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Stability Analysis of Fractional Duffing Oscillator II

M.O. Oyesanya and C.L. Ejikeme

Department of Mathematics, University of Nigeria, Nsukka 410001, Nigeria.

Abstract

In this second part of this paper we use the homotopy analysis method (HAM) to solve the nonlinear fractional Duffing Oscillator.

$$D^{\alpha}q(t)+\lambda \mathrm{D}^{\beta} q(t)-\mu q(t)+\nu q^{3}(t)= f(t)$$

We show that the fractional Duffing oscillator with two fractional derivatives can be seen as an appropriate model for earthquake prediction. We discuss the case for which $\alpha \neq 2\beta$ ***and*** $\alpha-\beta \neq n \in$ □ .

1.0 Introduction

This work is a continuation of the work in [63]. Therefore we maintain the format of that paper.

2.0 Continuing Analysis

In section 9.0 we considered the linear oscillator with forcing function typified by $f(t)=\lambda \sin \omega t$ and we got a solution (equation (50)) using Laplace transform. In figures 3 – 8 we exhibit simulation results with varying values of α, β, μ,δ. These simulations show the influence of the fractional order on the amplitudes of the solution. We will discuss this below.

3.0 Homotopy Analysis Method Applied

We now consider the problem (2) of section 9.0 namely

$$D^{\alpha}q(t)+\lambda \mathrm{D}^{\beta} q(t)-\mu q(t)+\nu q^{3}(t)= f(t) \qquad (65)$$

Using homotopy analysis method (HAM) we solve this problem. HAM is briefly treated in section 2.0 and further reading can be done with [46 – 50].

In solving this problem we utilize our Theorem 1 in section 10 and write (65) as the system (66)

If we follow theorem 3.1 of [57] equation (65) is equivalent to the system.

Corresponding author: M.O. Oyesanya, E-mail: moyesanya@yahoo.com, Tel.: +2348057025285

$$
\begin{aligned}
&D_*^{\gamma} x_{1_1}(t) = x_{1_2}(t)\\
&D_*^{\gamma} x_{1_2}(t) = x_{1_3}(t)\\
&D_*^{\gamma} x_{1_3}(t) = x_{1_4}(t)\\
&\quad .\\
&\quad .\\
&\quad .\\
&D_*^{\gamma} x_{1_{M\alpha_1}}(t) = f_1(t, \overline{x}(t))\\
&D_*^{\gamma} x_{2_1}(t) = x_{2_2}(t)\\
&D_*^{\gamma} x_{2_2}(t) = x_{2_3}(t)\\
&D_*^{\gamma} x_{2_3}(t) = x_{2_4}(t)\\
&\quad .\\
&\quad .\\
&\quad .\\
&D_*^{\gamma} x_{2_{M\alpha_2}}(t) = f_2(t, \overline{x}(t))\\
&\quad .\\
&\quad .\\
&\quad .\\
&D_*^{\gamma} x_{n_1}(t) = x_{n_2}(t)\\
&D_*^{\gamma} x_{n_2}(t) = x_{n_3}(t)\\
&D_*^{\gamma} x_{n_3}(t) = x_{n_4}(t)\\
&\quad .\\
&\quad .\\
&\quad .\\
&D_*^{\gamma} x_{n_{M\alpha_n}}(t) = f_n(t, \overline{x}(t))
\end{aligned} \tag{66}
$$

Together with the initial condition

$$
x_{i_j}(0) = \begin{cases} x_{i_1}(0), & \text{as } \mathrm{j}=1 \\ 0, & otherwise \end{cases} \tag{67}
$$

Noting that equation (65) can be written as

$$
D_*^{\alpha} y(t) = f\left(t, \pi y(t), D_*^{\beta_1} y(t), \ldots, D_*^{\beta_n} y(t)\right), \;\; y^{(k)}(0) = c_k, \;\; k = 0, \ldots., m \tag{68}
$$

Where $m < \alpha \le m+1,\ 0 < \beta_1 < \beta_2 < \ldots.. < \beta_n < \alpha$ and D_*^{α} denotes Caputo fractional derivative of order α. It should be noted of course that f can be nonlinear in general and

$$
\alpha, \beta_j \in \mathbb{R}, \alpha - \beta_n \le 1, \beta_j - \beta_{j-1} \le 1, \forall j, \quad 0 \le \beta \le 1 \tag{69}
$$

