Transactions of the Nigerian Association of Mathematical Physics
***Volume 3, (January, 2017*), pp 11 – 16**
© Trans. of NAMP

An Orthogonal Collocation Approximation Method for the Analytic Solution of Fredholm Integral Equations

Mamadu E.J.[1]and Ojobor S.A[2]

[1]Department of Mathematics, University of Ilorin, P.M.B 1515, Ilorin, Nigeria
[2]Department of Mathematics, Delta State University, Abraka, Nigeria.

Abstract

This paper considered orthogonal collocation approximation method for the analytic solution of Fredholm integral equations. The method of orthogonal collocation has been applied to many fields of mathematics-differential equations, boundary value problems, partial differential equations, el cetera. This paper seeks an extension of the method to integral equations. In this paper, orthogonal polynomials are constructed in the interval [-1,1] with respect to the weight function, $w(x) = 1 + x^2$, which are employed as trial functions in the approximation of the analytic solution. The resulting linear algebraic equations are solved for the unique determination of the unknown coefficients in the approximant. The method is implemented on some selected problems for experimentation, and the resulting numerical evidences show that the method is effective. Maple 18 software was employed for all computations in this work.

Keywords: Integral equation, Collocation, Orthogonality, Orthogonal Collocation, Fredholm integral equation.

1.0 Introduction

Let consider the integral equations of the form:

$$\int_a^b \mathrm{k}(\mathrm{x}, \mathrm{s})\mathrm{y}(s)\mathrm{ds} = \mathrm{f}(\mathrm{x}), \quad (1)$$

and

$$\int_a^b \mathrm{k}(\mathrm{x}, \mathrm{s})\mathrm{y}(s)\mathrm{ds} = \mathrm{f}(\mathrm{x}) + y(x)\ , \quad (2)$$

where the unknown function is $\mathrm{y}(s)$ in both cases, $\mathrm{y}(\mathrm{x})$ is a known function and $\mathrm{k}(\mathrm{x}, s)$ is the nucleus of the integrals. Basically, equations (1) and (2) are called the linear Fredholm integral equations of the first and second kinds. These kind of problems arise in a wide range of fields, which include, contact problems, heat transfer problems, astrophysics, etc, which are often modelled as partial differentials equations, but are however deformed into integral equations for standard mathematical resolution.

In distant years, few analytic methods have been developed and implemented, which nonetheless have been insufficient to handle complex integral mathematical algorithms to arrive at a compact solution that would interpret the physical attributes of the model in question. To this effect, many researchers have sought for methods which can be employed as the approximation of the analytic solution of the problem considered. For instance, Adeniyi [1] adopted the popular Tau method in the numerical solution of integral equations. Also, Elliott [2] applied the Chebychev series method for the numerical solution of Fredholm integral equations.

The orthogonal collocation method was developed by Michelsen and Villadsen (1972) [4]. This method remains one of the best methods in seeking the numerical solution of many mathematical problems due to its simplicity in application. The method has been subsequently studied and applied to many other fields in science and technology, including chemical engineering problems [3-5], linear differential equations [6], in the formulation of multisteps schemes [7] , etc. However, the orthogonal collocation method as an integral equation solver depends on the type of trial function adopted in the formulation of the scheme. Various trial functions includes, the canonical polynomials, the Chebychev polynomials, the legendre

Corresponding author: Mamadu, E.J., E-mail: mamaduebimene@hotmail.com, Tel.: +2348062650313, 08034029170(OSA)

polynomials, [8-9]. In recent times, the orthogonal collocation method has been applied in many areas such as; the determination of temperature distribution in cylindrical conductors [10], the solution of Pellet equation [11],the finite elements in dynamic optimization [12], the estimation and predictive control in APMonitor[13], the solution of Fredholmintegro-differentials (FIDEs) [14], the solution of Volterra integral equations [15],etc.

In this paper, the orthogonal polynomials are constructed in the interval [-1,1], with respect to the weight function, $w(x) = 1 + x^2$. These polynomials are employed as trial functions for the approximation of the analytic solution of the integral equation via an orthogonal collocation approximation method. The method is highly valuable as it converges rapidly to the exact solution even employing few polynomials as trial functions. For comparison of the method, readers should see [1-2, 8-9].
This paper is organized as follow. In section 2, the concept of orthogonal polynomials is given. Also, orthogonal polynomials are constructed. The mathematical formulation of the method is presented in section 3. Section 4 is devoted to the discussion of numerical evidences. Finally, the conclusions is given in section 5.

