Transactions of the Nigerian Association of Mathematical Physics
***Volume 2, (November, 2016*), pp 291 – 298**
© Trans. of NAMP

Bingham Fluid Model for Steady Flow in a Multi-Irregular Stenosed Artery

E.O. Oghre & F.M. Okoronkwo

Department of Mathematics, University of Benin, Benin City.

Abstract

The effect of multi-irregular shaped stenoses on non-Newtonian fluid through an axially symmetric, laminar, steady, one-dimensional flow of blood in the artery has been investigated. Blood has been treated as a non-Newtonian fluid obeying Bingham plastic fluid equation in which fluid does not flow beyond the yield stress value, and above the yield value, the fluid flow is possible. The artery is model as having irregular stenoses.The problem is solved using analytical techniques. The expressions for the flow characteristics, namely, the blood pressure gradient, the skin friction, the blood flow rate and velocity are obtained. Variations in volumetric flow rate with increasing axial variable for different blood viscosity and yield stress parameter have been shown graphically. It is observed that the flow rate increases with increasing yield stress values and decreases for increasing viscosity. We conclude that the multi-irregular shaped stenosis has more effect on non-Newtonian flow of blood than Newtonian fluid.

Keywords: Non-Newtonian fluid flow rate, pressure gradient, shear stress, stenosis, wall shear stress.

1.0 Introduction

The unnatural and abnormal growth at various location along the conduits of the cardiovascular system under diseased conditions is known as stenosis [1]. This formation of lipids in the intima may cause stenosis of the lumen leading to hardening and thickening of the arterial walls[2].Pathology within the artery wall could be the cause of stenosis since the exact causes of stenosis have not been ascertained. When this is the case, serious complications may occur due to the significant changes in blood pressure, wall shear stress, flow rate and velocity within the artery [3, 4].

In this study we investigate the flow of blood in a multi-irregular shaped stenosed artery and regarding blood as a Bingham fluid. Several researchers have considered mathematical modeling of blood flow in narrow arteries. Singh and Singh [5] analyzed a fully developed one dimensional Bingham plastic flow of blood through a small artery having multiple stenosis and post dilation. Sanjeev and Diwaka [6] studied the effect of a post dilation and multiple stenosis on an arterial blood flow. They observed that the numerical values of the resistance to flow ratio vary from maximum radii of the two abnormal segments to the minimum radii.Pankaj *et.al* [7] investigated the non-Newtonian flow of blood through a stenosed artery segment using power law model. They concluded that the thermodynamic behavior of blood flow is influenced by the presence of the arterial stenosis.Yadav and Kuma [8] have studied the behavior of non-Newtonian blood flow through a stenosed artery using power law fluid model. It was ascertained that resistance to flow increases with stenosis size for different index values of flow index behavior. Sanjeev and Chandrashekhar [9] demonstrated the effect of size of stenosis on blood flow through an artery where blood behaves like a power law fluid in a uniform circular tube with axially non-symmetric but radially symmetric stenosis. Ranadhir *et. al* [10] examine the characteristics of unsteady blood flow in an artery with a time dependent stenosis using power law fluid model.Sankar *et. al* [11] developed a mathematical analysis of single and two phase flow of blood in narrow arteries with multiple constriction. They analyze the pulsatile flow of blood in narrow arteries with multiple stenosis under body acceleration mathematically treating blood as single and two phase Herschel Bulkley fluid model. Kumar [12] considered the mathematical model for blood flow through a narrow artery with multiple stenosis. They discovered that shear stress increases as the stenosis height increases. Misra and Shit [13] investigated blood flow through arterial segment assuming blood asHerschel Bulkley fluid. They concluded that the skin

