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An Application of Sturm-Liouville Equation to the Solution of the Black–Scholes equation with Transaction cost and Portfolio Risk Measures

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Abstract

In this research work, we shall obtain the analytical solution of non-linear Black-Sholes equation with transaction cost measure and portfolio risk measure. We will reduce the Black-Scholes Partial Differential Equation (PDE) to a form of the Sturm–Liouville equation and obtain solutions given different boundary conditions. In each case we obtain a series solution which is a sequence of special functions. Furthermore we discuss the dynamic stability of equilibrium of the solution.It is observed that the equilibrium is dynamically stable if and only if the portfolio value (or the time path) $u(x,t)$ is convergent under the condition that the eigenvalue and minimized total risk$h_i < 0$.

Keywords: Black-Scholes PDE ,Eigenvalue problems,Transaction cost measure,Portfolio risk measure Sturm-Liouville Boundary Value Problem.

1.0 Introduction

Modern finance started with Black –Scholes Linear partial differential equation[1],which was obtained under several model restrictions.(e.g no transaction costs,agents are price takers and there are no feedback effects from the trading activity,the market is perfectly liquid, etc.)are relaxed then the linear Black-Scholes equation needs to be replaced by anonlinear one.Several models have been proposed to address the case of the price impact effect from large traders [2, 3, 4].In these last decades many different methods aimed at solving Black-Scholes partial differential equation have appeared.The PDEs arising from the generalized option pricing model pose three challenges to the numerical approximation: the degeneracy of the equation, the coefficients being time and space-dependent and also unbounded in the space variables. The aim of this research to obtain an analytical solution of Black-Scholes equation with transaction cost and volatile risk measures using the Sturm –Liouville method. Sturm-Liouville problem is a second-order ordinary differential equations problem where two boundary conditions are specified, but where no unique solution exists. These problems may be regular or singular at each endpoint of the underlying interval [1]. In [4] the nth eigenvalue as a function on the space of self-adjoint regular Sturm–Liouville problems with positive leading coefficient and weight functionswas considered. In [7] a method of computing accurate approximations to the eigenvalues and Eigen functions of regular Sturm–Liouville differential equationswas derived. The minimized sum of the optimal length of the hedge interval –time lag is the time-lag between two consecutive transactions which the transaction costs as well as the volatile portfolio risk depends on [8]. If transaction costs are taken into account perfect replication of the contingent claim is no longer possible. Modeling the short rate $\mathrm{r} = \mathrm{r}(\mathrm{t})$ by a solution to a one factor stochastic differential equation,

$$\mathrm{dS} = \mu(\mathrm{s},\mathrm{t})\mathrm{dt} + \sigma(\mathrm{x},\mathrm{t})\mathrm{dw}, \qquad (1.1)$$

where $\mu(\mathrm{S},\mathrm{t})\mathrm{dt}$ represent a trend or drift of the process and $\sigma(\mathrm{x},\mathrm{t})$ represents volatility part of the process, the risk adjusted Black-Scholes equation can be viewed as an equation with a variable volatility coefficient

$$\partial_t \mathrm{u} + \frac{\sigma^2(\mathrm{x},\mathrm{t})}{2}\mathrm{x}^2\left(1 - \mu(\mathrm{x}\,\partial_{\mathrm{x}}\mathrm{u})^{\frac{1}{3}}\right)\partial_{\mathrm{x}}^2\mathrm{u} + \mathrm{rx}\,\partial_{\mathrm{x}}\mathrm{u} - \mathrm{ru} = 0, \qquad (1.2)$$

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where $\sigma^2(x,t)$ depends on a solution u= $u(x,t)$ and $\mu = 3\left(\frac{C^2R}{2\pi}\right)^{\frac{1}{3}}$, since

$$\widehat{\sigma}^2(x,t) = \sigma^2(1-\mu(x\,\partial_x^2 u(x,t))^{\frac{1}{3}}. \tag{1.2a}$$

Incorporating both transaction costs and risk arising from a volatile portfolio into equation (1.2) we have the change in the value of portfolio to become.