Following the scheme as in [52] we write equation (65) as the system

$$
\begin{aligned}
&D_*^{\beta_1} q = q_1\\
&D_*^{\beta_2} q_1 = q_2\\
&D_*^{\beta_3} q_2 = q_3\\
&\quad .\\
&\quad .\\
&\quad .\\
&D_*^{\beta_n} q_{n-1} = -\delta q_{n-1} + \mu q_n - \gamma q_1^3 + f(t)
\end{aligned} \tag{70}
$$

Applying homotopy analysis method to the above system of FDE leads to the following so called zero order deformation equations:

$$
\begin{aligned}
&(1-p) D^{\beta_i}[\varphi_i(t:p) - q_{i0}(t)] = \mathrm{ph}_i\, H_i [D^{\beta_i} \varphi_i(t;p) - \varphi_{i+1}(t;p)],\ i = 1, 2, \ldots., n-1,\\
&(1-p) D^{\beta_n}[\varphi_n(t;p) - q_{n0}(t)] = \mathrm{ph}_i\, H_n [D^{\beta_i} \varphi_n(t;p) - f(t, \varphi_1, \varphi_2, \ldots, \varphi_n)]
\end{aligned} \tag{71}
$$

Where $p \in [0,1]$ is an embedding parameter, $h_i \ne 0$ are non-zero auxiliary parameters for which $H_i(t) \ne 0,\ i = 1, 2, \ldots, n$ arenon-zero functions, $q_{i0}(t)$ are initial guess of $q_i(t)$, $\varphi_i(t;p)$ are unknown functions..

We note from (71) that when p = 0 we have that $\varphi_i(t;0)=q_{i0}(t)$ and when p = 1 we have that $\varphi_i(t;1)=q_i(t), \quad i=1,2,3,....,n$ which shows that as p goes from 0 to 1 the solution varies from the initial guess $q_{i0}(t)$ to the solution $q_i(t)$. Thus if we expand $\varphi_i(t;p)$ in Taylor's series with respect to p we have

$$\varphi_i(t;p)=q_{i0}(t)+\sum_{m=1}^{\infty}q_{im}(t)p^m, \; i=1,2,...,\mathrm{n} \tag{72}$$

Where

$$q_{im}(t)=\frac{1}{m!}\frac{\partial^m\varphi_i(t;\mathrm{p})}{\partial p^m}\Big|_{p=0}, \quad i=1,2,...,n \tag{73}$$

Differentiating equation (71) m times with respect to p and setting p = 0 and finally dividing them by m! we obtain the mth – order deformation equation for i = 1, 2,…, n as (74) below

$$\begin{aligned}&D^{\beta_i}[q_{im}(t)-\chi_m q_{im-1}(t)]=h_iH_i(t)R_{im}(\bar{q}_{1m-1},...,\bar{q}_{nm-1},t), \; i=1,2,...,\mathrm{n}-1,\\&D^{\beta_n}[q_{nm}(t)-\chi_m q_{nm-1}(t)]=h_iH_n(t)R_{nm}(\bar{q}_{1m-1},...,\bar{q}_{nm-1},t)\end{aligned} \tag{74}$$

Where

$$\begin{aligned}&R_{im}(\bar{q}_{1m-1},...,\bar{q}_{nm-1},t)=\frac{1}{(m-1)!}\frac{\partial^{m-1}(D^{\alpha_i}\varphi_i(t;p)-\varphi_{i+1}(t;p))}{\partial p^{m-1}}\Big|_{p=0}\\&R_{nm}(\bar{q}_{1m-1},...,\bar{q}_{nm-1},t)=\frac{1}{(m-1)!}\frac{\partial^{m-1}(D^{\alpha_i}\varphi_n(t;p)-f(t,\varphi_1,\varphi_2,...,\varphi_n))}{\partial p^{m-1}}\Big|_{p=0}\end{aligned} \tag{75}$$