2.0 The Orthogonal Polynomials

In this section, we shall adopt a class of continuous scheme to setup their discrete counterpart to generate orthogonal polynomials for the weight function, $w(x) = 1 + x^2$, $x \in [-1,1]$. This we do by approximating the analytic solution by

$$y_n(x) = \sum_{i=0}^{n} a_i \varphi_i(x) \cong y(x), \quad (3)$$

with the orthogonal polynomials, $\varphi_i(x)$, defined below and which are derived.
Let $\varphi_n(x)$ be an orthogonal polynomial satisfying the orthogonality relation (see [14-15])

$$\int_{-1}^{1} (1 + x^2)\varphi_r(x)\varphi_s(x)\mathrm{dx} = \mathrm{h_s}\delta_{\mathrm{rs}} \quad (4)$$

Where

$$\delta_{\mathrm{rs}} = \begin{cases} 0, & \mathrm{r} \neq \mathrm{s} \\ 1, & \mathrm{r} = \mathrm{s} \end{cases}$$

Also, $w(x) > 0, x \in [-1,1]$ such that the moment exist.
If $\varphi_r(x)$ *and* $\varphi_s(x)$ are two orthogonal polynomials with reference to equation (4), then an inner product of $\varphi_r(x)$ *and* $\varphi_s(x)$ is given as;

$$\langle \varphi_r, \varphi_s \rangle = \int_{-1}^{1} (1 + x^2)\varphi_r(x)\varphi_s(x)\mathrm{dx} \quad (5)$$

By definition of orthogonality in a real functional space, we have that,

$$\langle \varphi_r, \varphi_s \rangle = 0, \ \mathrm{r} \neq \mathrm{s} \quad (6)$$

Hence, the construction of $\varphi_n(x), n \geq 0$, now follows:
For this purpose, we introduce additional properties such as,

$$\varphi_n(1) = 1, \quad (7)$$

$$\varphi_n(x) = \sum_{r=0}^{n} C_r^{(n)} x^r \quad (8)$$

Hence, using the equations (6) – (8), we generate the orthogonal polynomials are as follows(see [14-15]):
When $n = 0$ in (8), we have

$$\varphi_0(x) = C_0^{(0)} .$$

By definition, $C_0^{(0)} = 1 \Rightarrow \varphi_0(x) = 1$
When $n = 1$ in (8), we have

$$\varphi_1(x) = C_0^{(1)} + C_1^{(1)} x \quad (9)$$

By (7), we have

$$\varphi_1(1) = C_0^{(1)} + C_1^{(1)} = 1 \quad (10)$$

Also,

$$\langle \varphi_0, \varphi_1 \rangle = \int_{-1}^{1} (1 + x^2)\varphi_0(x)\varphi_1(x)\mathrm{dx} \quad (11)$$

Solving (10) and (11), we have,

$$\frac{8}{3} C_0^{(1)} = 0 \ , C_1^{(1)} = 1 \Rightarrow \varphi_1(x) = x$$

Following the above procedure, the first six orthogonal polynomials are presented in **Table 1** below.

Table 1: First six orthogonal polynomials

n	$\varphi_n(x)$
0	1
1	x
2	$\frac{1}{3}(5x^2-2)$
3	$\frac{1}{5}(14x^3-9x$
4	$\frac{1}{648}(333-2898x^2+3213x^4)$
5	$\frac{1}{136}(325x-1410x^3+1221x^5)$
6	$\frac{1}{1064}(-460+8685x^2-24750x^4+17589x^6)$

2.1 Properties of $\varphi_n(x)$

(i) $\varphi_n(x)$must have distinct roots.

(ii) $\varphi_n(x)$can be employed as trial or basis functions.

(iii) $\varphi_n(x)$depends on (i), weight function and the interval of orthogonality.

3.0 Mathematical Formulation of the Problem

We consider the Fredholm integral equation of the form

$$y(x)+\int_{-1}^{1}\mathrm{k}(\mathrm{x},\mathrm{s})\mathrm{y}(\mathrm{s})\mathrm{ds}=\mathrm{f}(\mathrm{x}) \quad (12)$$

Where-1≤ x, s ≤ 1. We write the approximate solution as:

$$y_n(x)=\sum_{i=0}^{n}a_i\varphi_i(x)\cong y(x) \quad (13)$$

Where$\varphi_i(x)$, i≥ 0 is theith orthogonal polynomial with weight function, w(x) = 1 + x^2, x ϵ [-1,1], which are derived and will be employed as basis function in the approximate solution.