Corresponding author: E.O. Oghre, E-mail: eoghre@yahoo.com, Tel.: +2348033551266

friction and the resistance to flow is maximum at the throat of the stenosis and minimum at the end. Sreenadh *et.al* [14] demonstrated the problem of steady flow of the Casson fluid through an inclined tube of non-uniform cross section with multiple stenosis. The analytical solution was evaluated for velocity, flow rate and resistance to flow. Venkatesan *et. al*, [15] investigated the flow of blood through a narrow artery with bell- shaped stenosis, treating blood as Casson fluid. The result was compared with the results of Herschel-Bulkley fluid model obtained by Misra and Shit [13] for the same geometry. In their study they observed that the resistance to flow and skin friction increase with the increase of maximum depth of the stenosis but these flow quantities when normalized, decrease with the increase of the yield stress as obtained by Misra and Shit [13].Srivastava*et. al* [16] considered the effects of an overlapping stenosis on blood flow characteristics in a narrow artery. To account for the non-Newtonian behavior, blood has been represented by a Casson fluid. They discovered that the impedance increases with the non-Newtonian behavior of blood as well as with the stenosis size.The shear stress at the stenosis two throats assumes the same magnitude. The shear stress at the stenosis critical height assumes significantly lower magnitude than its corresponding value at the throats. With respect to any given parameter, the nature of the variations of shear stresses at the throats and at the critical height of the stenosis is similar to that of the flow resistance.Lokendra *et. al* [17] investigated blood flow through an axially symmetric but radially stenosed artery representing blood by a non-Newtonian fluid obeying Casson fluid equation. They observed that the wall shear stress increases for the increases stenosis height. Wall shear stress decreases with increasing trend of shape parameter. Verma [18] examined the axially symmetric, laminar, steady, one-dimensional flow of blood through narrow stenotic vessel by considering blood as Bingham plastic fluid. It was shown that the wall shear stress and resistance to flow increase with the size of stenosis but this increase are smaller due to non-Newtonian behavior of the blood.

2.0 Mathematical Formulation

Let us consider an axially symmetric, laminar, steady flow of a non – Newtonian viscous fluid (blood). We consider the flow of blood in a multi irregular shaped stenosis when the blood is taken as a non – Newtonian. The blood flow is characterized by Bingham plastic model. The radius of the tube is represented by:

$$\frac{R(z)}{R_0} = \begin{cases} 1 - \frac{2h}{lR_0} \quad (z-d), & d \le z \le d + \frac{l}{2} \\ 1 + \frac{2h}{lR_0}(z-d-l), & d + \frac{l}{2} < z \le d + l \\ 1 - \frac{h}{R_0} + \frac{4h}{l^2 R_0}\left(z - d - \frac{3l}{2}\right)^2, & d + l < z \le d + 2l \end{cases} \qquad (1)$$

Where $R(z)$ is the radius of the artery with constriction, R_0 is the constant radius, h the maximum height of the stenosis , d indicate the location and 2L is the total length of the stenosis.

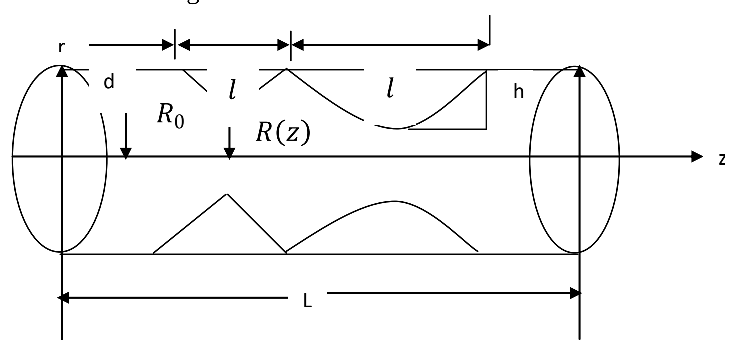

Fig 1: Geometry of Multi-Irregular Shaped Stenosed Artery.

