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Modified Variation Iteration Homotopy Perturbation Method for the Approximate Solution of Burgers Equation

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Abstract

The Burgers' equation is relevant in modelling wave processes in hydrodynamics and in acoustics. Hence, obtaining the approximate solutions of these problems is of great importance. This paper seeks the approximate solution of the nonlinear one-dimensional Burger's equation via the modified variation iteration homotopy perturbation method (MVIHPM). Themethod is an elegant mixture of variation iteration and homotopy perturbation method. The original structure of the variationiteration homotopy perturbation method is being modified in the approximate solution as the addition of the coefficients of the least and highest power of the homotopy parameter, p. The resulting numerical evidences show that the method is very effective and reliable for solving nonlinear partial differential equations (Burgers' equation) and nonlinear ordinary differential equations as compared to thevariation iteration homotopy perturbation method. Also, the convergence of the solution is discussed. All computations are performed with maple 18 software.

Keywords: Burgers Equation, VariationIteration Method, Homotopy Perturbation Method,Nonlinear PDEs, Variation theory.
MSC2010: 65M99

1.0 Introduction

Nonlinear equations are actively relevant in the mathematical modelling of real life situations. Existing analytic solvers for these equations such as the d-expansion method, the perturbation method, the Lyapunov parameter method, etc., are difficult to handle. Thus, iterative methods have been designed by various researchers in recent years to effectively handle these equations. These include; the reconstruction of variation iteration method (RVIM) [1], the variation iteration method (VIM) [2] and [3], the homotopy perturbation method [4,5], the homotopy analysis method (HAM)[6], The composition method[7], the piecewise-adaptive decomposition method[8]etc.

Underlying shock waves theories and fluid dynamic model such as the Burgers' model turbulence have been studied by various researchers for comprehensive understanding of physical flow by the use of various iterative schemes. The Burgers' equation in particular is a modelling tool in various fields such as gas dynamic, shock waves, turbulence, dispersive water waves, [9-12] etc. Different iterative methods have been developed by various researchers as earlier said.

In this paper, we investigate the approximate solution of the nonlinear one-dimensional Burgers' equation of the form:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2}, \qquad (1)$$

subject to the initial condition:

$$u(x,t) = \frac{1}{2} - \frac{1}{2}\tanh\left(\frac{x}{4}\right),$$

whose analytic solution is,

$$u(x,t) = \frac{1}{2} - \frac{1}{2}\tanh\left(\frac{1}{4}(x - \frac{t}{2})\right)$$

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Using the modified variation iteration homotopy perturbation method.
In this method, the correction functional is constructed from which the Lagrange multiplier is calculated via variation theory. Thereafter, the He's polynomials are introduced and comparison of like powers of the homotopy parameter, p, gives the solution of various order. Finally, the best approximate is presented as the sum of the least and highest power of the homotopy parameter, p. This method requires no discretization, linearization or perturbation. Also, the method requires less computational effort since it focuses on just two coefficients of the homotopy parameter, p to obtain the best approximate solution, which is major advantage over the VIHPM where all coefficients of the homotopy parameter, p are considered for the best approximate solution. The initial approximation is freely chosen by imposing the boundary and initial conditions. The method is illustrated through a precise analysis of equation (1) to compute numerical solutions.

2.0 Variation Iteration Method (VIM)

Consider the general differential equation

$$L[u(x,t)] + N[u(x,t)] = f(x,t), \tag{2}$$

where L is a linear operator and N a non linear term, and f is the source term.

A correction functional for equation (2) can be constructed as

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^x \lambda(s)\left[Lu_n(x,s) + N\tilde{u}_n(x,s) - f(x,s)\right]ds,\ n \geq 0, \tag{3}$$

where $\lambda(s)$ is a general Lagrange multiplier, $\tilde{u}_n(x,s) = 0$, i.e, $\tilde{u}_n(x,s)$ is a restricted variable.

