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A Stochastic Algorithm and Multiple Scale For Solution to PDE With Financial Application

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Abstract

This paper presents an application of multiple scale and stochastic approximation method to discretize generic financial PDE. The multiple scale method was adopted in calculating the periodic solutions resulted from a Hopf bifurcation of a discretized generic PDE to monitor and stabilize the oscillatory movement of the market price of stock. Thereafter a stochastic algorithm was formulated to price an American option under the Black-Scholes model through a drifted financial derivative system. With finer discretization, positive periodic solution, space nodes and time nodes, we demonstrate that the drifted financial derivative system can be efficiently and easily solved with high accuracy, by using a stochastic approximation method which proves to be faster in pricing an American option. Multiple scale method is used to obtain the periodic solution which in turn is used to identify the movements of two parameters responsible for oscillatory behavior of the system to be risk-free in conjunction with the stock price. It is discovered that as the stock price decreases, the periodic solution fluctuates according to risk-free rate. An illustrative example is given in concrete setting. An illustrative example is given in a concrete setting.

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1.0 Introduction

As it is well known, the iteration methods play a fundamental role in numerical analysis due to their simple structure and flexibility in practical computation. Theoretically, all kinds of equations including functional equation(s) can be solved by using iteration method. For instance, the solution of first-order ordinary differential equation(s) (ODE) can be defined as the limit of the Picard iteration sequence. However the Picard sequence actually does not work in solving differential equations except for the very simple ODE, because it requires computing integration repeatedly. An iteration method works in solving differential equations only if special features or special classes of equations are addressed, for which the solution can be obtained by a small number of iterations.

For numerical approximations, the most popular numerical methods for pricing American options can be classified to lattice method, Monte Carlo simulation and finite difference method. Sure, besides finite difference methods, there are other popular numerical methods based on discretization for solving PDEs like finite element method, boundary element method, spectral and pseudo-spectral methods and etc. In fact, finite difference method ranks as the most popular one among its kind in financial engineering. The lattice method is simple and still widely used for evaluating American options. It was first introduced in [1] and the convergence of the lattice method for American options is proved in[2]. Monte Carlo method requires some further modification due to the early exercise feature. Fu [3, 4] priced American-style options by using Monte Carlo method in conjunction with gradient-based optimization techniques. Duck et al. [5] proposed a technique which generates monotonically varying data to enhance the accuracy and reliability of Monte Carlo-based method in handling early exercise features.

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The finite difference method for pricing American options was first presented in [6,7, 8]. Jaillet et al. [9] showed the convergence of the finite difference method. A comparism of different numerical methods for American options pricing was discussed in [10,11]. Generally, there still exist some difficulties in using these numerical methods. For finite difference method, the difficulty arises from the early exercise property, which changes the original Black-Scholes equation to an inequality that cannot be solved via fractional finite difference process. Therefore, finding the early exercise boundary prior to spatial discretization (discretization on underlying asset) is a must in each time step. Horng and Tien [12] proposed a simple numerical method base on finite difference and method of lines to overcome this difficulty in American option valuation. To those who have ever dealt with Black-Scholes equations, the instabilities and oscillatory behavior modeled by Black-Scholes equation are all too well known. What perhaps not so familiar is that the Black-Scholes equations (PDE) could be discretized into delay differential equation (DDE) which have the opposite effect in the financial market, namely that they could suppress oscillations and stabilize equilibria which would be unstable in the absence of delays. For instance, oscillatory behavior can often be connected to a Hopf bifurcation of an equilibrium solution under the variation of some parameter, and such local bifurcations share common qualities expressed in terms of the behavior on a low-dimensional center manifold. Hence, an analysis of stabilization near a generic Hopf bifurcation would yield general results applicable to any system near a Hopf instability and serve as a useful guide for understanding the behavior of a financial market systems under time delays.

Although the financial derivatives are governed by the celebrated parabolic partial differential Black-Scholes formula, but it is not clear how derivatives are controlled and stabilized.Also the early exercise boundary prior to spatial discretization in each time step has been established for American option valuation.Another approach used in[13] is proposed in this work based on the fact that financial derivative experience a drift which hardly can be brought to equilibrium state. In the case of ordinary differential equation (ODE's), a very popular method for obtaining transient behavior is the two way variable expansion method (also known as multiple scales) in [14,15,16] proposed to understand the behavior of a financial market systems under time delays.

The analyses are made based on the discretization of Black-Scholes equation using central finite-difference approximation into first-order ordinary differential equation, which was transformed to a delay differential equation to monitor and stabilize the oscillatory movement of the market price of stock, using the multiple scale method and later transformed to a drifted financial derivative system. We solve the resulting drifted financial derivative system by employing a stochastic algorithm described and analyzed in [13], where each iteration requires the adjustment of the drift parameter based on the dividend yield.

The outline of the work is the following: In section 2 we presented a scalar DDE and the properties it must satisfy for the existence of Hopf bifurcation. Illustrative example was also given. We review modeling of Black-Scholes, the partial differential equation which financial derivative have to satisfy and formulate Linear Complementarity Problem (LCP) for an American option in section 3. In section 4, we discretize the generic PDE into LCP and drift financial derivative system and later transformed to DDE to check the stability of the market, while in section 5, multiple scale method were presented, applied and analyzed. A stochastic algorithm is formulated in section 6. Numerical experiments are presented in section 7 and conclusions are given in section 8.