$$\partial_t u + \frac{\widehat{\sigma}^2(x,t)}{2} x^2\,\partial_x^2 u + rx\,\partial_x u - ru = (r_{TC} + r_{VP})x, \tag{1.3}$$

where $r_{TC} = \frac{C|\Gamma|\widehat{\sigma}x}{\sqrt{2\pi}}\frac{1}{\sqrt{\Delta t}}$ is the transaction costs measure, $r_{Vp} = \frac{1}{2}R\widehat{\sigma}^4x^2\Gamma^2\Delta t$ is the volatile portfolio risk measure and $\Gamma = \partial_s^2 V$. Minimizing the total risk with respect to the time lag Δt yields;

$$\min_{\Delta t}(r_{TC} + r_{uP}) = \frac{3}{2}\left(\frac{C^2R}{2\pi}\right)^{\frac{1}{3}}\widehat{\sigma}^2|x\,\partial_x^2 u|^{\frac{4}{3}}.$$

For simplicity of solution and without loss of generality, we choose the minimized risk as

$$\left\{\min_{\Delta t}(r_{TC} + r_{uP})\right\}^{\frac{3}{2}} = Ax^2\,\partial_x^2 u, \tag{1.4a}$$

with

$$A = (\tfrac{3}{2})^{\frac{3}{2}}\left(\frac{C^2R}{2\pi}\right)^{\frac{1}{2}}\widehat{\sigma}^3. \tag{1.4b}$$

They change in the value of the portfolio after minimizing the total risk with respect to time lag is given as

$$\partial_t u + \frac{\widehat{\sigma}^2(x,t)}{2} x^2\,\partial_x^2 u + rx\,\partial_x u - ru = Ax^2 u. \tag{1.5}$$

For simplicity of solution and without loss of generality, let

$$f(x) = \frac{\sigma^2}{2}x^2, rx = g(x),\ k(t)u = \frac{3}{2}\left(\frac{C^2R}{2\pi}\right)\sigma^2|x\,\partial_x^2 u|^{\frac{4}{3}}.$$

Then equation (1.5) becomes

$$f(x)\,\partial_x^2 u + g(x)\,\partial_x u - r(x)V = -\partial_t u + k(t)u\,. \tag{1.6}$$

In this work we investigate equation (1.6) with the view of obtaining the analytical solution subject to the conditions

$$\left.\begin{array}{c} u(a,t) = u(b,t) = 0 \\ u(x,0) = s(x) \end{array}\right\} \tag{1.7}$$

by the method of Sturm- Liouville Equation.

The strength, and beauty, of the approach developed by Sturm and Liouville is that considerable general information, and some specific detail, can be obtained without ever finding the solution to the problem.

2.0 Sturm-Liouville Equation

A classical "Sturm- Liouville equation", is a real second –order linear differential equation of the form

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y = \lambda r(x)y, \tag{2.1}$$

In the simplest of cases all coefficients are continuous on the finite closed interval $[a,b]$, $p(x)$ has continuousderivative. In this case y is called a ''solution'' if it continuously differentiable on (a,b) and satisfies the equation (2.1) at every point in (a,b). In addition, the unknown function y is required to satisfy boundary conditions. The function $r(x)$,is called the ''weight'' or ''density'' function.

We introduce the Sturm-Liouville operator as

$$L[y] = \frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y$$

and consider the Sturm-Liouville equation

$$L[y] + \lambda r(x)y = 0, \tag{2.2}$$

where $p > 0, r \geq 0$ and $p, q,$ and r are continuous functions on the interval [a,b]: along the with BC

$$B_a[y] = \alpha_1 y(a) + \alpha_2 p(a)y'(a) = 0 \qquad B_b[y] = \beta_1 y(b) + \beta_2 p(b)y'(b) = 0 \tag{2.3}$$

where $\alpha_1^2 + \alpha_2^2 \neq 0$ and $\beta_1^2 + \beta_2^2 \neq 0$

The problem of finding a complex number $\lambda = \mu$ such that BVP (2.2)-(2.3) has a non –trivial solution is called Sturm Liouville problem.The value $\lambda = \mu$ is called an eigenvalue and the corresponding solution $y(:,\mu)$ is called an eigenfunction.