And

$$\chi_m=\begin{cases}0, & m\le 1\\ 1, & m>1\end{cases}$$

Applying the Riemann-Liouville integral operator J^{β_i} on both sides of equation (74), we have

$$\begin{aligned}&q_{im}(t)=\chi_m q_{im-1}(t)-\chi_m\sum_{j=0}^{n-1}q_{im-1}^j(0^+)\frac{t^j}{j!}+J^{\beta_i}h_iH_i(t)R_{im}(\bar{q}_{1m-1},...\bar{q}_{nm-1},t)\\&q_{nm}(t)=\chi_m q_{nm-1}(t)-\chi_m\sum_{j=0}^{n-1}q_{nm-1}^j(0^+)\frac{t^j}{j!}+J^{\beta_i}h_iH_n(t)R_{nm}(\bar{q}_{1m-1},...\bar{q}_{nm-1},t)\end{aligned} \tag{76}$$

4.0 Homotopy Analysis Method for Fractional Duffing Oscillator

In this section we now applied the above to the problem fractional Duffing oscillator. Consider the forced Duffing oscillator with fractional order derivatives given by the equation

$$\begin{aligned}&D^{\alpha}u+\delta D^{\beta}u+\rho u-\mu u^3=\lambda sin\omega t, \qquad \mathrm{t}\in[0,\infty)\\&\mathrm{u}(0)=\mathrm{a}\neq, \; \mathrm{D}^{\beta}u(0)=b\neq 0, \quad a,b \; are \; real \; or \; complex \; constants\end{aligned} \tag{77}$$

Where u = u(t) $\in C^m[0,b], \; b>0, \; m=[\alpha]$ is the displacement function, $0<\beta\le 1, 1<\alpha\le 2, \; \mu>0, \; \rho>0, \; \lambda>0$ where δ > 0 is the damping factor, ω is the frequency and a the initial displacement.

We now convert (77) to an equivalent system as depicted in section (17.0) above. With α and β in Q there exist $(p,q),(t,s)\in(\square^+)^2, \quad (p,q)=1,(t,s)=1$. Setting $\alpha=\frac{p}{q}, \; and \; \beta=\frac{t}{s}$ we pick an M (least common multiple) of q and s so that we have $\frac{1}{M}=\gamma \; and \; N:=M\alpha$ where N is the dimension of the equivalent system obtained. We state our result in the following theorem:

Theorem: The system (77) is equivalent to the N-dimensional system with u = u_0.

$$\begin{cases} D^{\gamma}u_0 = u_1 \\ D^{\gamma}u_1 = u_2 \\ D^{\gamma}u_2 = u_3 \\ \quad . \\ \quad . \\ \quad . \\ D^{\gamma}u_{N-2} = u_{N-1} \\ D^{\gamma}u_{N-1} = \lambda \sin \omega t - \delta u_{\beta/\gamma} - \rho u - \mu u^3 \end{cases} \tag{78}$$

With initial conditions $\begin{cases} u_0(0) = a \neq 0 \\ u_{\beta/\gamma} = b \neq 0 \end{cases}$ (79)

The proof of this theorem is in the same vein as in [57] and is omitted. We now proceed to solving (78, 79) using the homotopy analysis as outlined in section 17 above. We set $H_i(t) = 1$ *and* $h_i = h$. We choose our initial guess a

$$u_{0_0} = E_{\alpha,\beta}(-\delta t^{\alpha}) . \tag{80}$$

Our m-th order deformation is now given by

$$u_{0_m} = \chi_m u_{0_{m-1}} + hJ^{\gamma}\left(D^{\gamma}u_{0_{m-1}} - u_{1_{m-1}}\right)$$
$$\chi_m = \begin{cases} 0 & m \le 1 \\ 1 & m > 1 \end{cases} \tag{81}$$

This leads us using (80) in (81) to the solutions

$$u_{0_1} = hJ^{\gamma}\left(D^{\gamma}u_{0_0} - u_{1_0}\right) = hu_{0_0} = hE_{\alpha,\beta}\left(-\delta t^{\alpha}\right)$$
$$u_{0_2} = \left(h + h^2\right)E_{\alpha,\beta}\left(-\delta t^{\alpha}\right) - J^{\gamma}u_{1_1} \tag{82}$$
$$u_{0_3} = \left(h + 2h^2 + h^3\right)E_{\alpha,\beta}\left(-\delta t^{\alpha}\right) - \left(h + h^2\right)J^{\gamma}u_{1_1} - hJ^{\gamma}u_{1_2}$$