2.0 Hopf Bifurcation of Time-Delay Systems

Consider the following scalar DDE's with a parameter q^* (the delay or some other physical parameter)

$$x^{(n)}(t) = F(x(t), x'(t), \dots, x^{n-1}(t), x(t-\tau), x'(t-\tau), \dots, x^{n-1}(t-\tau), q^*), x \in R \quad (1)$$

where F has at least up to fourth order continuous derivatives satisfying $F(0,0,\dots,0,q^*) \equiv 0$. Equation (1) is assumedto admit

a Hopf bifurcation at $q^* = {q^*}_0$. The existence of the bifurcation can be characterized by the root location of the characteristic function $D(\lambda, q^*)$ of the linearized equation at $x = 0$ of (1) as follows:

b. For a small $\varepsilon := q^* - {q^*}_0$. $D(\lambda, q^*)$ has exactly one pair of simple complex roots $\lambda(\varepsilon) = \alpha(\varepsilon) \pm i\beta(\varepsilon)$ such that at $\varepsilon = 0$, one has $\alpha(0) = 0$, $w_0 = \beta(0) > 0$, and all the other characteristics roots have negative real parts.

c. $\alpha'(0) = R\frac{d\lambda}{d\varepsilon}(0) \neq 0$ (the transversality condition), where $R(z)$ stands for the real part of $z \epsilon \mathbb{C}$.

Due to the Hopf bifurcation Theory, the bifurcated nontrivial periodic solution has a period approximately ${}^{2\pi}/_{\beta(\varepsilon)}$, and ${}^{2\pi}/_{\beta(\varepsilon)} \to {}^{2\pi}/_{w_0}$ as $\varepsilon \to 0$. Thus, in the vicinity of the Hopf bifurcation, namely for a sufficiently small $|\varepsilon|$, the stationary solution of (1) has a form

$$x(t) = r(\varepsilon t)\cos(w(\varepsilon)t + \theta) + 0(\varepsilon t))$$
$$= r\cos(w_o t + \theta) + 0(\varepsilon), \quad (2)$$

as done in applications of method of multiple scales, where

$r \coloneqq r(0), \theta \coloneqq \theta(0)$ for short. Therefore, it is expected that the time-delay system near the Hopf bifurcation behaves similar to the Black-Scholes differential equation involving a term $x''(s)$. The key features of the Hopf bifurcation of (1) can be preserved if the right hand function F is approximated with the third or fifth order Taylor expansion, which is required in the computation of the averaged power function, defined in [17]. That is to say, the local dynamics near the Hopf bifurcation of (1) can be determined from the averaged power function [17].

Example1; Let us study the following scalar DDE arising from laser physics Schanz and Pelster [18].

$$\dot{x}(t) = -\left(\frac{\pi}{2} + \varepsilon\right) \sin x(t-1), \quad (3)$$

where $|\varepsilon| \ll 1$ is a small parameter. Equation (3) undergoes a Hopf bifurcation at $\varepsilon = 0$, because the following conditions hold [16,17]:

1. For small $\varepsilon < 0$, the zero solution $x = 0$ of (3) is asymptotically stable.
2. At $\varepsilon = 0$, the characteristic function $p(\lambda) \coloneqq \lambda + (\frac{\pi}{2} + \varepsilon)e^{-\lambda}$ has a pair of complex conjugate roots $\lambda = \pm i \pi/2$, and the other roots of $p(\lambda)$ have negative real parts.
3. $R\left[\frac{d\lambda}{d\varepsilon}\right]_{\varepsilon=0} \neq 0$, where R(z) stands for the complex conjugate of z.

The key features near the Hopf bifurcation can be determined from

$$\dot{x}(t) = -\left(\frac{\pi}{2} + \varepsilon\right) \times \left(x(t-1) - \frac{x^3(t-1)}{6} + \frac{x^5(t-1)}{120}\right), \quad (4)$$

because Hopf bifurcation is a local property of dynamical systems and also refers to the analysis or evaluation of market conditions based on two distinct scenarios.

3.0 Option Pricing Model

Here, we consider the Black and Scholes Model [14,19] and the partial differential equation which financial derivative (stock) have to satisfy. The Black-Scholes Model assumes a market consisting of a single risky asset (S) and a risky-free bank account (r). This market is given by the equations;

$$dS = \mu S dt + \sigma S dz \quad (5)$$

$$dB = rBdt, \quad (6)$$

here (5) is a geometric Brownian-Motion and (6) a non-stochastic. S is a Brownian-Motion, Z is a Wiener process μ is a constant parameter called the drift. It is a measure of the average rate of growth of the asset price. Meanwhile, σ is a deterministic function of time. When σ is constant, (5) is the original Black-Scholes Model of the movement of a security, S. In this form $\mu$is the mean return of S, and σ is a variance. The quantity dZ is a random variable having a normal distribution with mean 0 and variance dt.

$dZ \propto N(0, (\sqrt{dt})^2)$.

For each interval dt, dZ is a sample drawn from the distribution $N(0, (\sqrt{dt})^2)$, this is multiplied by σ to produce the term σdZ. The value of the parameters μ and σ may be estimated from historical data.

Under the usual assumptions, [14,19] have shown that the worth V of any contingent claim written on a stock, whether it is American or European, satisfies the famous Black-Scholes equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV = 0, \quad (7)$$

where volatility σ, the risk-free rate r, and dividend yield q are all assumed to be constants. The value of any particular contingent claim is determined by the terminal and boundary conditions. For an American option, notice that the PDE (7) only holds in the not-yet-exercised region. At the place where the option should be exercised immediately, the equality sign in (7) would turn into an inequality one. That means the option value $V(S,t)$ at each time follows either

$V(S,t) = \wedge(S,t)$ for the early exercised region or (7) for the not-yet-exercised region, where

$\wedge(S,t)$ is the payoff of an American option at time t.