3.0 Application

In this section we solve equation (1.6) by the proposed method under different boundary conditions.

Let us apply separation of variable, which in doing we must impose that

$u(x,t) = X(x)T(t)$

The Equation (1.3) may be written as

$$\frac{\hat{L}X(x)}{X(x)} = \frac{\hat{M}T(t)}{T(t)}$$

where

$$\hat{L} = f(x)\frac{d^2}{dx^2} + g(x)\frac{d}{dx} + r(x) \text{ and } \hat{M} = \frac{d}{dt} + K$$

Since ,by definition,$\hat{L}$ and $X(x)$ are independent of time and $\hat{M}$ and $T(t)$ are independent of position x,then both sides of the above equation must be equal to a constant that we call λ .In such case:

$$\hat{L}X(x) = \lambda X(x) \quad (3.1)$$

$$X(0) = X'(\pi) = 0 \quad (3.2)$$

$$\hat{M}T(t) = \lambda T(t) \quad (3.3)$$

The first of these equation must be solved as a Sturm Liouville problem(SLP) to be able to obtain the eigenvalue $\lambda_n$and eigenfunctionsX_n.

We call the operator M the adjoint to L, and $M(t) = 0$ is then the adjoint equation. Ingeneral, the differential operators M and L are not identical, but if $M \equiv L$ then the equation $L(x) = 0$ is said to be self –adjoint.

The eigenvalue λ have to be non-negative.We consider their separate cases $\lambda < 0, \lambda = 0, \lambda > 0$

Solving for

$\hat{L}X(x) = \lambda X(x)$.

The problem is the eigenvalue problem and since we seeking for bounded solutions, then the eigenvalue λ_n could be written as

$$\lambda_n = \frac{(2n-1)^2}{4}, n = 0, \pm1, \pm2, ..., \quad (3.4)$$

and the eigenfunctions are

$$X_n = sin\left(\sqrt{\lambda_n}x\right), n = 0, \pm1, \pm2, ... \quad (3.5)$$

Equation (3.3) issolved analytically to obtain

$$\frac{d}{dt}T_n(t) = (\lambda_n - k)T_n(t)$$

or

$$T_n(t) = a_n e^{-\left(\lambda_n t - \int_0^t kd\tau\right)}. \quad (3.6)$$

For each $n \in Z^+$, the L- equations $\hat{L}X(x) + \lambda X(x) = 0$ can also have another independent eigenfunctions

$$X_n = cos\left(\sqrt{\lambda_n}x\right), n = 0, \pm1, \pm2, ... \quad (3.7)$$

So that the general series solution to equation (1.6)is the linear combination of (3.5) and (3.7)together with (3.6)

$$u(x,t) = \sum_n \left(a_n sin\left(\sqrt{\frac{(2n-1)^2}{4}}\right)x + b_n cos\left(\sqrt{\frac{(2n-1)^2}{4}}\right)x\right) exp\left\{-\left(\left(\frac{(2n-1)^2}{4}t - \int_0^t kd\tau\right)\right)\right\}. \quad (3.8)$$

We have by the boundary condition equations (1.7)

$$u(0,t) = u(\pi,t) = \sum_n b_n\, exp\left\{-\left(\left(\frac{(2n-1)^2}{4}t - \int_0^t kd\tau\right)\right)\right\} = 0,$$

$$u(x,0) = \sum_n \left(a_n sin\left(\sqrt{\frac{(2n-1)^2}{4}}\right)x + b_n cos\left(\sqrt{\frac{(2n-1)^2}{4}}\right)x\right) = S(x).$$