The constitutive equation for Bingham fluid model is expressed as:

$$\dot{e} = f(\tau) = -\frac{dv_r}{dr} = \frac{1}{\mu}(\tau_{rz} - \tau_b) \quad , \quad \tau_{rz} \ge \tau_b$$

$$f(\tau) = 0 \qquad \tau_{rz} \le \tau_b \qquad (2)$$

The approximation equation of motion governing the flow field in the tube is:

$$0 = -k - \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}) \qquad (3)$$

Where $k = \frac{\partial p}{\partial z}$, z and r are axial and radial coordinate, v_r is the axial velocity, τ_{rz} and τ_b are the shear and yield stresses, p is the blood pressure and μ is the velocity.

The appropriate boundary conditions are:

$v_r = 0$ at r = R(Z) (4)

τ_{rz} is finite at r = 0 (5)

$v_r = 0$ at r = 0 (6)

The volumetric equation as given by Robinowitsch equation is:

$$Q = \int_0^R 2\pi r v_r dr \quad (7)$$

Now solving equation (3) under boundary conditions (4) and (6) and then using we get,

$$Q = 2\pi \int_0^R \left(-\frac{dv_r}{dr}\right)\frac{r^2}{2} dr \quad (8)$$

Putting the value of $-\frac{dv_r}{dr}$ in (2) we have:

$$Q = \pi \int_0^R f(\tau)\frac{R^2}{\tau_w^2}\tau_{rz}^2 \frac{R}{\tau_w} d\tau_{rz} \quad (9)$$

Since $\tau_w = -\frac{rk}{2}$ (10)

Substituting the value of $f(\tau)$ from equation (2) into equation (9) and then integrating , we have

$$Q = \frac{\pi R^3}{\mu}\left[\frac{\tau_w}{4} - \frac{\tau_b}{3}\right] \quad (11)$$

Putting equation (10), we have the flow rate which is dependent upon $R^4(z)$ and τ_b

$$Q = -\frac{\pi R^4(z)}{8\mu}\left(\frac{dp}{dz}\right) - \frac{\pi R^3 \tau_b}{3\mu} \quad (12)$$

since

$$\tau_w = \frac{4\mu Q}{\pi R^3} + \frac{4\tau_b}{3} \quad (13)$$

invoking equation (10) into equation (13), we obtain the pressure gradient for stenotic region.

$$k = -\frac{2}{R}\left[\frac{4\mu Q}{\pi R^3} + \frac{4\tau_b}{3}\right] \quad (14)$$

The skin friction for normal arteryat (R = R_0 ,$\tau_b = 0$ and h = 0) is obtained from (12)

$$\tau_N = -\frac{4\mu Q}{\pi {R_0}^3} \quad (15)$$

If there is stenosis, skin friction will depend on the geometry of the artery. Using equation (1) and (13) we obtain:

$$\tau_{w1} = \left[\frac{4\mu Q}{\pi\left\{R_0 - \frac{2h}{l}\ (z-d)\right\}^3} + \frac{4\tau_b}{3}\right] \quad \text{at} \quad d \le z \le d + \frac{l}{2}$$

$$\tau_{w2} = \left[\frac{4\mu Q}{\pi\left\{R_0 + \frac{2h}{l}\ (z-d-l)\right\}^3} + \frac{4\tau_b}{3}\right] \quad \text{at} \quad d + \frac{l}{2} < z \le d + l$$

$$\tau_{w3} = \left[\frac{4\mu Q}{\pi\left\{R_0 - h + \frac{4h}{l^2}\ \left(z-d-\frac{3l}{2}\right)^2\right\}^3} + \frac{4\tau_b}{3}\right] \quad \text{at} \quad d + l < z \le d + 2l \quad (16)$$

$$\bar{\tau}_{w1} = \frac{\tau_w}{\tau_N} = \frac{{R_0}^3}{\left\{R_0 - \frac{2h}{l}\ (z-d)\right\}^3} + \frac{\pi\tau_b {R_0}^3}{3\mu Q}$$

$$\bar{\tau}_{w2} = \frac{\tau_w}{\tau_N} = \frac{{R_0}^3}{\left\{R_0 + \frac{2h}{l}\ (z-d-l)\right\}^3} + \frac{\pi\tau_b {R_0}^3}{3\mu Q}$$

$$\bar{\tau}_{w3} = \frac{\tau_w}{\tau_N} = \frac{{R_0}^3}{\left\{R_0 - h + \frac{4h}{l^2}\ \left(z-d-\frac{3l}{2}\right)^2\right\}^3} + \frac{\pi\tau_b {R_0}^3}{3\mu Q} \quad (17)$$