The generic form of (7) is derived by the change of variable $\tau = T - t$ to

$$\frac{\partial V}{\partial \tau} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - (r-q)\, S\, \frac{\partial V}{\partial S} + rV = LV\,, \quad (8)$$

where $V(.,\tau) \equiv V(.,T-\tau),\ \sigma(.,\tau) \equiv \sigma(.,T-\tau),\ at\ \tau = 0\ to\ \tau = T$

$S_{min} < S < S_{max}$, subject to the initial condition $V(S,0) = \wedge(S)$.

For the computations, the unbounded domain is truncated to

$$(S,t) \in (0,S)\ x\ (0,T] \quad (9)$$

with sufficiently large $S \equiv S_{max}$.

The worth V of an American option under Black-Scholes model satisfies an LCP

$$\begin{cases} LV \geq 0 \\ V \geq \wedge \\ (LV)(V - \wedge) = 0\,, \end{cases} \quad (10)$$

we impose the boundary conditions

$$\begin{cases} V(0,t) = 0 \\ V(S,t) = \wedge(S),\ S \in (0, S_{max}) \end{cases} \quad (11)$$

Beyond the boundary $S = S_{max}$, the worth V is approximated to be the same as the payoff $\wedge$, that is $V(S,t) = \wedge(S)$ for $S \geq S_{max}$.

4.0 Discretizing the Financial PDE

American options can be exercised at any time before expiry. Formally, the value of an American put option with a strike price k is

$V(0,k) = sup\ (0 \leq \tau^* \leq T: E(e^{-\tau\tau^*}(k - S_{\tau*})^+)$.

The optimal exercise time τ^* is the value that maximizes the expected payoff - any scheme to price an American must calculate this.

For American options with payoff $\wedge(s)$, the equivalent of equation (8) is

$$\begin{bmatrix} \frac{\partial V}{\partial \tau} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - (r-q)S\frac{\partial V}{\partial S} + rV \geq 0 \\ V(S,T) \geq \wedge(S) \end{bmatrix} \quad (12)$$

$\left[\frac{\partial V}{\partial \tau} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - (r-q)s\frac{\partial V}{\partial s} + rV\right][V(S,T) - \wedge(S)] = 0$.

Consider a uniform spatial mesh on the interval $[s_{min}, s_{max}]$:

$S_j = S_{min} + j\delta S,\ j = O, 1, \ldots, n+1$, where

$$\delta s = \frac{S_{max} - S_{min}}{n+1},\ \ S_{max} = S_0\, exp\left[\left(r - q - \frac{\sigma^2}{2}\right)T + 6\sigma\sqrt{T}\right]. \quad (13)$$

The truncated domain (9) has the lower bound $S_{min} = 0$ and upper bound S_{max} as in (13).

Replacing all derivatives with respect to S by their central finite-difference approximations, we obtain the following approximation to the Black-Scholes PDE (12)

$$\frac{\partial V(\tau,S)}{\partial \tau} = \frac{1}{2}\ \sigma^2(S)\, S^2 \frac{V(\tau,\ S+\delta S) - 2V(\tau,S) + V(\tau, S - \delta S)}{\delta S^2}$$
$$+ (r-q)\, S\, \frac{V(\tau,\ S+\delta S) - V(\tau,\ S-\delta S)}{2\delta S} - rV(\tau,S) + O(\delta S^2). \quad (14)$$

Let $V_j(\tau)$ denote the semi-discrete approximation to $V(\tau, S_j)$. Applying (14) at each internal node S_j, we obtain the following system of first-order ordinary differential equations;

$$\frac{dV_j(\tau)}{d\tau} = \frac{1}{2}\left(\left(\frac{\sigma(S_j)S_j}{\delta S}\right)^2 - \frac{(r-q)S_j}{\delta S}\right)V_{j-1}(\tau) - \left(-\left(\frac{\sigma(S_j)S_j}{\delta S}\right)^2 - r\right)V_j(\tau) + \frac{1}{2}\left(\left(\frac{\sigma(S_j)S_j}{\delta S}\right)^2 + \frac{(r-q)S_j}{\delta S}\right)V_{j+1}(\tau),$$

$j = 1, 2, \ldots, n$; (15)

with discretized form given as

$\frac{dV_j(\tau)}{d\tau} = L_{j,j-1}V_{j-1}(\tau) - L_{j,j}V_j(\tau) + L_{j,j+1}V_{j+1}(\tau)$.

System (15) has n equation in $n+2$ unknown functions,

$V_0(\tau), V_1(\tau), \ldots, V_n(\tau), V_{n+1}(\tau)$. Using the boundary conditions we have the functions $V_0(\tau)$ and $V_{n+1}(\tau)$ which respectively approximate the solution at the boundary nodes $S_0 = S_{min}$ and $S_{n+1} = S_{max}$ .As a result, the system of differential equations (15) can be written as the following matrix-vector differential equation with an n-by-n tri-diagonal coefficient matrix L whose entries are defined in (15)

$$\frac{dV(\tau)}{d\tau} = LV(\tau) + G(\tau), \quad (16)$$

subject to the initial condition

$$V(0) = \Lambda := [\Lambda(S_1), \Lambda(S_2), \ldots, \Lambda(S_n)]^T. \quad (17)$$