And we have

$$u_0 = u_{0_0} + u_{0_1} + u_{0_2} + u_{0_3} + \ldots$$
$$= \left(1 + 3h + 3h^2 + h^3\right)E_{\alpha,\beta}\left(-\delta t^{\alpha}\right) - \left(2h + h^2\right)J^{\gamma}u_{1_1} - hJ^{\gamma}u_{1_2} + \ldots$$

In the same way we solve for $u_1, u_2, \ldots, u_{N-1}$ having

$$u_{(N-1)m} = u_{(\mathrm{N}-1)_{m-1}} + hJ^{\gamma}\begin{Bmatrix} D^{\gamma}\left(u_{(N-1)_{m-1}}\right) - \lambda \sin \omega_i t + \delta u_{\beta/\gamma(m-1)} + \rho u_{0_{m-1}} + \\ \mu[u_{0_2}\left(u_{0_0}u_{0_0}\right) + u_{0_1}\left(u_{0_1}u_{0_0} + u_{0_0}u_{0_2}\right) + u_{0_0}\left(u_{0_2}u_{0_2} + u_{0_1}u_{0_1} + u_{0_0}u_{0_2}\right) \end{Bmatrix} \tag{83}$$

A Special Case:

A special case can be seen in the following example:

Consider the fractional Duffing oscillator problem given by

$$D^{\frac{3}{2}}u + \delta D^{\frac{1}{2}}u + \rho u + \mu u^3 = \lambda \sin \omega t \qquad t \in [0, \infty)$$
$$u(0) = \tfrac{2}{\Gamma(\frac{1}{2})}, \quad D^{\frac{1}{2}}u_0 = 1 \tag{84}$$

The equivalent system is given by

$$D^{\frac{1}{2}}u_0 = u_1, \quad u_0(0) = \tfrac{2}{\Gamma(\frac{1}{2})}$$
$$D^{\frac{1}{2}}u_1 = u_2, \quad u_1(0) = 1 \tag{85}$$
$$D^{\frac{1}{2}}u_2 = u_3 = \lambda \sin \omega t - \delta D^{\frac{1}{2}}u_0 - \rho u_0 - \mu u_0^3, \quad u_3(0) = 0$$

This can be solved in the same way as done in the general case above. We show this below
Consider the equation (85i)
$D^{\frac{1}{2}}u_0 = u_1.$

$(1 \quad q)(D^{\frac{1}{2}}u_0(t) \quad u_{0_0}(t)) = q\hbar H(t)(D^{\frac{1}{2}}u_0(t) \quad u_1(t))$ $D^{\frac{1}{2}}u_0(t) = u_{0_0}(t)$ and
Denote by $u_{00}(t)$, the initial guess for the equation (0.3). The zeroth order deformation equation will be of the form
where $q = 0$ gives
$q = 1$ gives the original equation. Choosing our initial guess
$u_{0_0}(t) = E_{\frac{3}{2},\frac{1}{2}}(-\delta t^{\frac{3}{2}}) = u_{0_0}.$

Thus the zeroth order deformation equation for (85) is
$(1-q)(D^{\frac{1}{2}}u_0 - E_{\frac{3}{2},\frac{1}{2}}(-\delta t^{\frac{3}{2}})) = q\hbar H(t)(D^{\frac{1}{2}}u_0 - u_1)$
To obtain the *mth*−order deformation equation, we use the procedure described in section 6.0 by taking
$L = D^{\frac{1}{2}}u_0$ -u_1
and

$$N = D^{\frac{1}{2}}u_0 - u_1$$

so that we have the *mth*−order deformation equation to be

$$u_{0m} = \chi_m u_{0_{m-1}} + hHJ^{\frac{1}{2}}\left(D^{\frac{1}{2}}u_{0_{m-1}} - u_{1_{m-1}}\right)$$