So that

$$u(x,t) = \sum_n a_n s(x)\, exp\left\{-\left(\left(\frac{(2n-1)^2}{4} - \frac{3}{2}\left(\frac{C^2R}{2\pi}\right)\sigma^2|x\partial_x^2 u|^{\frac{4}{3}}\right)t\right)\right\}$$

$$= \sum_n \left(a_n sin\left(\sqrt{\frac{(2n-1)^2}{4}}\right)x + b_n cos\left(\sqrt{\frac{(2n-1)^2}{4}}\right)x\right)e^{h_1(t)}, \quad (3.9)$$

where$a_n = \frac{(X_n(x)s(x))}{(X_n(x),X_n(x))} = \frac{\int_a^b X_n(x)\,S(x)\,dx}{\int_a^b X_n^2 dx}$,

and $h_1(t) = \left(\frac{(2n-1)^2}{4} - \frac{3}{2}\left(\frac{C^2R}{2\pi}\right)\sigma^2|x\partial_x^2 u|^{\frac{4}{3}}\right)t$ and is $=, < or > 0$ depending whether $\frac{(2n-1)^2}{4} - \frac{3}{2}\left(\frac{C^2R}{2\pi}\right)\sigma^2|x\partial_x^2 u|^{\frac{4}{3}}$ is $=, < or > 0$.

If instead the boundary conditions becomes

$$\left.\begin{aligned} u(0,t) = u(L,t) = 0 \\ u(x,0) = s(x) \end{aligned}\right\}. \quad (3.10)$$

then the eigenvalues are given as

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \qquad n \in \{1,2,\dots\},$$

and the correspondingindependent eigenfunctions are

$$\varphi_n = sin\left(\frac{n\pi x}{L}\right), n \in \{1,2,\dots\}. \varphi_n = cos\left(\frac{n\pi x}{L}\right), n \in \{1,2,\dots\},$$

so that the general series solution will now be

$$u(x,t) = \sum_n \left(a_n sin\left(\frac{n\pi x}{L}\right)x + b_n cos\left(\frac{n\pi x}{L}\right)x\right) exp\left\{-\left(\left(\frac{n\pi}{L}t - \int_0^t kd\tau\right)\right)\right\}. \quad (3.11)$$

Therefore by the boundary condition of equations (3.10), we have;

$$u(0,t) = u(L,t) = \sum_n b_n\, exp\left\{-\left(\left(\frac{n\pi}{L}t - \int_0^t kd\tau\right)\right)\right\} = 0,$$

$$u(x,0) = \sum_n \left(a_n sin\left(\frac{n\pi}{L}\right)x + b_n cos\left(\frac{n\pi}{L}\right)x\right) = S(x).$$

And

$$u(x,t) = \sum_n a_n s(x)\, exp\left\{-\left(\left(\frac{n\pi}{L} - \frac{3}{2}\left(\frac{C^2R}{2\pi}\right)\sigma^2|x\partial_x^2 u|^{\frac{4}{3}}\right)t\right)\right\} \quad (3.12)$$

The eigenfunctions expansion of $S(x)$ is then Fourier cosine series expansion of $S(x): [0,L] \to R$

$$S(x) = a_0 + \sum_{n=1}^{\infty} a_n\, cos\left(\frac{n\pi x}{L}\right) \quad (3.13)$$

with

$$a_0 = \frac{\langle g,1\rangle}{\langle 1,1\rangle} = \frac{1}{L}\int_0^L S(x)dx.$$

And

$$a_{n>1} = \frac{\langle S,\ cos\left(\frac{n\pi x}{L}\right)\rangle}{\langle cos\left(\frac{n\pi x}{L}\right), cos\left(\frac{n\pi x}{L}\right)\rangle} = \frac{\langle s,\ cos\left(\frac{n\pi x}{L}\right)\rangle}{\int_0^L cos^2\left(\frac{n\pi x}{L}\right)dx}$$

$$= \frac{\langle S,\ cos\left(\frac{n\pi x}{L}\right)\rangle}{\frac{1}{L}\int_0^L \left(1 + cos\left(\frac{n\pi x}{L}\right)\right)dx}$$

$$= \frac{2\int_0^L S(x)\, cos\left(\frac{n\pi x}{L}\right)dx}{L}.$$

So that

$$S(x) = \frac{1}{L}\int_0^L S(x)dx + \sum_{n=1}^{\infty} \frac{2\int_0^L S(x)\, cos\left(\frac{n\pi x}{L}\right) cos\left(\frac{n\pi x}{L}\right)dx}{L}.$$