Here we use the notation:

$$L = \begin{pmatrix} L_{11} & L_{12} & 0 & \cdots & 0 & 0 \\ L_{21} & L_{22} & L_{23} & \cdots & 0 & 0 \\ 0 & L_{32} & L_{33} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & L_{n-1,n-1} & L_{n-1,n} \\ 0 & 0 & 0 & \cdots & L_{n,n-1} & L_{n,n} \end{pmatrix}, V(\tau) = \begin{pmatrix} V_1(\tau) \\ V_2(\tau) \\ \vdots \\ V_{n-1}(\tau) \\ V_n(\tau) \end{pmatrix}.$$

The vector $G(\tau) \in R^n$ is given by

$$\left[\left(\frac{\sigma^2(S_0)S_0^2}{2\delta S^2} - \frac{(r-q)S_0}{2\delta S}\right)V_0(\tau), 0, \ldots, 0, \left(\frac{\sigma^2(S_{n+1})S_{n+1}{}^2}{2\delta S^2} + \frac{(r-q)S_{n+1}}{2\delta s}\right) V_{n+1}(r)\right]^T.$$

$G(\tau)$ contains boundary values of the mesh solution.

The spatial discretization leads to:

a. Semi-discrete LCP

According [20] from (13), (16) and (17), we have

$$\begin{cases} L^j V^{j+1} \geq g^j \\ V^{j+1} \geq \Lambda \\ (V^{j+1} - \Lambda)^T (L^j V^{j+1} - g^j) = 0 \end{cases}, \quad (18)$$

where L is n-by-n tri-diagonal coefficient matrix, g is a vector resulting from the second term in equation (15) V and Λ are vectors containing the grid point values of the worth V and the pay off Λ, respectively. This again must be solved at every time step. A crude approximation is to solve the system $L^j X = g^j$, then set $L^{j+1} = max(X, \Lambda)$.

b. Drifted financial derivative system

According to [21], $G(\tau)$ term in (16) can be treated as an enforced input to the financial derivative system, resulted from boundary condition, defined in (11). With zero boundary condition, equation (16) yields.

$$\dot{V} = Lv \ , \quad (19)$$

which represents a pfaffian differential constraints as in [22] but not of kinematic nature arises from the conservation on non-zero financial derivatives. The transformed financial derivative system (19) can be re-expressed as

$$Lv = d \ . \quad (20)$$

System (20) represents a drifted financial derivative system with a drift term d. In such a system the derivative value V can be solved by computing the stochastic algorithm used in [13].

In general, equation (16) can be denoted in the form

$$\frac{dv}{dt} = -f\ (v_t\ ; \alpha), \quad (21)$$

where $v_t \in R^n, and\ \alpha \in R$ is a parameter. This usual notation v_t denotes the values of the system state over a time windows of finite length τ, that is $v_t(\theta) = v(t+\theta) \in R^n, \theta \in [-\tau, 0]$, and $v_t \in C$, where $C \coloneqq c\ ([-\tau, 0], R^n)$ denotes the Banach space of continuous functions over the interval $[-\tau, 0]$ equipped with the supremum norm. It is assumed that $f: C\ x\ R \longrightarrow R^n$ is twice continuously differentiable in its arguments and $f(0; \alpha) = 0$ for all α. Assume further that the origin undergoes a supercritical Hopf bifurcation at $\alpha = 0$. Hence, for small positive α the origin is unstable and there exist a small amplitude limit cycle. To study the behavior near the origin, it is convenient to scale the variable $v \longrightarrow \varepsilon v\ and\ \alpha \longrightarrow \varepsilon \alpha$, where ε is a small positive parameter. This transforms (21) into a weekly nonlinear system of the form

$$\frac{dv}{dt} = -\big(L\ v_t + \varepsilon\ f\ (v_t;\ \varepsilon)\big), \quad (22)$$

where $L: C \longrightarrow R^n$is a linear operator and f is a C^2 function with$f(0; \varepsilon) = 0$ for all ε. Equation (22) is a perturbation of the linear equation [15]

$$dv/_{dt} = -L\ v_t. \quad (23)$$

5.0 Multiple Scale and Financial Derivatives

In this section, we show how the method of multiple scale can be applied to the discretized PDE in equations (19) and (23).

Example (2)

Consider the DDE problem, one that has an exact solution, namely;

$$\frac{dx}{dt} = -x(t-T), T = \pi/_2 + \varepsilon\mu. \quad (24)$$

Equation (24) undergoes a Hopf bifurcation at $\varepsilon = 0$, because the following conditions in example 1 hold .Hence, equation (24) behaves similar to Black-Scholes differential equation involving a term $X''(S)$.

Lemma 5.1

Let t be replaced by two time variable: regular time $B = t$ (A riskless Bond (cash) as in (5)) and slow time $S = \varepsilon t$ (underlying security which evolves in accordance with stock price S, as in (6)), then the solution of (24) is given by

$$X_0 = R_0\, exp\left(\frac{4\mu(S_0 e^{\left(\mu - \sigma^2/_2\right)t + \sigma w_t}}{\pi^2+4}\right) cos\left(r^{-1}\, ln\, B{B_0}^{-1} - \left(\frac{2\pi\mu\left(S_0 e^{\left(\mu-\sigma^2/_2\right)t+\sigma w_t}\right)}{\pi^2+4} + \theta_0\right)\right). \quad (25)$$

Proof.