1
where

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases}$$

$h \neq 0$ and $H(t) \neq 0$ are the auxiliary parameter and function
respectively. Taking $H(t) = 1$ and $m = 1,2,3$ we have

$$u_{0_1} = \hbar E_{\frac{3}{2},\frac{1}{2}}(-\delta t^{\frac{3}{2}}) - \frac{h}{\sqrt{\pi}} - \frac{2h}{\sqrt{\pi}}t^{\frac{1}{2}}$$

$$u_{02} = (h+h^2)E_{\frac{3}{2},\frac{1}{2}}(-\delta t^{\frac{3}{2}}) - \frac{(h+h^2)}{\sqrt{\pi}} - \frac{(2h+2h^2)}{\sqrt{\pi}}t^{\frac{1}{2}}$$

$$u_{03} = (h+2h^2+h^3)E_{\frac{3}{2},\frac{1}{2}}(-\delta t^{\frac{3}{2}}) - \frac{(\hbar+2\hbar^2+\hbar^3)}{\sqrt{\pi}} - \frac{2(\hbar+2\hbar^2+\hbar^3)}{\sqrt{\pi}}t^{\frac{1}{2}} + 2\hbar^2 t - \frac{4\delta\hbar^3}{3\sqrt{\pi}}t^{\frac{3}{2}}$$

$$-\rho\hbar^3\sum_{k=0}^{\infty}\frac{(-\delta)^k\Gamma(2+k)\Gamma(\frac{3k}{2}+1)}{(\frac{3k}{2}+\frac{3}{2})\Gamma(\frac{3k}{2}+\frac{1}{2})\Gamma(\frac{3k}{2}+\frac{3}{2})}t^{\frac{3k}{2}+\frac{3}{2}} - \frac{3\mu\hbar^4}{2}\sum_{k=0}^{\infty}\frac{(-\delta)^k\Gamma(3+k)\Gamma(\frac{3k}{2}+1)}{k!(\frac{3k}{2}+\frac{3}{2})\Gamma(\frac{3k}{2}+\frac{1}{2})\Gamma(\frac{3k}{2}+\frac{3}{2})}t^{\frac{3k}{2}+\frac{3}{2}}$$

$$+\frac{3\mu\hbar^4}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{(-\delta)^k\Gamma(2+k)\Gamma(\frac{3k}{2}+1)}{k!(\frac{3k}{2}+\frac{3}{2})\Gamma(\frac{3k}{2}+\frac{1}{2})\Gamma(\frac{3k}{2}+\frac{3}{2})}t^{\frac{3k}{2}+\frac{3}{2}} + \frac{6\mu\hbar^4}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{(-\delta)^k\Gamma(2+k)(\frac{3k}{2}+\frac{1}{2})}{k!(\frac{3k}{2}+2)\Gamma(\frac{3k}{2}+2)}t^{\frac{3k}{2}+2}$$

$$-\frac{h^3\lambda}{2}\sum_{k=0}^{\infty}\frac{(-)^k\omega^{2k+1}\Gamma(2k+2)}{(2k+1)!\Gamma(2k+\frac{7}{2})}t^{\frac{3k}{2}+\frac{3}{2}}.$$

$$u_{04} = (h+3h^2+3h^3+h^4)E_{\frac{3}{2},\frac{1}{2}}(-\delta t^{\frac{3}{2}}) - \frac{(h+4h^2+5h^3+h^5)}{\sqrt{\pi}}$$

$$-\frac{2(\hbar+3\hbar^2+3\hbar^3+\hbar^4)}{\sqrt{\pi}}t^{\frac{1}{2}} + (3\hbar^2+4\hbar^3+\hbar^4)t \quad -\frac{4\delta\hbar^3}{3\sqrt{\pi}}t^{\frac{3}{2}} - \frac{4\hbar^4}{3\pi}t^{\frac{3}{2}} \quad +\frac{\delta\hbar}{2}t^2 - \frac{\hbar^4}{2}t^2$$

$$+\rho\hbar^3\sum_{k=0}^{\infty}\frac{(-\delta)^k\Gamma(\frac{3k}{2}+1)}{(\frac{3k}{2}+\frac{3}{2})\Gamma(\frac{3k}{2}+\frac{1}{2})\Gamma(\frac{3k}{2}+\frac{3}{2})}t^{\frac{3k}{2}+\frac{3}{2}} + \frac{(\hbar^3+2\hbar^4)\lambda}{2}\sum_{k=0}^{\infty}\frac{(-)^k\omega^{2k+1}\Gamma(2k+2)}{(2k+1)!\Gamma(2k+\frac{7}{2})}t^{2k+\frac{5}{2}}$$