This implies that the particular series solution based on the given boundary condition is

$$u(x,t) = \sum_n \frac{2\int_0^L S(x)\, cos\left(\frac{n\pi x}{L}\right) cos\left(\frac{n\pi x}{L}\right)dx}{L} exp\left\{-\left(\left(\frac{n\pi}{L} - \frac{3}{2}\left(\frac{C^2R}{2\pi}\right)\sigma^2|x\partial_x^2 u|^{\frac{4}{3}}\right)t\right)\right\}$$

$$= \sum_n \frac{\int_0^L S(x)(1 - cos\, 2x)dx}{L} exp\left\{-\left(\left(\frac{n\pi}{L} - \frac{3}{2}\left(\frac{C^2R}{2\pi}\right)\sigma^2|x\partial_x^2 u|^{\frac{4}{3}}\right)t\right)\right\}.$$

$$= \sum_n \frac{\int_0^L S(x)(1 - cos\, 2x)dx}{L} e^{h_2(t)}. \quad (3.14)$$

Here $h_2(t) = \left(\frac{n\pi}{L} - \frac{3}{2}\left(\frac{C^2R}{2\pi}\right)\sigma^2|x\partial_x^2 u|^{\frac{4}{3}}\right)$ and is equal to, greater than or less than zero depending where $\left(\frac{n\pi}{L} - \frac{3}{2}\left(\frac{C^2R}{2\pi}\right)\sigma^2|x\partial_x^2 u|^{\frac{4}{3}}\right) =, >, or < 0.$

Again if instead the boundary conditions becomes

$$\left.\begin{aligned} u(1,t) + u'(1,t) = 0 \\ u(x,0) = f(x) \end{aligned}\right\}. \quad (3.15)$$

Thus we obtain an infinite number of eigenvalues

$\lambda = \mu_n^2, n \in \{1,2,\dots\},$

with corresponding eigenfunctions

$\varphi_n = sin(\mu_n x).$

A function $f(x): [0,1] \to R$ that is piecewise continuous with piecewise continuous derivative $f'(x)$ has an eigefunction expansion;

$$\sum_{n=1}^{\infty} c_n \sin(\mu_n x),$$

where

$$c_n = \frac{\langle f(x), \sin(\mu_n x)\rangle}{\langle \sin(\mu_n x), \sin(\mu_n x)\rangle} = \frac{\int_0^1 f(x)\sin(\mu_n x)dx}{\int_0^1 \sin^2(\mu_n x)dx}$$

$$= \frac{\int_0^1 f(x)\sin(\mu_n x)dx}{\frac{1}{2}\int_0^1 (1-\cos(2\mu_n x))dx} = \frac{\int_0^1 f(x)\sin(\mu_n x)dx}{\frac{1}{2}-\frac{1}{4\mu_n}\sin(2\mu_n x)}$$

$$= \frac{\int_0^1 f(x)\sin(\mu_n x)dx}{\frac{1}{2}(1-\frac{1}{\mu_n}\sin(\mu_n x)\cos(\mu_n x)} = \frac{\int_0^1 f(x)\sin(\mu_n x)dx}{\frac{1}{2}(1+COS^2(\mu_n x))}.$$