Dividing equation (17) by ${R_0}^3$ and using the interval at maximum height of the stenosis $\left(z = d + \frac{l}{2}\right)$ we obtain the dimensionless skin friction:

$$\bar{\tau}_{w1} = \frac{1}{\left(1 - \frac{h}{R_0}\ \right)^3} + \frac{\pi\tau_b}{3\mu Q}$$

$$\bar{\tau}_{w2} = \frac{1}{\left(1 - \frac{h}{R_0}\ \right)^3} + \frac{\pi\tau_b}{3\mu Q}$$

$$\bar{\tau}_{w3} = \frac{1}{\left(1 - \frac{h}{R_0} + \frac{4h}{R_0}\ \right)^3} + \frac{\pi\tau_b}{3\mu Q} \quad (18)$$

When $\tau_b = 0$ in equation (13) the skin friction is same with Das *et. al* [19] for Newtonian fluid.

Now, integrating equation (2) in the region $\tau_{rz} \ge \tau_b$ and since $\frac{\tau_{rz}}{\tau_w} = \frac{r}{R}$, we obtain the blood velocity within the stenotic region.

$$-v_r = \frac{R}{\mu\tau_w}\left[\frac{\tau_{rz}^2}{2} - \tau_b\tau_{rz}\right]_0^{\tau_w} \quad (19)$$

$$-v_r = \frac{R}{2\mu}\left[\tau_w - 2\tau_b\right] \quad (20)$$

Substituting equation (13) into equation (20) we obtain

$$v_r = \frac{R^2}{4\mu}\left(\frac{dp}{dz}\right) + \frac{R\tau_b}{\mu} \quad (21)$$

3.0 Results and Discussion

In order to illustrate the analytic expression of blood graphically, the following parameter values was used for the purpose of computation.

Table 1: Parameter values.

Parameter	Nominal value	Reference
μ	1	[19]
R_0	2	[20]
τ_B	0.2- 0.6	Assumed
l	1	[19]
μ	0.2 – 0.6	Assumed
d	1	[19]
h/R_0	0.2 – 0.5	Assumed
L	10	[19]

3.1 Skin Friction

Figure 2: described the effect of stenosis size on the skin friction (wall shear stress) at the surface of the stenosis from equation (18). The wall shear stress was plotted against the axial variable for different values of stenosis size (h/R_0 = 0.2, 0.3, 0.4 and 0.5) taking the yield stress as constant. It is noticed that the wall shear stress increases with an increase in the value of stenosis size. It also increases for increasing values of the axial variable up to the mid-point of the stenosis and then decreases after the stenotic region.

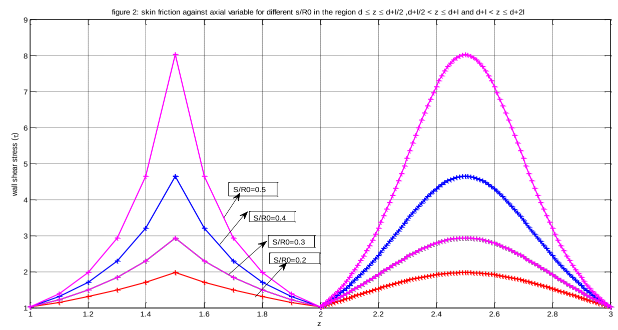

Fig. 2:

3.2 Pressure Gradient

Figure 3: Illustrate the effect of viscosity of blood on the blood pressure. The graph of the pressure gradient $\left|\frac{dp}{dz}\right|$ against axial variable z with different values of blood viscosity μ has been plotted from equation (14). Taking volumetric flow rate Q and the yield stress τ_b as constant (Q=1 and $\tau_b = 0.02\text{n}/m^2$). It is clear that the pressure gradient decreases within the stenotic region and then increases after the stenotic region for increasing values of blood viscosity.

