{array}\right\} \tag{2.21}$$

Consideration on $2V_4$ suggests the following as Lyapunov functions for the system (2.19).

$$\left.\begin{aligned}
2V_6 &= xh(x) + \left(y + f(x)x\right)^2 \\
2V_7 &= xh(x) + \left(y + F(x)\right)^2 \\
F(x) &= \int_0^x f(s)\,ds \\
2V_8 &= 2\int_0^x h(s)\,ds + \left(y + f(x)x\right)^2 \\
2V_9 &= 2\int_0^x h(s)\,ds + \left(y + F(x)\right)^2
\end{aligned}\right\} \qquad (2.22)$$

However eligibility for status of Lyapunov functions demands that the time derivative of each of the above functions in equation (2.22) be examined, The evaluation of time derivative of $2V_6$ demands that h and f be C^1 function for $2V_7$ that h should be a C^1 function. For $2V_8$, the demand is that f should be a C^1 function while $2V_9$ demands that f and h should be C^0 functions. From these considerations $2V_9$ has the least of constraints on h and f . Thus $2V_9$ is our choice. Also considering $2V_5$, we have the following as Lyapunov functions for the system (2.19)

$$2V_{10} = y^2 + xh(x)$$

and

$$2V_{11} = y^2 + 2\int_0^x h(s)\,ds.$$

Consideration on $2V_{10}$ demands that h should be C^1 function, while $2V_{11}$ is that h is C^0 .Thus based on these considerations. We limit our choice to $2V_9$ and $2V_{11}$. Next $2V_9$ and $2V_{11}$ are all positive definite functions and their time derivatives along their solution paths of (2.19) are

$$\left.\begin{aligned}
\dot{V}_9 &= -h(x)F(x) \\
\dot{V}_{11} &= -f(x)y^2
\end{aligned}\right\} \qquad (2.23)$$

Thus (1) subject to the conditions that f , h are continuous functions and $h(x)x > 0, x \neq 0, f(x) \geq 0$. $2V_{11}$ is an appropriate Lyapunov function for that system (2.19). (2) Subject to the conditions that f , h are continuous functions and $xh(x) > 0, \forall(x \neq 0), h(x)F(x) \geq 0, \forall x, 2V_9$ is a Lyapunov function for the sytem (2.19).

3.0 Applications of Lyapunov functions to Stability/Instability of nonlinear second order ordinary differential Equations

We shall examine the following examples below to illustrate the usefulness of Lyapunov functios in direct determination of stability or instability of nonlinear second order solutions of ordinary differential equations. This is done without solving the given nonlinear differential equations, [22 -24].First and foremost, we state theorems due A.M. Lyapunov which are the key in establishing our claims.

Given the differential equation

$$\dot{x} = f(t,x),\ f(t,0) = 0 \qquad (3.1)$$

f is continuous in (t,x).

Theorem 3.1 (Lyapunov)

Suppose there exists a C^1 function $V: R^n \to R$ with the following properties:

(1) V is positive definite ,

(2) The time derivative $\dot{V}$ along the solution paths of equation (3.1) is negative semi definite. Then the trivial solution $x \equiv 0$ is stable in the sense of Lyapunov.

Theorem 3.2 (Lyapunov)

Suppose there exists a C^1 function $V: R^n \to R$ with the following properties:

(1) V is positive definite,

(2) The time derivative $\dot{V}$ along the solution paths of equation (3.1) is negative definite. Then the trivial solution $x \equiv 0$ is asymptotically stable in the sense of Lyapunov.

Theorem 3.3 (Lyapunov)

Suppose there exists a C^1 function $V: R^n \to R$ with the following properties:

(1) V is positive definite,

(2) The time derivative $\dot{V}$ along the solution paths of equation (3.1) is negative definite and in every neighbourhood of the origin there exists x_0 such that $V(x_0) > 0$, then the trivial solution $x \equiv 0$ of equation (3.1) is unstable on the sense of Lyapunov.

Example 3.1: Consider the equation

$$\ddot{x} + g(x) = 0 \tag{3.2}$$

where $x \equiv$ displacement, $\dot{x} \equiv$ velocity, $g(0) = 0, K.E = \frac{1}{2}\dot{x}^2, P.E = \int_0^x g(s)ds$.

By considering the function

$$V = \frac{1}{2}\dot{x}^2 + \int_0^x g(s)ds$$

Show that the trivial solution of equation (3.2) or the equivalent system

$$\dot{x} = y, \dot{y} = -g(x) \tag{3.3}$$

Is stable in the sense of Lyapunov. What sufficient condition on g(x) would ensure the stability of the system (3.3)?

Solution: The given function

$$V(x, y) = \frac{1}{2}y^2 + \int_0^x g(s)ds$$

Is positive definite if $\int_0^x g(s)ds > 0.$ The time derivative $\dot{V}$ along the solution paths of (3.3) is

$$\dot{V} = y\dot{y} + yg(x)$$
$$= -yg(x) + yg(x) = 0$$

Thus the time derivative $\dot{V}$ is negative semi definite. Thus by Theorem 3.1 the system (3.3) is stable in the sense of Lyapunov. The condition $xg(x) > 0, x \neq 0$ is the sufficient condition required for stability of the trivial solution $x \equiv 0$ of equation (3.1)

Example 3.2: By considering the function

$$2V = x^2 + y^2 \tag{3.4}$$

Show that the system

$$\dot{x} = -x - \frac{x^3}{3} - x\sin y, \dot{y} = -y - \frac{y^3}{3} \tag{3.5}$$

is asymptotically stable in the sense of Lyapunov

Solution: The function

$$2V = x^2 + y^2$$

Is positive definite if $x \neq 0, y \neq 0$. The time derivative $\dot{V}$ along the solution paths of equation (3.5) is

$$\dot{V} = x\dot{x} + y\dot{y}$$
$$= x\left(-x - \frac{x^3}{3} - x\sin y\right) + y\left(-y - \frac{y^3}{3}\right)$$
$$= -x^2 - x^4 - x^2\sin y - y^2 - \frac{y^4}{3}$$
$$= -\left(x^2 + y^2 + \frac{x^4}{3} + \frac{y^4}{3} + x^2\sin y\right)$$
$$= -\left(2x^2 + y^2 + \frac{x^4}{3} + \frac{y^4}{3}\right)$$

Thus $\dot{V}$ is negative definite. Therefore by Theorem 3.2 the system (3.5) is asymptotically stable in the sense of Lyapunov.

Example 3.3: Consider the scalar equation

$$\ddot{x} - x^3 = 0 \qquad (3.6)$$

and $V(x_{,1}\, x_2)$ dedined by $V(x_1, x_2) = x_1 x_2$. Show that the scalar equation (3.6) is unstable in the sense of Lyapunov.

Solution: The scalar equation (3.6) is equivalent to the system

$$\dot{x}_1 = x_2, \dot{x}_2 = x_1^3$$

Our $V(x_1, x_2) = x_1 x_2$ is positive semi definite. The time derivative

$$\dot{V}(x_1, x_2) = \dot{x}_1 x_2 + x_1 \dot{x}_2 = x_2^2 + x_1^4 > 0.$$

Since $\dot{V}(x_1, x_2) > 0$ by Theorem 3.3 the equation (3.6) is unstable.

4.0 Discussion

The eligibility of our Lyapunov functions demand that the time derivative of these functions be examined. The smoothness of these functions is a priority in our consideration and we always go for one with the least constraint. When the constraints are stiff, we may reject the functions.

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