Substituting (13) in (12), we have,

$$\sum_{i=0}^{n}a_i\varphi_i(x)+\sum_{i=0}^{n}a_i\int_{-1}^{1}\mathrm{k}(\mathrm{x},\mathrm{s})\varphi_i(s)\mathrm{ds}=\mathrm{f}(\mathrm{x}) \quad (14)$$

Equation (14) is then collocated at the zeros of the polynomial, $\varphi_i(x)$.This is selected to have i which must match the number of the unknown coefficients, a_i in the trial function. Also, it is chosen to have i to avoid over determined or under determind cases. Hence, we obtain a set of (n + 1) equations in (n + 1) unknown, a_i. The unknown parameters are determined with a solver, which in this case is the Gaussian elimination method, and substituting this parameters in equation (13), we get the approximate solution$y_n(x)$of the integral equation (12).

4.0 Error Estimation

In this section an error estimation for the approximate solution of (12) is obtained. Let us call $E_n(x) = y(x) – y_n(x)$ as the error function of the approximate solution $y_n(x)$ to y(x), where y(x) is the exact solution of (12) . Hence, $y_n(x)$ satisfies the perturbed problem:

$$y(x)+\int_{-1}^{1}\mathrm{k}(\mathrm{x},\mathrm{s})\mathrm{y}(\mathrm{s})\mathrm{ds}=\mathrm{f}(\mathrm{x})+H_n(x). \quad (15)$$

The perturbation term H_n(x) can be obtained by substituting the computed solution y_n(x) into the equation such that

$$H_n(x)=y_n(x)+\int_{-1}^{1}\mathrm{k}(\mathrm{x},\mathrm{s})\mathrm{y}_\mathrm{n}(\mathrm{s})\mathrm{ds}-\mathrm{f}(\mathrm{x}) \quad (16)$$

We proceed to find an approximate $E_{n,N}$(x) to the error function $E_n(x)$in the same way as we did for the solution of (12) . Hence, the error function, $E_n(x)$ satisfy the problem

$$\mathrm{E_n}(\mathrm{x})+\int_{-1}^{1}\mathrm{k}(\mathrm{x},\mathrm{s})\mathrm{E_n}(\mathrm{s})\mathrm{ds}-\mathrm{f}(\mathrm{x})=-H_n(x). \quad (17)$$

5.0 Numerical Applications

In this section, the method is applied to solve some selected Linear Fredholm integral equations for experimentation.

Example 5.1: Consider the linear integral of the first kind

$$\int_{0}^{1}\mathrm{xe^{xs}}\mathrm{y}(\mathrm{s})\mathrm{ds}=\mathrm{e^x}-\mathrm{y}(\mathrm{x}) \quad (18)$$

The analytic solution is y(x) = 1.

Substitute the approximate solution given by equation (13), for n=3 we obtain,

$$\frac{1}{2}\int_{-1}^{1} x e^{x\left(\frac{y+1}{2}\right)}\left[\sum_{r=0}^{3} a_r\varphi_r\left(\frac{y+1}{2}\right)\right]dy = e^x - \sum_{r=0}^{3} a_r\varphi_r(x) \quad (19)$$

Substituting the values of $\varphi_r(x)$ (r = 0,1,…, 3) in (19) and evaluating using maple 18 software, the resulting equation is solved by collocating at the zeros of $\varphi_4(x)$ to obtain the unknown parameters as

a_0 =1.0000, a_1 = 0.000000, a_2 = 0.000000 and a_3 =0.000000.

Therefore,

$y_3(x) = \sum_{i=0}^{3} a_i\varphi_i(x) = 1.000000$,

Which is the same exact solution of equation (18). Similarly, $y_4(x) = y_5(x) = y_3(x)$.

Example 5.2

Consider the linear Fredholm integral of the first kind.

$$y(x) = 1 + \int_0^1 x^2 s^3 y(s)ds \quad (20)$$

The exact solution is $y(x) = 1 + \frac{3}{10}x^2$.

Using the proposed method for equation (20) and solving for $n = 3$, we have,

$$\sum_{r=0}^{3} a_r\varphi_r(x) - \frac{1}{8}\int_{-1}^{1} x^2(y+1)^3\left[\sum_{r=0}^{3} a_r\varphi_r\left(\frac{y+1}{2}\right)\right]dy = 1. \quad (21)$$

Substituting the values of $\varphi_r(x)$ (r = 0,1,…, 3) in (21) and evaluating using maple 18 software, the resulting equation is solved by collocating at the zeros of $\varphi_4(x)$ to obtain the unknown parameters as

a_0 =1.120000, a_1 = 0.000000, a_2 = 0.180000 and a_3 =0.000000.

Therefore,

$y_3(x) = \sum_{i=0}^{3} a_i\varphi_i(x) = 1 + \frac{3}{10}x^2$,

Which is the same exact solution of equation (20). Similarly, $y_4(x) = y_5(x) = y_3(x)$.