From equation (3), the correction functional ([2] and [3]) for equation (1) is given a

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(s)\left[\frac{\partial^2 u_n(x,s)}{\partial x^2} - \frac{\partial u_n(x,s)}{\partial t} - u_n(x,s)\frac{\partial u_n(x,s)}{\partial x}\right]ds,\ n \geq 0, \tag{4}$$

Solving equation (4) for the optimal value of$\lambda(s)$via the variationtheory, we get

$\lambda(s) = -1.$

Hence, equation (4) can be rewritten as

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left[\frac{\partial^2 u_n(x,s)}{\partial x^2} - \frac{\partial u_n(x,s)}{\partial t} - u_n(x,s)\frac{\partial u_n(x,s)}{\partial x}\right]ds, \qquad n \geq 0 \tag{5}$$

which is the variation iteration method (VIM) for the Burgers' equation.

3.0 Homotopy Perturbation Method (HPM)

Considering the Burgers' equation (1)and its correction functional in equation (5), we define the structure of the homotopy perturbation method He [4,5] as

$$H(v,p) = (1-p) * \left[\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 u_0}{\partial x^2}\right] + p\left[\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} - u\frac{\partial u}{\partial x}\right] = 0, \tag{6}$$

where p is an embedded homotopy parameter and lies in the closed interval $[0,1]$. The HPM defines the approximate solution as a power series in p, and is given as

$$v(x,t) = \sum_{n=0}^{\infty} p^n v_n \tag{7}$$

By comparisons of like powers of p give the solution of various orders and the best approximate solution is given as

$$u(x,t) = \sum_{n=0}^{\infty} v_n \tag{8}$$

The HPM treats the nonlinear part $N[u(x,t)]$ as

$$N[u(x,t)] = \sum_{i=0}^{+\infty} p^i \left(\frac{1}{n!}\frac{\partial^n}{\partial p^n}\left(N\left(\sum_{i=0}^{n} p^i u_i\right)\right)_{p=0}\right),\ n = 0,1,2, \ldots$$

3.1 Variation Iteration Homotopy Perturbation Method (VIHPM)

From equation (1) and its correction functional given in equation (5), we obtain by applying the HPM the iterative scheme,

$$\sum_{n=0}^{\infty} p^n v_n(x,t) = u_0(x,t) - \int_0^t \left[\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{\infty} p^n v_n(x,s)\right) - \frac{\partial}{\partial t}\left(\sum_{n=0}^{\infty} p^n v_n(x,s)\right) - \left(\sum_{n=0}^{\infty} p^n v_n(x,s)\right)\frac{\partial}{\partial x}\left(\sum_{n=0}^{\infty} p^n v_n(x,s)\right)\right]ds, n \geq 0, \tag{9}$$

which is thevariation iteration homotopy perturbation method. By comparisons of like powers of p give the solution of various orders and the best approximate solution is given as in equation (8).

3.2 Modified Variation Iteration Homotopy Perturbation Method (MVIHPM)

From equation (9), the modified variation iteration homotopy perturbation method defines the approximate solution as a power series in p, and is given as

$$v(x,t) = \sum_{n=0}^{\infty} p^n v_n \tag{10}$$

By comparisons of like powers of p give the solution of various orders and the best approximate solution is given as the sum of the coefficients of the least and highest occurring powers of the homotopy parameter, p. Therefore

$u(x,t) = v_0 + v_n,$ (11)

where v_0 is the coefficient of the least occurring homotopy parameter, and v_n is the coefficient of the highest occurring homotopy parameter. The method requires less computational effort since it focuses on just two coefficients of the homotopy parameter to obtain the best approximate solution. This is a major advantage of the MVIHPM over the VIHPM where all coefficients of the homotopy parameter are considered for the best approximate solution. The convergence of the method is maximized by the primary variable, $t(x$ is the secondary variable) in equation (11), where it fluctuates as the number of iterations, n increases.

Theorem 3.1

The rate of convergence of solutions of the Burgers' equation is maximized for $t = \alpha 10^{-n}$, where α assumes any positive integer or decimal number, and n is the number of iterations.

Proof

By MVIHPM, the best approximate solution be given as

$u_n(x,t) = v_0 + v_n$,

satisfying the initial condition,

$u(x,t) = \frac{1}{2} - \frac{1}{2}\tanh\left(\frac{x}{4}\right).$

Let exact solution of equation (1) be also given ,

$u(x,t) = \frac{1}{2} - \frac{1}{2}\tanh\left(\frac{1}{4}(x - \frac{t}{2})\right).$

Then, the absolute error,

$e_r = |u(x,t) - u_n(x,t)| \leq \delta,$

where δ is the maximum error .