$$+\rho\hbar^4\sum_{k=0}^{\infty}\frac{(-\delta)^k\Gamma(\frac{3k}{2}+1)}{(\frac{3k}{2}+\frac{3}{2})\Gamma(\frac{3k}{2}+\frac{1}{2})\Gamma(\frac{3k}{2}+\frac{3}{2})}t^{\frac{3k}{2}+\frac{1}{2}} + \frac{3\mu(\hbar^4+\hbar^5)}{2}\sum_{k=0}^{\infty}\frac{(-\delta)^k\Gamma(3+k)\Gamma(\frac{3k}{2}+1)}{k!(\frac{3k}{2}+\frac{3}{2})\Gamma(\frac{3k}{2}+\frac{1}{2})\Gamma(\frac{3k}{2}+\frac{3}{2})}t^{\frac{3k}{2}+\frac{3}{2}}$$

$$+\frac{3\mu\hbar^5}{2}\sum_{k=0}^{\infty}\frac{(-\delta)^k\Gamma(\frac{3k}{2}+1)}{(\frac{3k}{2}+\frac{3}{2})\Gamma(\frac{3k}{2}+\frac{1}{2})\Gamma(\frac{3k}{2}+\frac{3}{2})}t^{\frac{3k}{2}+\frac{3}{2}}-\frac{3\mu(\hbar^4+\hbar^5)}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{(-\delta)^k\Gamma(2+k)\Gamma(\frac{3k}{2}+1)}{k!(\frac{3k}{2}+\frac{3}{2})\Gamma(\frac{3k}{2}+\frac{1}{2})\Gamma(\frac{3k}{2}+\frac{3}{2})}t^{\frac{3k}{2}+\frac{3}{2}}$$
$$-\frac{3\mu(\hbar^4+\hbar^5)}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{(-\delta)^k\Gamma(2+k)(\frac{3k}{2}+1)}{k!\Gamma(\frac{3k}{2}+\frac{1}{2})\Gamma(\frac{3k}{2}+\frac{5}{2})}t^{\frac{3k}{2}+\frac{3}{2}}+\frac{3\mu}{2}(\hbar^4+\hbar^5)\sum_{k=0}^{\infty}\frac{(-\delta)^k\Gamma(3+k)\Gamma(\frac{3k}{2}+1)}{\Gamma(\frac{3k}{2}+\frac{1}{2})\Gamma(\frac{3k}{2}+\frac{5}{2})}t^{\frac{3k}{2}+\frac{3}{2}}$$
$$+\frac{3\mu\hbar^5}{2\pi}\sum_{k=0}^{\infty}\frac{(-\delta)^k\Gamma(\frac{3k}{2}+1)}{k!\Gamma(\frac{3k}{2}+\frac{1}{2})\Gamma(\frac{3k}{2}+\frac{5}{2})}t^{\frac{3k}{2}+\frac{3}{2}}+\hbar^4\sum_{k=0}^{\infty}\frac{(-\delta)^k\Gamma(\frac{3k}{2}+1)}{\Gamma(\frac{3k}{2}+\frac{1}{2})\Gamma(\frac{3k}{2}+\frac{5}{2})}t$$
$$-\frac{6\mu}{\sqrt{\pi}}(\hbar^4+\hbar^5)\sum_{k=0}^{\infty}\frac{(-\delta)^k(\frac{3k}{2}+\frac{1}{2})}{\Gamma(\frac{3k}{2}+3)}t^{\frac{3k}{2}+2}+\frac{12\mu\hbar^5}{\pi}\sum_{k=0}^{\infty}\frac{(-\delta)^k\Gamma(\frac{3k}{2}+2)}{\Gamma(\frac{3k}{2}+\frac{1}{2})\Gamma(\frac{3k}{2}+\frac{7}{2})}t^{\frac{3k}{2}+\frac{5}{2}}$$
$$-\frac{12\mu\hbar^5}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{(-\delta)^k\Gamma(2+k)(\frac{3k}{2}+\frac{1}{2})}{k!\Gamma(\frac{3k}{2}+3)}t^{\frac{3k}{2}+2}+\frac{12\mu\hbar^5}{\pi}\sum_{k=0}^{\infty}\frac{(-\delta)^k(\frac{3k}{2}+\frac{1}{2})}{\Gamma(\frac{3k}{2}+3)}t^{\frac{3k}{2}+2}$$
$$+\frac{6\mu(\hbar^4+\hbar^5)}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{(-\delta)^k\Gamma(2+k)(\frac{3k}{2}+\frac{1}{2})}{k!(\frac{3k}{2}+2)\Gamma(\frac{3k}{2}+2)}t^{\frac{3k}{2}+2}-\frac{6\mu\hbar^5}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{(-\delta)^k\Gamma(2+k)(\frac{3k}{2}+1)}{k!(\frac{3k}{2}+2)\Gamma(\frac{3k}{2}+\frac{5}{2})}t^{\frac{3k}{2}+\frac{3}{2}}$$