Example 5.3

Consider the linear Fredholm integral of the first kind.

$$\int_0^1 (x+s)y(s)ds = y(x) - \frac{3}{2}x + \frac{5}{6} \quad (22)$$

Substitute the approximate solution given by equation (5), for case n=3 we obtain,

$$\frac{1}{4}\int_{-1}^{1}(2x+y+1)\left[\sum_{r=0}^{3} a_r\varphi_r\left(\frac{y+1}{2}\right)\right]dy - \sum_{r=0}^{3} a_r\varphi_r(x) = -\frac{3}{2}x + \frac{5}{6} \quad (23)$$

Substituting the values of $\varphi_r(x)$ (r = 0,1,…, 3) in (22) and evaluating using maple 18 software, the resulting equation is solved by collocating at the zeros of $\varphi_4(x)$ to obtain the unknown parameters as

a_0 =-1.000000, a_1 = 1.000000, a_2 = 0.000000 and a_3 =0.000000.

Therefore,

$y_3(x) = \sum_{i=0}^{3} a_i\varphi_i(x) = -1 + x$,

Which is the same exact solution of equation (22). Similarly, $y_4(x) = y_5(x) = y_3(x)$.

6.0 Discussion of Results

The numerical evidences clearly reflect the efficiency and reliability of the method. The results obtained show the rapid convergence of the approximant to the analytic solution using few polynomials (n=3) in all examples considered. Also, extending the number of polynomials (n=4 and 5), same approximate solution is obtained. This shows the stability of solution at various grid points.

7.0 Conclusion

The orthogonal collocation approximation method has been successively applied to obtain the analytic solution of the integral equations. The method is effective and reliable as shown in the examples considered in this paper. The method can be extended to solve problems in dynamical systems in analyzing flow lines.

8.0 References

[1] Adeniyi, R.B. (2004). The tau method in the numerical of integral equations. Abacus, **10**:261-274.

[2] Elliott, D. (1963). A Chebychev series method for the numerical solution of Fredholm integral equations. Computer Journal,**6**: 102 -112.

[3] Villadsen, J.V. and Stewart, W.E. (1967). Solution of boundary value problems by orthogonal collocation. Chemical Engineering Sci,**22**(1): 483-501.

[4] Finlayson, B.A. (1980). Orthogonal collocation on finite elements - progress and potential. Mathematics and Computer in stimulation, **22(1):**11 - 17.

[5] Biscaia, E.C. Junior, Mansur, M.B., Salum, A. and Castro, R.M.Z (2001). A moving boundary problem and orthogonal collocation in solving a dynamic liquid surfacant membrane model including osmosis and breakage. Brazilian Journal of Chemical Engineering, **18**: 2.

[6] Lambert, D.J. (1973). Computational methods in ordinary differential equations. New York: John Willey.

[7] Adeniyi, R.B. Adeyefa, E.O and Alabi, M.O (2006). A Continuous formulation of some non-perturbed multistep collocation approach using Chebyshev polynomials as basis functions. Journal of the Nigerian Association of Mathematical Physics**, 10**:261-274.

[8] Fox, L and Parker, I.B. (1968). Chebychev polynomials in numerical analysis. Oxford University Press, Oxford, London.

[9] Lanczos, C. (1938).Trigonometric interpolation of empirical analytical functions . J. Math. Phys., **17:** 123-199.

[10] Fortini, M.A., Stamoulis, M.N., Ferreira, F.M., Claubia, P., Costa, A.L. and Silver, C.A.M. (2008). Application of the orthogonal collocation method to determination of temperature distribution in cylindrical conductor. Annals of Nuclear Energy, **35(9):**1681-1685.

Doi:10.1016/j.anucene.2008.02.006.

[11] Jannike, S. and Hugo, J. (2012).Solution of the Pellet equation by use of the orthogonal collocation and least square methods: Effects and different orthogonal Jacobi polynomials. International Journal of Chemical Reactor Engineering10(1):1542-6580.

[12] Cizinar, M., Salhi,D., Fikar, M., and Latifi, M.A. (2015). A matlab package for orthogonal collocations on finite elements in dynamic optimization. 15th Int. Conference Process Control, Slokavia.

[13] Hedengen, J.D. and Asgharzadeh, S.R., Powee, K.M. and Edgar, T.F. (2014). Nonlinear modeling, estimation and predictive control in APmonitor. Computers and Chemical Engineering, **70**:133-148.

[14] Mamadu,E.J. and Njoseh, I.N. (2016).Certain orthogonal polynomials in orthogonal collocation methods of solving Fredholmintegro-differential equation (FIDEs). Trans. of NAMP, 2: 59-64.

[15] Mamadu, E.J. and Njoseh, I.N. (2016). Numerical solutions of Volterra equation with certain orthogonal polynomials. Journal of Applied Mathematics and Physics, JAMP, **4**:367-382