Let the dependent variable $x(t)$ be replace by $x\ (B, S)$. Hence, if $x\ (B, S)$

$$dx = \frac{\partial x}{\partial B}\, dB + \frac{\partial x}{\partial s}\, ds$$

$$\frac{dx}{dt} = \frac{\partial x}{\partial B}\frac{dB}{dt} + \frac{\partial x}{\partial s}\frac{ds}{dt}$$

$$\frac{dx}{dt} = \frac{\partial x}{\partial B} + \varepsilon\frac{\partial x}{\partial s}\, .$$

From equation (24), we have

$$\frac{\partial x}{\partial B} + \varepsilon\frac{\partial x}{\partial s} = -x(B-T, S-\varepsilon T), \tag{26}$$

since $T = {}^{\pi}/_{2} + \varepsilon\mu$, the delayed term may be expanded for small ε as follows;

$$x\left(\mathrm{B} - {}^{\pi}/_{2} - \epsilon\mu, S - \varepsilon\, {}^{\pi}/_{2} - \varepsilon^2\mu\right)$$

$$= x\left(\beta - {}^{\pi}/_{2,} S\right) - \varepsilon\,\mu\,\frac{\partial x_d}{\partial B} - \varepsilon\, {}^{\pi}/_{2}\frac{\partial x_d}{\partial S} + 0\,(\varepsilon^2), \tag{27}$$

where x_d is an abbreviation for $x\left(B - {}^{\pi}/_{2}, S\right)$. Next we expand

$$x = x_0 + \varepsilon x_1 + 0\,(\varepsilon^2), \tag{28}$$

so

$$\frac{\partial x}{\partial B} = \frac{\partial x_0}{\partial B} + \varepsilon\frac{\partial x_1}{\partial B} + 0(\varepsilon^2) \tag{29}$$

$$\frac{\partial x}{\partial S} = \frac{\partial x_0}{\partial S} + \varepsilon\frac{\partial x_1}{\partial S} + 0(\varepsilon^2). \tag{30}$$

By substituting equation (27), (29) and (30) into (26), gives

$$\frac{\partial x_0}{\partial B} + \varepsilon\frac{\partial x_1}{\partial B} + \varepsilon\frac{\partial x_0}{\partial S} + \varepsilon^2\frac{\partial x_1}{\partial S} = -x_0\left(B - {}^{\pi}/_{2}, S\right) - \varepsilon x_1\left(B - {}^{\pi}/_{2}, S\right) + \varepsilon\mu\,\frac{\partial x_{0d}}{\partial B} + \quad \varepsilon^2\mu\,\frac{\partial x_{1d}}{\partial B}\,\varepsilon\, {}^{\pi}/_{2}\frac{\partial x_{0d}}{\partial S} + \varepsilon^2\, {}^{\pi}/_{2}\frac{\partial x_{1d}}{\partial S} + 0(\varepsilon^2), \tag{31}$$

$$\frac{\partial x_0}{\partial B} + x_0\left(B - {}^{\pi}/_{2}, S\right) = 0, \tag{32}$$

$$\frac{\partial x_1}{\partial B} + x_1\left(B - {}^{\pi}/_{2}, S\right) = \mu\,\frac{\partial x_{0d}}{\partial B} + {}^{\pi}/_{2}\frac{\partial x_{0d}}{\partial S} - \frac{\partial x_0}{\partial S}. \tag{33}$$

Equation (32) has a periodic solution (since (24) is autonomous)

$$x_0 = R\,(S)\,cos(B - \theta(S)), \tag{34}$$

where as usual in this method $R(S)$ (the approximated amplitude of a periodic motion of stock) and $\theta(S)$ (the frequency of the bifurcated periodic solution) are yet undetermined function of slow times S.

Taken $x_{0d} = -\frac{\partial x_0}{\partial B}$ in equation (33)

$$\frac{\partial x_1}{\partial B} + x_1\left(B - {}^{\pi}/_{2}, S\right) = -\frac{\mu\partial^2 x_0}{\partial B^2} - {}^{\pi}/_{2}\frac{\partial^2 x_0}{\partial B\partial S} - \frac{\partial x_0}{\partial S}. \tag{35}$$

Substitute equation (34) into (35)

$$\frac{\partial x_0}{\partial B} = -R\,sin(B-\theta),$$

$$\frac{\partial x_0}{\partial S} = R\theta'\,sin(B-\theta) + R'\,cos\,(B-\theta),$$

$$\frac{\partial^2 x_0}{\partial B^2} = -R\,cos\,(B-\theta), \text{ and}$$

$$\frac{\partial^2 x_0}{\partial B\partial S} = -[-R\theta'\,cos(B-\theta) + R'\,sin(B-\theta)]$$

$$= R\theta'\,cos(B-\theta) - R'\,sin(B-\theta).$$

Let $B - \theta = \psi, by\ substituting\ \frac{\partial x_0}{\partial B}, \frac{\partial x_0}{\partial S}, \frac{\partial^2 x_0}{\partial B^2}\ and\ \frac{\partial^2 x_0}{\partial B\partial S}$ in (35), we have

$$\mu R\,cos(\psi) - {}^{\pi}/_{2}R\theta'\,cos(\psi) + {}^{\pi}/_{2}R'\,sin(\psi) - R\theta'\,sin(\psi) - R'\,cos(\psi) = \frac{\partial x_1}{\partial B} + x_1\left(B - {}^{\pi}/_{2}, S\right). \tag{36}$$