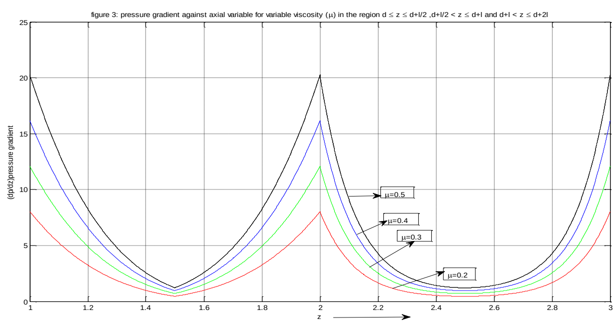

Fig. 3:

3.3 Flow Rate

Figure 4: depicts the effect of yield stress on blood flow rate. From equation (12), the flow rate has been plotted against axial variable for different values of yield stress. Considering blood viscosity and pressure gradient as constant ($\mu = 1,\ \frac{dp}{dz} = 1$). It is obvious that the flow rate |Q| decreases with an increase in the value of the axial variable and decreases after the mid-point of the stenosis and then increases again. It is also clear that the flow rate increases for increasing values of yield stress.

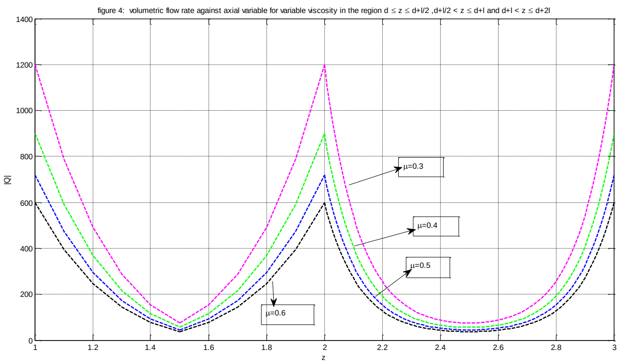

Fig. 4:

Figure 5: indicate the effect of viscosity on the flow rate for stenotic region for different values of viscosity. Here, we have taken the yield stress and pressure gradient has constant (τ_b = 0.02n/m^2, $\frac{dp}{dz} = 1$). We have plotted the absolute value of blood flow rate within the stenotic region from equation (12) against the axial variable (z). We put the different values of z in the geometry to obtain the relation for flow rate. It is obvious that |Q| decreases with the increase of z within the stenotic region and then increases after the stenotic region. Flow rate also decreases for increasing viscosity of blood within the stenotic region.

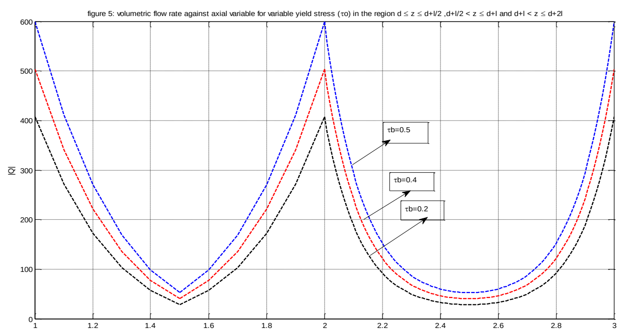

Fig. 5:

3.4 Velocity

Figure 6: demonstrate the graph of axial velocity of blood against the axial variable for different values of viscosity (μ= 0.2, 0.3, and 0.4). We have taken the pressure gradient and the yield stress as constant ($\frac{dp}{dz}$=1 andτ_b = 0.02n/m^2). It is observed that axial velocity of blood decreases within the stenotic region with the increase in axial variable up to the mid-point and then increases again. It is also notice that as viscosity increases, velocity of blood also decreases.