If

$x = \frac{m}{10},\ m = 0,1,2,3, \ldots n,$ and $t = \alpha 10^{-n},$

then for $\alpha = 0.2$ *and* 0.6 chosen arbitrarily, $\mathrm{m} = 0$ and $n = 1$, we obtain $\delta = 10^{-6}$ and 10^{-5} respectively. Similarly, for $\alpha = 0.2$ and 0.6 , $\mathrm{m} = 0$ and $n = 2$, we obtain $\delta = 10^{-8}$ and 10^{-7}.

Hence, by mathematical induction $(n + 1)$ iterations will see the approximate solution coincide with the exact solution.

4.0 Applications

In order to ensure the effectiveness and accuracy of MVIHPM for solving nonlinear partial differential equations, we will solve the nonlinear one-dimensional Burgers' equations as given in equation (1). Results are compared with VIHPM.

4.1 Nonlinear One-dimensional Burgers' Equation

Using the correction functional as in equation (5) and following the MVIHPM with the initial approximation,

$u_0(x,t) = \frac{1}{2} - \frac{1}{2}\tanh\left(\frac{x}{4}\right),$

we obtained the **Tables** below showing the comparison of absolute errors for $n = 1$and 2.

Table 1: Comparison of absolute error obtained by VIHPM and MVIHPM using first approximation, $n = 1$

X	$t = 0.02$		$t = 0.06$		$t = 0.09$		$t = 0.12$		$t = 0.15$	
	E_{VIHPM}	E_{MVHPM}	E_{VIHPM}	E_{MVHPM}	E_{VIHPM}	E_{MVHPM}	E_{VIHPM}	E_{MVHPM}	E_{VIHPM}	E_{MVHPM}
0.00	2.5016e-03	1.5651e-06	7.5141e-03	1.4133e-05	1.1282e-02	3.1878e-05	1.5057e-02	5.6812e-05	1.8839e-02	8.8989e-05
0.10	2.4376e-03	2.9663e-05	7.3228e-03	7.9571e-05	1.0995e-02	1.0870e-04	1.4675e-02	1.3064e-04	1.8363e-02	1.4536e-04
0.20	2.3709e-03	6.0742e-05	7.1229e-03	1.7288e-04	1.0696e-02	2.4873e-04	1.4277e-02	3.1745e-04	1.7866e-02	3.7898e-04
0.30	2.3017e-03	9.1520e-05	6.9156e-03	2.6532e-04	1.0385e-02	3.8752e-04	1.3863e-02	5.0267e-04	1.7349e-02	6.1071e-04
0.40	2.2303e-03	1.2184e-04	6.7018e-03	3.5645e-04	1.0065e-02	5.2440e-04	1.3437e-02	6.8540e-04	1.6817e-02	8.3941e-04
0.50	2.1571e-03	1.5157e-04	6.4825e-03	4.4583e-04	9.7364e-03	6.5868e-04	1.2999e-02	8.6475e-04	1.6270e-02	1.0640e-03
0.60	2.0825e-03	1.8055e-04	6.2589e-03	5.3302e-04	9.4012e-03	7.8975e-04	1.2552e-02	1.0399e-03	1.5712e-02	1.2833e-03
0.70	2.0068e-03	2.0866e-04	6.0319e-03	6.1764e-04	9.0608e-03	9.1699e-04	1.2099e-02	1.2099e-03	1.5145e-02	1.4964e-03
0.80	1.9303e-03	2.3578e-04	5.8025e-03	6.9930e-04	8.7169e-03	1.0398e-03	1.1640e-02	1.3742e-03	1.4572e-02	1.7024e-03
0.90	1.8534e-03	2.6178e-04	5.5719e-03	7.7767e-04	8.3709e-03	1.1578e-03	1.1179e-02	1.5320e-03	1.3996e-02	1.9003e-03
1.00	1.7764e-03	2.8658e-04	5.3408e-03	8.5243e-04	8.0244e-03	1.2704e-03	1.0717e-02	1.6827e-03	1.3418e-02	2.0893e-03

Table 2: Comparison of absolute error obtained by VIHPM and MVIHPM using second approximation,$n = 2$.