Equating coefficient of sin (ψ) and cos (ψ) to zero

$$\mu R - {}^{\pi}/_{2}R\theta' - R' = 0 \tag{37}$$

$${}^{\pi}/_{2}R' - R\theta' = 0. \tag{38}$$

From equation (38)

$$\theta' = \frac{\pi R'}{2R}. \tag{39}$$

Plug equation (39) in equation (37)

$$R' = \frac{4\mu R}{\pi^2+4}. \tag{40}$$

Plug (40) in equation (39)

$$\theta' = \frac{2\pi\mu}{\pi^2+4}. \tag{41}$$

From equation (40), we have;

$$\frac{1}{R}\,dR = \frac{4\mu}{\pi^2+4}\,ds,$$

so that

$$ln\,R = \frac{4\mu S}{\pi^2+4} + K,$$

and

$$R(S) = R_0\, exp\left(\frac{4\mu S}{\pi^2+4}\right) . \quad (42)$$

Similarly

$$\theta(S) = \frac{2\pi\mu S}{\pi^2+4} + \theta_0. \quad (43)$$

Substitute R(s) and θ(s) in equation (34)

$$x \approx x_0 = R_0\, exp\left(\frac{4\mu S}{\pi^2+4}\right) cos\left(t - \left(\frac{2\pi\mu S}{\pi^2+4} + \theta_0\right)\right) . \quad (44)$$

It is not difficult to show from (5) and (6) that

$$B = B_0 e^{rt}, \quad (45)$$

and

$$S = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma\, w_t} , \quad (46)$$

respectively.

Solving for t in (45) and plugging it in (44) using (46), we have (25) as required.

Theorem 5.1

An exact solution of DDE (Delay Differential Equation) which undergoes a Hopf bifurcation at $\varepsilon = 0$ behaves similar to Black-Scholes differential equation involving a term $x''(S)$.

Proof.

Let $S = \varepsilon t$ in equation (46), we have

$$S_0 = \varepsilon t exp\left[\left(\frac{\sigma^2}{2} - \mu\right) t - \sigma W_t\right]. \quad (47)$$

Plugging (47) into (25) we arrive at;

$$X_0 = R_0\, exp\left(\frac{4\mu\varepsilon t}{\pi^2+4}\right) cos\left(r^{-1}\, ln\, BB_0{}^{-1} - \left(\frac{2\pi\mu\varepsilon t}{\pi^2+4} + \theta_0\right)\right). \quad (48)$$

Notice that $S_0 \to 0$ as $\varepsilon \to 0$ and X_0 fluctuate according to r.

Remark 1

The periodic solution derived from two way variable expansion method (25) proves that, if the time is delayed, the oscillatory movement of the price of stock can be monitored and instability controlled and stabilized using the slow time $S = \varepsilon t$.

6.0 Formulation of Stochastic Algorithm

We consider the finite dimensional variation problem: find $v \in D\,(\varphi)$ such that

$$Lv + \partial\varphi(v) \ni b , \quad (49)$$

subject to equation (18), where (φ) is convex function,

$D\,(\varphi) = [v \in R^n: \varphi(v) < \infty] \neq \emptyset$, then for $v \in D\,(\varphi)$, the sub gradient $\partial\varphi$ of $\varphi: R^n \to R$ at v is defined as;

$$\partial(\varphi) = \langle g \in R^n: f\,(v+t) - f(v) \geq \langle g, t\rangle\rangle\ \forall\, v + t \in D(\varphi). \quad (50)$$

It is well known that if a function $f\ on\ R^n$ is differentiable, then there exists $d\, \epsilon\, R^n$ such that $f(v) - f(v_0) = (d\ , v - v_0) + ||\, v - v_0||$,

where $d = \frac{\partial f(v)}{\partial v}$ is the gradient of the function f.

Denote $\partial f^k = \frac{\partial f(v^k)}{\partial v}$, $\frac{\partial^2 f(v^k)}{\partial v_r \partial v_s} = \partial^2_{r,s} f^k$, as in [23], we constructed a sequence of random vector $d^k \in R^n$ that strongly approximate $\partial f^k = \partial f(v^k)$ for each k in the sense that

$$E\left\|d_j{}^k - \partial f^k\right\| = 0,$$

and their expected Euclidean distance

$$E\left\|d_j{}^k - \partial f^k\right\|^2 = M^{-1}\sigma^2,$$

is minimum so that a search in the direction of the random sequence $\langle d_j{}^k\rangle$ approximate a search through the true gradient ∂f^k and this is expected to lead to the non-zero global minimizing factor if it exists. To this end, we consider the natural Taylor's expansion of a quadratic function f about point v_0 given by

$$f(v) - f(v_0) = \langle \partial f(v_0), v - v_0\rangle + {}^1\!/_2\,(v - v_0)H(v_c)(v - v_0), \quad (51)$$

where v_c is on the line segment between v and v_0 and $H(v_c)$ is the Hessian of f at v_c.