Therefore

$$\sum_{n=1}^{\infty} c_n \sin(\mu_n x) = \sum_{n=1}^{\infty} \frac{2}{1+COS^2(\mu_n x)} \int_0^1 f(x)\sin(\mu_n x)\sin(\mu_n x)dx,$$

so that

$$u(x,t) = \sum_n \frac{2}{1+COS^2(\mu_n x)} \int_0^1 f(x)\sin^2(\mu_n x)dx\, e^{-\left(\mu_n t - \int_0^t k d\tau\right)}$$

$$= \sum_n \left(2\int_0^1 \sin(\mu_n x)\,dx\right) exp\left\{-\left(\mu_n - \frac{3}{2}\left(\frac{C^2R}{2\pi}\right)\sigma^2|x\partial_x^2 u|^{\frac{4}{3}}\right)t\right\}$$

$$= \sum_n \frac{2(\cos\mu_n - 1)}{\mu_n} exp\left\{-\left(\mu_n - \frac{3}{2}\left(\frac{C^2R}{2\pi}\right)\sigma^2|x\partial_x^2 u|^{\frac{4}{3}}\right)t\right\}$$

$$= \sum_n \frac{2(\cos\mu_n - 1)}{\mu_n} e^{h_3(t)}, \qquad (3.16)$$

where

$$h_3(t) = \left(\mu_n - \frac{3}{2}\left(\frac{C^2R}{2\pi}\right)\sigma^2|x\partial_x^2 u|^{\frac{4}{3}}\right)t.$$

Notice that $h_3(t) = 0$ when $\mu_n = \frac{3}{2}\left(\frac{C^2R}{2\pi}\right)\sigma^2|x\partial_x^2 u|^{\frac{4}{3}}$,$h_3(t) < 0$ if $\mu_n < \frac{3}{2}\left(\frac{C^2R}{2\pi}\right)\sigma^2|x\,\partial_x^2 u|^{\frac{4}{3}}$ and $h_3(t) > 0$ if $\mu_n > \frac{3}{2}\left(\frac{C^2R}{2\pi}\right)\sigma^2|x\,\partial_x^2 u|^{\frac{4}{3}}$.

4.0 Discussion and Conclusion

We obtained in this work the solution of Black- Scholes equation with transaction cost measure and volatile portfolio risk measure using Sturm- Liouville solution method. Here we described first the nature of the eigenvalues and then, turn to the associated solutions (the eigenfunctions).

It is noteworthy to observe that $a_n \sin\left(\sqrt{\frac{(2n-1)^2}{4}}\right)x$, $a_n \sin\left(\frac{n\pi}{L}\right)x$ and $c_n \sin(\mu_n x)$ and the sum $\left(a_n \sin\left(\sqrt{\frac{(2n-1)^2}{4}}\right)x + b_n \cos\left(\sqrt{\frac{(2n-1)^2}{4}}\right)x\right)$are circular functions of $\left(\sqrt{\frac{(2n-1)^2}{4}}\right)x$,$\left(\frac{n\pi}{L}\right)$xand $\mu_n x$ respectively, with period 2π and amplitude 1. That is the value function $u(x,t)$ will repeat its configuration every time$\left(\sqrt{\frac{(2n-1)^2}{4}}\right)x$, $\left(\frac{n\pi}{L}\right)xor\mu_n x$ increases by 2π. So equations (3.9), (3.12), (3.14) and (3.16) will also display a repeating cycle every time x increases.

Had $u(x,t)$ consisted only of the expression $a_n \sin(.)x$ in each case, the implication would have been that the time path of u would be a never-ending, constant amplitude fluctuation around the equilibrium value of u. But there is also a multiplication term $e^{h_i(t)}$, $(i = 1, 2, 3$, for the equation under analysis).

For instance $h_i > 0$ implies that $e^{h_i(t)}$ increases continually ast increases then $u(x,t)$ will deviate from the equilibrium value and the time path will be characterized by explosive fluctuation. If $h_i = 0$, then $e^{h_i(t)} = 1$. This implies constant amplitude. That is $u(x,t)$ will display a uniform pattern of deviation from the equilibrium and there will be uniform fluctuations. On the other hand if $h_i < 0$, then $e^{h_i(t)}$will continuously decrease as t increases and each successive cycle will have smaller amplitude than the preceding one and $u(x,t)$ be characterized by damped fluctuations.

The condition of convergence of $u(x,t)$ is when $h_i < 0$. Therefore the equilibrium is dynamically stable if and only if $u(x,t)$ is convergent.

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