Neglecting the values of m >3, we investigate the influence of hon the solution series and since we still have freedom to choose the auxiliary parameter; we first consider the convergence of the related series such as u_0. These curves contain a horizontal line segment and this horizontal line segment denotes the valid region of hwhich guaranteed the convergence of related series. It is observed that the valid region for his $-2.0 < h < 0.0$ as shown in the figures 1 and 2 below:

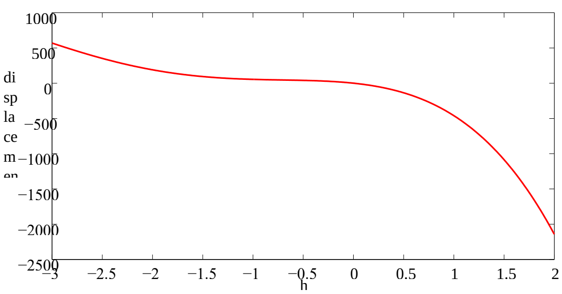

Figure 3: Graph of u_0 against h (h curve) for t = 5

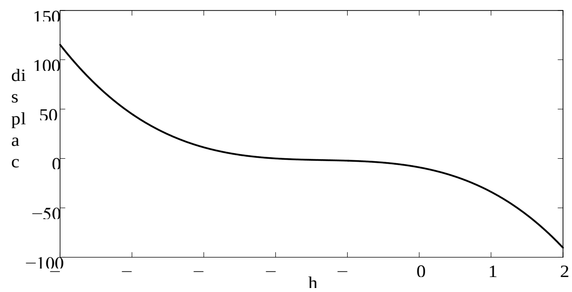

Figure 4: Graph of u_0 against h (h curve) for t = 8

Thus, the middle point of this interval, that is, −1.0, is an appropriate selection for ~ in which the numerical solution converges. Thus, the approximate analytical solution of system (0.2) is

$u_0 = u_{00} + u_{01} + u_{02} + u_{03} + u_{04} + \cdots$

which gives

$$u_0=\frac{2}{\sqrt{\pi}}+\frac{2}{\sqrt{\pi}}t^{\frac{1}{2}}+2t-\frac{4}{3\pi}t^{\frac{3}{2}}-\frac{1}{2}t^2$$
$$+\frac{\delta}{2}t^2+\lambda\sum_{k=0}^{\infty}\frac{(-)^k\omega^{2k+1}\Gamma(2k+2)}{(2k+1)!\Gamma(2k+\frac{7}{2})}t^{2k+\frac{5}{2}}-\frac{3\mu}{2}\sum_{k=0}^{\infty}\frac{(-\delta)^k\Gamma(3+k)\Gamma(\frac{3k}{2}+1)}{k!(\frac{3k}{2}+\frac{3}{2})\Gamma(\frac{3k}{2}+\frac{1}{2})\Gamma(\frac{3k}{2}+\frac{5}{2})}t^{\frac{3k}{2}+\frac{3}{2}}$$
$$-\frac{3\mu}{2}\sum_{k=0}^{\infty}\frac{(-\delta)^k\Gamma(3+k)\Gamma(\frac{3k}{2}+1)}{k!(\frac{3k}{2}+\frac{3}{2})\Gamma(\frac{3k}{2}+\frac{1}{2})\Gamma(\frac{3k}{2}+\frac{5}{2})}t^{\frac{3k}{2}+\frac{5}{2}}+\frac{3\mu}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{(-\delta)^k\Gamma(2+k)\Gamma(\frac{3k}{2}+1)}{k!(\frac{3k}{2}+\frac{3}{2})\Gamma(\frac{3k}{2}+\frac{1}{2})\Gamma(\frac{3k}{2}+\frac{3}{2})}t^{\frac{3k}{2}+\frac{3}{2}}$$
$$+\frac{6\mu}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{(-\delta)^k\Gamma(2+k)(\frac{3k}{2}+1)}{k!(\frac{3k}{2}+2)\Gamma(\frac{3k}{2}+\frac{5}{2})}t^{\frac{3k}{2}+\frac{3}{2}}+\frac{6\mu}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{(-\delta)^k\Gamma(2+k)(\frac{3k}{2}+\frac{1}{2})}{k!(\frac{3k}{2}+2)\Gamma(\frac{3k}{2}+2)}t^{\frac{3k}{2}+2}$$