Equation (51) can also be represented as

$$f(v) = {}^1\!/_2\,(Lv, v) - (d, v). \quad (52)$$

Given that

$E\ \left(e(v_j)\right) = 0$ for each j,

and
$E\left(e(v_i)\, e(v_j)\right) = \sigma^2\delta_{ij}, 0 < \sigma^2 < \infty.$
Let $Y(v_1), Y(v_2), \ldots, Y(v_m)$ be real-valued independent observable random variable performed on $v_1, v_2, \ldots, v_n$, $n+2 < m < \frac{1}{2}n(n+1)$ chosen in the neighbourhood of v^k for a fixed K, then
$Y_j = Y(v_j) = f(v + t_j) - f(v_j),$
$= \langle \partial f(v^k), t_j \rangle + \frac{1}{2}\sum_{k=1}^{m}\sum_{r=1}^{m} t_{kj}t_{rj}\partial_{kr}^2 f + e(v_j),$
is identifiable with (49) so the fixed $t_j \in R^n$ satisfying $\Sigma_{i=1}^{m} t_{ij} = 0$, $M^{-1}\sum_{i=1}^{\infty} t_{ij}^2$linearizes f, [24] and hence the least square approximation.

$$d^k = M^{-1}\sum_{j=1}^{m} t_j Y_j, \quad M = \sum_{j=1}^{m} t_j t_j', \qquad (53)$$

exist and is adequate for approximating ∂f such that Euclidean distance
$E||d^k - \partial f(v^k)|| = 0$ for each k, also yield
$E|| d^k - \partial f(v^k)||^2 = M^{-1}\sigma^2.$
In the sequel we assume without loss of generality that $\sigma^2 = 1. \langle d^k \rangle$ is thus, a sequence of independently and identically distributed random vector and determines the direction of search.
It follows that by letting v^0 be an initial point, the sequence of path produce by $\{v^k\}_{k=0}^{\infty}$ through its definition

$$V^{k+1} = V^k - p^k d^k, \qquad (54)$$

by successive iteration, is the trajectory of the point v^0 and any limiting point of the sequence is therefore attractor of v^0.

6.1 Getting the Domain of Attraction

Let $R_t^n - N(O)$ be petitioned into exclusive segment, S_j,
$j = 1,2, \ldots, t,\ n < t \leq 2^n$. Let v_j be chosen randomly in S_j, such that $f(v_j) > 0\ \forall\ j$. Let $P_j = P(v_j = \alpha)$ be the probability that $v_j = \alpha$ so that
$P_j \geq 0,\ \sum_{j=1}^{t} P_j = 1,$
put

$$P_j = \frac{f(v_j)}{\sum_{j=1}^{t} f(v_j)},$$

so that $\bar{v} = \sum_{j=1}^{t} v_j P_j$

$$= \sum_{j=1}^{t} \frac{v_j f(v_j)}{\sum_{j=1}^{t} f(v_j)}.$$

It is shown in [23] that if

$$\hat{v} = \bar{v} - pd,\ p > 0, \qquad (55)$$

where d is as (53), then
$f(\hat{v}) = min\{f(v_j): v_j \in S\}$. It follows that the segment S_T if when $\hat{v} \in S_T$ contains $v > 0$ for which $f(v)$ is minimum and hence we have
$\varphi(U_v) \subset S_T$ so that if $\langle 0 \rangle$ is the attractor of the point $\bar{v}$ and
$\varphi(0) \cap \varphi(U_{\bar{v}}) = \emptyset$ then $N(0) \cap N(U_{\hat{v}}) = \emptyset$ or else $N(0) = N(U_{\hat{v}})$ with global domain of attraction $\varphi(0) = \varphi(U_{\bar{v}})$.
Where $U_{v^*} = \{v^* \in R^n : v^* > 0 : \partial f(v^*) = 0\}$ (56)
is a way of stochastically solving problem (53). Thus we have

Lemma 6.1 [23]

Suppose that $U_{\hat{v}} \neq \emptyset$. Thus there exist a neighborhood $N(U_{\hat{v}}) \subseteq D(\partial f)$ of $U_{\hat{v}}$ such that for any initial guess $\hat{v} \in \varphi(U_{\bar{v}})$, the non-negative minimizer $U_{\hat{v}}$ is obtained as a limit of iteratively constructed sequence $\langle v^j \rangle_{j=1}^{\infty}$ generated from $\hat{v}$by $V^{j+1} = V^j - p^j d^j$. Then with$\hat{v}$ as our starting point we search for the minimizer of f as follows:
Starting at $\hat{v}$ as in equation (55)

1. Compute the d^k as in equation (53)
2. Compute the corresponding p as specified below
3. Compute $V^{k+1} = V^k - p^k d^k$.

Has the process converge? i.e. $\|V^{k+1} - V^k\| < \sigma, \sigma > 0$, if yes then
$V^{k+1} = V^k$. If no return to (1)

Theorem 6.1 [23]
Let (p^k) be a real sequence such that
i. $p^0 = 1,\ 0 < p^k < 1, \forall\ k > 1$
ii. $\sum_{k=0}^{\infty} p^k = \infty$
iii. $\sum_{k=0}^{\infty} p^{2k} < \infty$.
Then the sequence $\langle v^k \rangle_{k=0}^{\infty}$ generated by $\hat{v} \in \varphi\ (U_{\hat{v}}) \subseteq D\ (\partial f)$ and defined iteratively by $V^{j+1} = V^j - p^j d^j$ remain in $D\ (\partial f)$ and coverage strongly to $U_{\hat{v}}$.
Proof:
Let $b^k = p^k\ ||d^k - \partial f^k||$
Then $\langle b_k \rangle_{k=1}^{\infty}$is a sequence of independent random variable and $E(b_k) = 0$, for each k.
Noticing that the sequence of partial sums $\langle S_k \rangle_{k=1}^{\infty}, S_k = \sum_{j=1}^{k} b_j$ is a martingale. Therefore
$E(S_k^2) = \sum_{j=1}^{k} E\left(b_j^2\right) = \sum_{j=1}^{k} P^{2j}\ E||\ d^j - \partial f^j||^2 = M^{-1}\sigma^2 \sum_{j=1}^{k} p^{2j}$,
,,
and
$\sum_{j=1}^{k} E\left(b_j^2\right) < \infty$,since $\sum_{j=1}^{k} p^{2j} < \infty$.
Hence by a version of martingale convergence theorem in [25], we have
$\underset{k\to\infty}{log}\ S_k = \sum_{j=1}^{\infty} b_j < \infty$,
so that
$\underset{k\to\infty}{log}\ p^k||\ d^k - \partial f^k|| = 0.$
Noticing that in (49), L is positive definite so that $f(v)$ is convex and hence ∂f is monotone. But an earlier result in theory of monotone operators, due to [26], shows that the sequence $\langle V^k \rangle$ generated by $V^0 \in D(\partial f)$ and defined iteratively by
$V^{k+1} = V^k - p^k\ \partial f^k$,
remain in $D(\partial f)$ and converges strongly to $\langle v^* : \partial f(v^*) = 0 \rangle$. It follows from this result that our sequence converges strongly to $U_{v^*}\ if\ U_{v^*} \neq 0$.