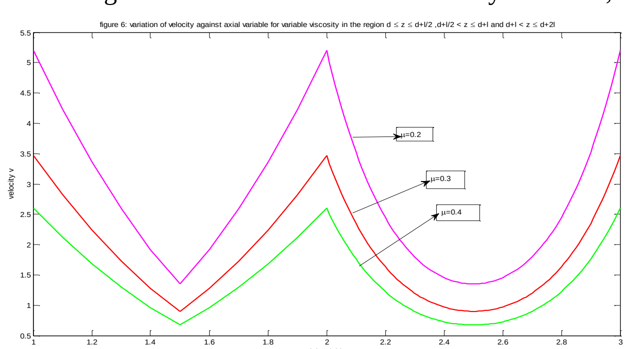

Fig. 6:

In figure 7: We have examined the effect of yield stress on the blood velocity. The axial velocity obtained from equation (21) has been plotted against axial variable z for various values of the yield stress ($\tau_b = 0.2, 0.4\ and\ 0.6$).It is true that the velocity of blood decreases with an increase in the axial variable up to the mid-point of the stenosis and then increases after the stenosis with the increasing yield stress values.

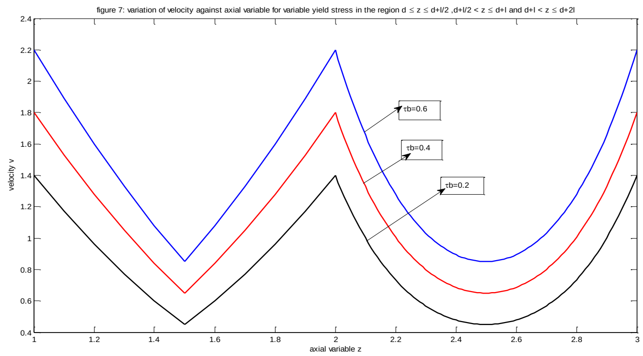

Fig. 7:

4.0 Conclusion

The present study analyzed the steady flow of blood in a narrow arterial segment with multi-irregular shaped stenosis regarding blood to act as a non-Newtonian fluid, possessing a finite yield stress that is a Bingham plastic fluid in which the shear stress is proportional to the shear rate, and the result are compared with Das *et al*,2014 [18] for Newtonian fluid. We discovered that for Newtonian and non –Newtonian fluid:

- Flow rate decreases with the increase in dynamic viscosity of blood.
- Wall shear stress or skin friction increases for increase in stenosis size.
- Blood pressure increases and decreases within the stenotic region for increasing blood viscosity.
- Axial velocity of blood increases with the increase in axial variable within the stenotic region.

The major findings of the present study for non-Newtonian fluid are listed below:

I. Blood flow rate Q increases significantly with the increasing values of yield stress.

II. The blood velocity increases with the increasing yield stress values.

Therefore, the differences between the Newtonian and non-Newtonian models (Bingham fluid) show that the non-Newtonian behavior is an important factor and should not be neglected.

Hence, the present analysis may be useful in analyzing blood flow in disease state. We conclude that the multi-irregular shaped stenosis have more effect on non-Newtonian fluid than Newtonian fluid.

5.0 References

[1] Young D.F.(1979).”Fluid mechanics of arterial stenosis”. ASME J. Biomech. Eng. 101, 157-175.

[2] Fang-Bao Tiana , Luoding zhub, n, Pak-Wing Fokc, Xi-Yun Lud (2013). ”Simulation of a pulsatile non-Newtonian Flow Post Stenosed 2D artery with atherosclerosis”. Computers in Biology and Medicine 43, 1098-1113.

[3] Young D.F. and Tsai F.Y. (1973).”Flow characteristics in model of arterialstenosis-steady flow”. Journal of Biomechanics 6, .395-410.

[4] Young D.F.(1968). Effect of a time-dependent stenosis of flow through a tube. Journal of Eng. Ind. 90,.248-254.

[5] Singh A.K. and Singh D. P. (2012). A Computational study of BinghamPlastic Flow of Blood through an artery by multiple stenoses and post dilatation.Advance in Applied.Science Research. (ISSN: 0976:8610).