Some Simulations:
Figures 5 – 10 below show some simulations that was done plotting u(t) against t

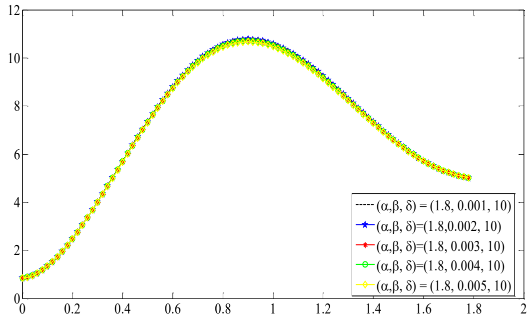

Figure 5: Influence of β on the solution as $\beta \to 0,\ 1.5 < \alpha < 2$

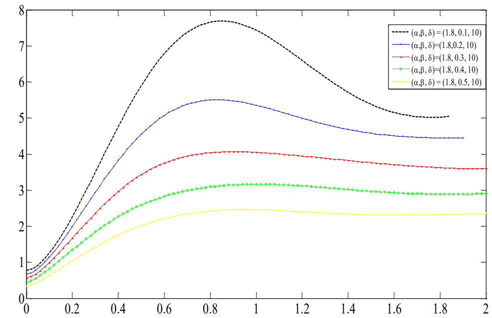

Figure 6: The case for $\beta \to 1,\ 0 < \beta < 1$

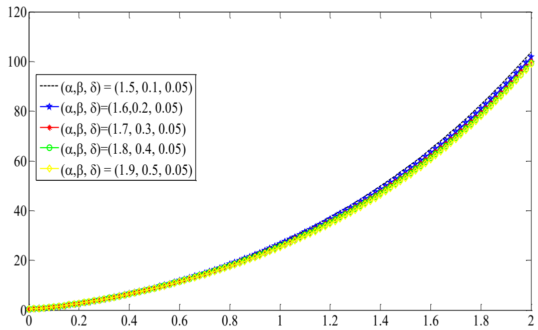

Figure 7: The case for $\alpha \to 2$, with $0 < \beta < 0.5$

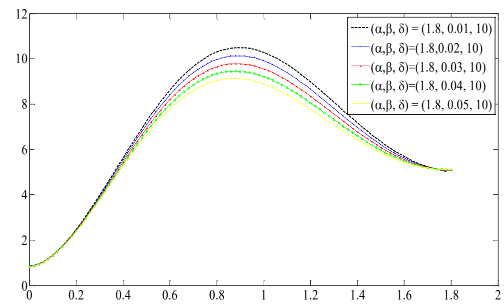

Figure 8: Influence of β in the rage $0.01 < \beta < 0.05$

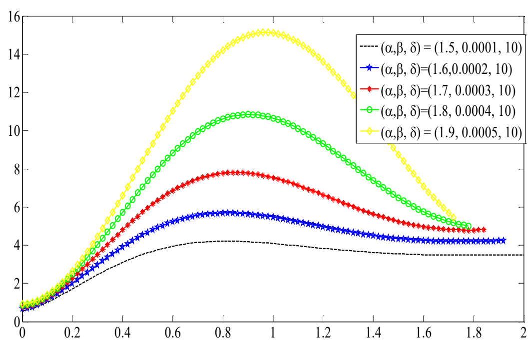

Figure 9: The case of $\alpha \to 2$ and $\beta \to 0$

5.0 Remarks

This research is ongoing. We are still working on the problem on the issue of stability of solutions using Lyapunov Mittag-Leffler stability procedure [64], phase space analysis which will help us show very clearly that the fractional Duffing is a more appropriate model for predicting earthquake.

It is obvious from simulations that the fractional Duffing oscillator analysis is more robust than the integer case.

6.0 References

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