7.0 Numerical Experiment

In our numerical example, we price American put options using stochastic algorithm. The parameters we used for the Black-Scholes model are the same as in [20] and they are defined below:

Table 1: Estimated Parameters for the Black-Scholes Model

Parameter	Notation	Value
Risk free interest rate	r	0.2
Dividend yield	q	0.1
Strike price	k	7
Volatility	σ	0.3
Time to expiry	T	2
Spot price	S_0	10
Ratio of Nodes	ϑ	30

We illustrate the method in a concrete setting, by first plugging the parameter in table 1 in (44) which gives $X_0 \approx 17$ a positive periodic solution, satisfying that the stock price is positively correlated with the market. Hence substituting in (13) and (15), with time nodes $3x10^3$ and space nodes $9x10^4$ satisfying the ratio of nodes ϑ as stipulated, we have the financial matrix
(3 by 3 tri-diagonal coefficient matrix).

$$L = \begin{pmatrix} 0.2 & 0.05 & 0 \\ -0.1 & 0.2 & 0.1 \\ 0 & -0.15 & 0.2 \end{pmatrix}.$$

By using the equation of total investment return;

$$r = d + q, \tag{57}$$

where r is the risk adjusted discount rate for V (the worth); q is the dividend yield (or convenience yield in case of commodities) and d is the drift (or capital gain rate). Hence $d = 0.1$ for $q = 0.1$ and $d = 0.2$ for $q = 0.0$ (No dividend yield).
From (20), we have

$$\begin{pmatrix} 0.2 & 0.05 & 0 \\ -0.1 & 0.2 & 0.1 \\ 0 & -0.15 & 0.2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0.2 \\ 0.2 \\ 0.2 \end{pmatrix},$$

$$Lv=\begin{pmatrix} 0.2 & 0.05 & 0 \\ -0.1 & 0.2 & 0.1 \\ 0 & -0.15 & 0.2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix},$$

$$(Lv,v)=\begin{pmatrix} 0.2v_1+0.05v_2+0 \\ -0.1v_1+0.2v_2+0.1v_3 \\ 0-0.15v_2+0.2v_3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}. \quad (58)$$

From (52) with (58), we have ;

$$f(v)=\frac{1}{2}[0.2v_1^2-0.05v_1v_2+0.2v_2^2-0.05v_2v_3+0.2v_3^2]-[0.2v_1+0.2v_2+0.2v_3]. \quad (59)$$

From (55) with (59), we have

$$f(\hat{v})=\frac{1}{2}0.02p^2-0.12p, \quad (60)$$

$$\frac{\partial f(\hat{v})}{\partial p}=0.02p-0.12,$$

from (56) $\frac{\partial f(\hat{v})}{\partial p}=0$, hence $p=6$.

For $k=0$ in $(54) v^0=0,\ d^0=-0.2$ and $p^0=6$, hence $V^1=1.2$.

Given that the actual solution in [20] is $V(S,t)=1.171339$, the PDE result is 0.14459568, which [27] is given as0.14275. Approximations such as in [27] are not accurate enough to test the accuracy of the finite difference scheme.

The stochastic approximation method above (54) starting at $V_0=(0\ 0\ 0)$ gives a fixedpoint $V^*(S,t)=1.2$, after one iteration for both values of the drift. This solution is the same as in [20].

This shows that a stochastic approximation method can be used on a discretized financial PDE to price an American option with a considerable success.

8.0 Conclusion

In this work we considered a stochastic algorithm and multiple scale method on a drifted financial derivative system for pricing American options under the Black-Scholes model. For the Black-Scholes partial derivative, we employed central finite-difference approximation into first-order ordering differential equation and later transformed to a drifted financial derivative system. The multiple scale method was adopted in calculating the periodic solution resulted from a Hopf bifurcation of a discretized generic PDE to monitor and stabilize the oscillatory movement of the market price of stock.Thereafter a stochastic algorithm was formulated to price an American option under the Black-Scholes model through a drifted financial derivative system. In numerical experiment, we formed a financial matrix and the value of the drift parameter using Table 1. With finer discretization, positive periodic solution; space nodes, and time nodes, we have demonstrated that the drifted financial derivative system can be efficiently and easily solved with stochastic approximation method. This approach in turn, yields a fast method of pricing American option.We used the multiple scale method to obtain the periodic solution which in turn is used to identify the movements of two parameters responsible for oscillatory behavior of the system to be risk-free in conjunction with the stock price. It is discovered that as the stock price decreases, the periodic solution fluctuates according to risk-free rate.

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