[6] Sanjeev Kumar and Chandrashekhar Diwakar (2013).”Blood Flow Resistance For a Small Artery with the Effect of Multiple Stenoses and Post StenoticDilatation”. International Journal of Engineering Sciences and Emerging Technologies. Vol. 6, 57-64

[7] Pankaj Mathur and Surekha Jain (2013).”Mathematical Modelling of Non-**Applied** Newtonian Blood Flow through Artery in the Presence of Stenosis”. Advances in Mathematical Biosciences. ISSN 2248-9983 Volume 4, PP. 1-12.

[8] Yadav S. S. and Krishna Kumar (2012). Effect of Resistance to flow and wall shear stress on axially symmetric artery for Non-Newtonian fluid Model of blood flow through a stenosis. International Journal of Mathematics Trends and Technology-Volume 3.

[9] Sanjeev Kumar and Chandrashekhar Diwakar (2013).”A Mathematical Model Of Power Law Fluid with an Application of Blood Flow through an Artery with Stenosis. Advances in Applied Mathematical Biosciences.Volume 2, PP. 51-61.

[10] Daniel N.Riahi, Ranadhir Roy,and Nelson Carrasquero 2014.”Mathematical Modelling of Blood Flow in an Artery with an Unsteady Stenosis using Power-Law Fluid Model. Transactions on Applied Mathematics. Volume 1, Number 2. 3(1) (2015)335-345.

[11] Sankar D .S., Nagar A. K. and Kumar A.V,(2014).”Mathematical Analysis of Single and Two-Phase Flow of Blood in Narrow Arteries with MultipleConstrictions”. Journal of Applied Fluid Mechanics, Vol. 8, No. 4, 871- 883.

[12] Kumar Ratan Bose (2012).”A Mathematical Model for Blood Flow through a Narrow Artery with Multiple Stenoses”. Journalof Applied Mathematics and Fluid Mechanics. ISSN 0974-3170 Volume 4, 233-242.

[13] Mistra J. C. and Shit G.C. (2006).” Blood flow through arteries in a Pathological State. A theoretical study International Journal of Engineering Science, 44,662- 671.

[14] Sreenadh S., Raga Pallavi and Bh.Satyanarayana (2011).”Flow of a CassonFluid Through an Inclined Tube of Non- uniform Cross Section withMultiple Stenoses”. Advance in Applied Science Research.(ISSN: 0976- 8610).

[15] Venkatesan J., Sankar D. S., Hemalatha K. and Yazariah Yatim (2013). Mathematical Analysis of Casson Fluid Model for Blood Rheology inStenosed Narrow Arteries. Journal of Applied Mathematics. Article ID 583809,II Pages.

[16] Srivastava V. P., Rochana Vishnoi, Shailesh Mishra and Poonam Sinha (2011).” A Two layer Non-Newtonian Arterial Blood Flow through anOverlapping Constriction”. AAM: Intern. J., Vol. 6, 1781-1797.

[17] Lokendra Parmar, S. B Kulshreshtha and D. P. Singh (2013).”Effects of Stenoses on Casson Flow of Blood Through Arteries”. InternationalJournal of Advance Computer and Mathematics Sciences. Vol. 4, 257- 268.

[18] S.R Verma (2014). MATHEMATICAL Modeling of Bingham plastic Model of Blood Flow Through Stenotic Vessel. Journal of Engineering Researchand Applications. Vol. 4, 11-17. ISSN: 2248-9622.

[19] Satarupa Das, Soma Das, Satyasaran Changdar and Soumen De (2014). Analysis of blood flow through multi-irregular shaped stenosed artery”. International Journal of Pharmacy and Biological Sciences. Volume 4, (e- ISSN:2230-7605). 244

[20] Arun Kumar Maiti (2014).“Effect of Stenosis on Bingham Plastic Flow of Blood Through an Arterial Tube”. International Journal of Mathematics. Volume 13, Number 1