X	$t = 0.02$		$t = 0.06$		$t = 0.09$		$t = 0.12$		$t = 0.15$	
	E_{VIHPM}	E_{MVHPM}	E_{VIHPM}	E_{MVHPM}	E_{VIHPM}	E_{MVHPM}	E_{VIHPM}	E_{MVHPM}	E_{VIHPM}	E_{MVHPM}
0.00	7.5006e-04	1.5600e-08	2.2506e-03	1.4060e-07	3.3763e-03	3.1640e-07	4.5022e-03	5.6250e-07	7.1306e-03	1.4102e-06
0.10	7.2774e-04	3.1068e-06	2.1836e-03	9.2271e-06	3.2758e-03	1.3735e-05	4.3684e-03	1.8173e-05	6.9187e-03	2.8256e-05
0.20	7.0459e-04	6.2138e-06	2.1142e-03	1.8548e-05	3.1717e-03	2.7718e-05	4.2295e-03	3.6818e-05	6.6990e-03	5.7781e-05
0.30	6.8073e-04	9.2896e-06	2.0426e-03	2.7778e-05	3.0644e-03	4.1563e-05	4.0864e-03	5.5280e-05	6.4724e-03	8.7018e-05
0.40	6.5628e-04	1.2320e-05	1.9692e-03	3.6869e-05	2.9543e-03	5.5202e-05	3.9397e-03	7.3468e-05	6.2402e-03	1.1582e-04
0.50	6.3135e-04	1.5289e-05	1.8945e-03	4.5779e-05	2.8422e-03	6.8570e-05	3.7902e-03	9.1294e-05	6.0034e-03	1.4406e-04
0.60	6.0606e-04	1.8184e-05	1.8186e-03	5.4466e-05	2.7284e-03	8.1602e-05	3.6385e-03	1.0867e-04	5.7633e-03	1.7159e-04
0.70	5.8053e-04	2.0991e-05	1.7420e-03	6.2890e-05	2.6135e-03	9.4241e-05	3.4853e-03	1.2553e-04	5.5208e-03	1.9830e-04
0.80	5.5488e-04	2.3698e-05	1.6651e-03	7.1013e-05	2.4981e-03	1.0643e-04	3.3314e-03	1.4179e-04	5.2771e-03	2.2405e-04
0.90	5.2921e-04	2.6293e-05	1.5880e-03	7.8802e-05	2.3825e-03	1.1812e-04	3.1774e-03	1.5738e-04	5.0332e-03	2.4876e-04
1.00	5.0363e-04	2.8766e-05	1.5113e-03	8.6227e-05	2.2674e-03	1.2926e-04	3.0239e-03	1.7224e-04	4.7901e-03	2.7231e-04

5.0 Discussion of Results

As shown is **Tables 1** and **2**, the amount of errors depend on the primary parameter, t. To obtain better enhancement in results, we assigned the primary parameter $t = \alpha 10^{-n}$, where α is chosen arbitrarily to assume any positive real number, and n is the number of iterations. The absolute errors maximizes with more iterations in t. For instance, when $\alpha = 0.2$, then $t = 0.02$ yields the maximum error 10^{-6} using first approximation at $x = 0$,as shown in **Table 1**. Similarly, for $\alpha = 0.2$, then $t = 0.002$ yields the maximum error 10^{-8} using second approximation at $x = 0$, as shown in **Table 2**. Hence, for$(n + 1)$ iterations, the approximate solution will coincide with the exact solution.

6.0 Conclusions

In this paper, the modified variation iteration homotopy perturbation methodhas been applied to seek the approximate solution of the Burgers' equation. The result obtained from the MVIHPM is in agreement with the reconstruction invariation iteration method inEsfandyaripour*et al.* [1]. Hence, the method is very effective and reliable in seeking the approximate solution of nonlinear partial differential equations. In this work, the maple 18 software is use for our computations. The method requires less computational effort which focuses on just two coefficient of the homotopy parameter to obtain the best approximate solution, which is a major advantage over the VIHPM.

7.0 References

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